Representation of a Plane Wave.

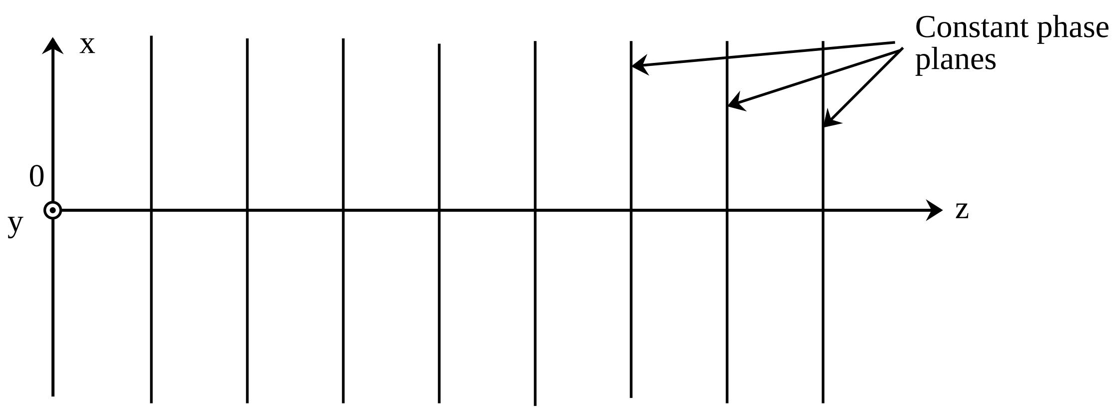
When $\nabla \cdot \mathbf{E} = 0$, the electric field satisfies the wave equation

$$\nabla^2 \mathbf{E} + \beta^2 \mathbf{E} = 0, \tag{1}$$

where $\beta^2 = \omega^2 \mu \epsilon$. We have learnt that one of the many possible solutions to the above equation is

$$\mathbf{E} = \hat{x} E_0 e^{-j\beta z}. \tag{2}$$

The expression $e^{-j\beta z}$, when viewed in three dimensions, has constant phase planes or wavefronts which are orthogonal to the z-axis.



To denote a plane wave propagating in other directions, we write it as

$$\mathbf{E} = \hat{a} E_0 e^{-j\beta_x x - j\beta_y y - j\beta_z z}, \tag{3}$$

where $\hat{a}$ is a constant unit vector, and E_0 a constant. If we substitute (3) into (1), we obtain

$$[-\beta_x^2 - \beta_y^2 - \beta_z^2 + \beta^2] E_0 = 0. \tag{4}$$

In order for (3) to satisfy (1) and that $E_0 \neq 0$, we require that

$$\beta_x^2 + \beta_y^2 + \beta_z^2 = \beta^2 = \omega^2 \mu \epsilon. \tag{5}$$

If we define a vector $\boldsymbol{\beta} = \hat{x}\beta_x + \hat{y}\beta_y + \hat{z}\beta_z$, and $\mathbf{r} = \hat{x}x + \hat{y}y + \hat{z}z$, then (3) can be written as

$$\mathbf{E} = \hat{a}\, E_0 e^{-j\boldsymbol{\beta} \cdot \mathbf{r}}, \tag{6}$$

where the magnitude of $\boldsymbol{\beta}$ is

$$|\boldsymbol{\beta}| = [\beta_x^2 + \beta_y^2 + \beta_z^2]^{\frac{1}{2}} = \beta. \tag{7}$$

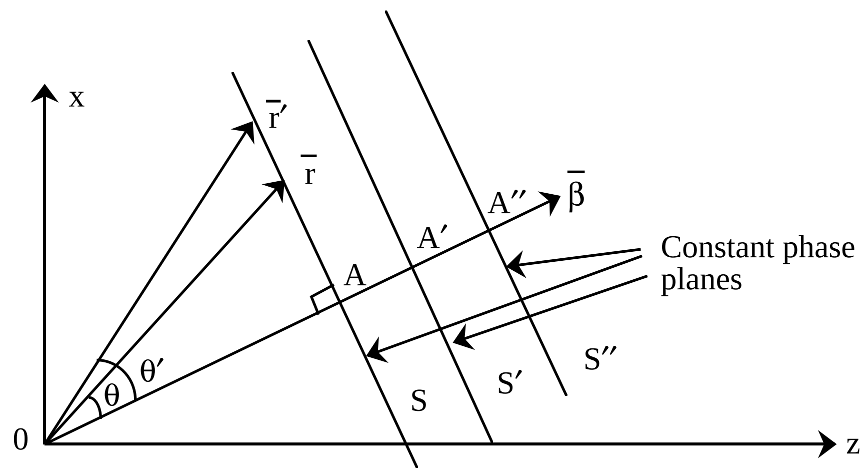
Equation (6) is a concise way to write a solution to (1). Since $\nabla \cdot \mathbf{E} = 0$ using (3), we note that

$$\nabla \cdot \mathbf{E} = -j[\hat{x}\beta_x + \hat{y}\beta_y + \hat{z}\beta_z] \cdot \hat{a}E_0 e^{-j\boldsymbol{\beta}\cdot\mathbf{r}}. \tag{8}$$

Therefore, in order for $\nabla \cdot \mathbf{E} = 0$, we require that

$$\boldsymbol{\beta} \cdot \hat{a} = 0. \tag{9}$$

To explore further how the function $e^{-j\boldsymbol{\beta}\cdot\mathbf{r}}$ look like, we assume $\boldsymbol{\beta}$ to be pointing in a direction as shown in the figure. The value of $\boldsymbol{\beta} \cdot \mathbf{r}$ is constant on a plane that is orthogonal to $\boldsymbol{\beta}$.



That is

$$\boldsymbol{\beta} \cdot \mathbf{r} = |\boldsymbol{\beta}|\,|\mathbf{r}| \cos\theta = \beta(OA), \tag{10}$$

for all $\mathbf{r}$ on the plane S that is orthogonal to $\boldsymbol{\beta}$. Hence, S is the constant phase plane of $e^{-j\boldsymbol{\beta}\cdot\mathbf{r}} = e^{-j\beta(OA)}$. As one moves progressively in the $\boldsymbol{\beta}$ direction, the function $e^{-j\boldsymbol{\beta}\cdot\mathbf{r}}$ has a phase that is linearly decreasing with distance. Therefore, $e^{-j\boldsymbol{\beta}\cdot\mathbf{r}}$ denotes a plane wave that is propagating in the $\boldsymbol{\beta}$ direction. When $\boldsymbol{\beta}$ is pointing in the z-direction, such that $\boldsymbol{\beta} = \hat{z}\beta$, then $e^{-j\boldsymbol{\beta}\cdot\mathbf{r}} = e^{-j\beta z}$, which is our familiar solution of a plane wave propagating in the z-direction.

An example of a plane wave electric field satisfying Maxwell's equations is

$$\mathbf{E} = \hat{y}E_0 e^{-j\beta_x x - j\beta_z z}, \tag{11}$$

where $\beta_x^2 + \beta_z^2 = \beta^2$. The corresponding magnetic field can be derived using Maxwell's equations.

$$\nabla \times \mathbf{E} = -j\omega\mu\mathbf{H}. \tag{12}$$

Hence,

$$\begin{aligned}\mathbf{H} &= \frac{-1}{j\omega\mu}\left(\hat{z}\frac{\partial}{\partial x}E_y - \hat{x}\frac{\partial}{\partial z}E_y\right) \\ &= (\hat{z}\beta_x - \hat{x}\beta_z)\,\frac{E_0}{\omega\mu}e^{-j\beta_x x - j\beta_z z}. \end{aligned} \tag{13}$$

In general, when ∇ operates on a plane wave phasor described by $e^{-j\boldsymbol{\beta}\cdot\mathbf{r}}$, it transforms into $-j\boldsymbol{\beta}$. This is obvious also from Equation (8). Therefore, from (12), we can express

$$\mathbf{H} = \frac{1}{\omega\mu}\boldsymbol{\beta} \times \mathbf{E}. \tag{14}$$

Therefore, $\mathbf{H}$ is orthogonal to both $\mathbf{E}$ and $\boldsymbol{\beta}$, or that $\mathbf{H} \cdot \mathbf{E} = 0$, and that $\mathbf{H}\cdot\boldsymbol{\beta} = 0$, in addition to $\mathbf{E}\cdot\boldsymbol{\beta} = 0$. Furthermore, $\mathbf{E}\times\mathbf{H}$ points in the direction of $\boldsymbol{\beta}$. Therefore, for a plane electromagnetic wave, $\mathbf{E}$, $\mathbf{H}$, and $\boldsymbol{\beta}$ form a right-handed orthogonal system. It is also a transverse electromagnetic (TEM) wave.