

Wave Polarization.

We learnt that

$$\mathbf{E} = \hat{x}E_x = \hat{x}E_1 \cos(\omega t - \beta z), \quad (1)$$

is a solution to the wave equation because $\nabla \cdot \mathbf{E} = 0$. Similarly,

$$\mathbf{E} = \hat{y}E_y = \hat{y}E_2 \cos(\omega t - \beta z + \phi), \quad (2)$$

is also a solution to the wave equation. Solutions (1) and (2) are known as **linearly polarized waves**, because the electric field or the magnetic field are polarized in only one direction. However, a linear superposition of (1) and (2) are still a solution to Maxwell's equation

$$\mathbf{E} = \hat{x}E_x(z, t) + \hat{y}E_y(z, t). \quad (3)$$

If we observe this field at $z = 0$, it is

$$\mathbf{E} = \hat{x}E_1 \cos \omega t + \hat{y}E_2 \cos(\omega t + \phi). \quad (4)$$

When $\phi = 90^\circ$,

$$E_x = E_1 \cos \omega t \quad E_y = E_2 \cos(\omega t + 90^\circ), \quad (5)$$

$$\text{When } \omega t = 0^\circ, \quad E_x = E_1, \quad E_y = 0. \quad (6)$$

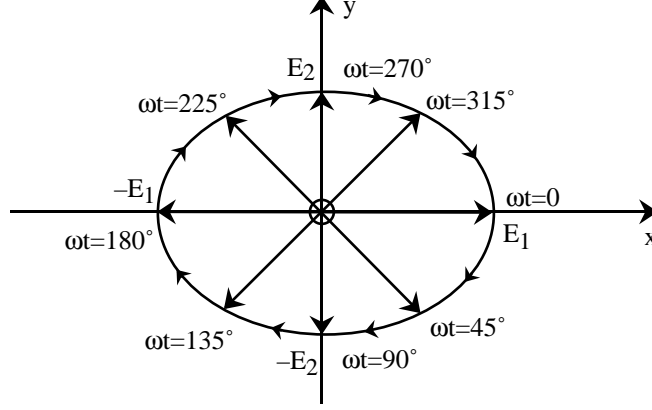
$$\text{When } \omega t = 45^\circ, \quad E_x = \frac{E_1}{\sqrt{2}}, \quad E_y = -\frac{E_2}{\sqrt{2}}. \quad (7)$$

$$\text{When } \omega t = 90^\circ, \quad E_x = 0, \quad E_y = -E_2. \quad (8)$$

$$\text{When } \omega t = 135^\circ, \quad E_x = -\frac{E_1}{\sqrt{2}}, \quad E_y = -\frac{E_2}{\sqrt{2}}. \quad (9)$$

$$\text{When } \omega t = 180^\circ, \quad E_x = -E_1, \quad E_y = 0. \quad (10)$$

If we continue further, we can sketch out the tip of the vector field \mathbf{E} . It traces out an ellipse as shown when $E_1 \neq E_2$. Such a wave is known as an **elliptically polarized wave**.



When $E_1 = E_2$, the ellipse becomes a circle, and the wave is known as a *circularly polarized wave*. When ϕ is -90° , the vector \mathbf{E} rotates in the counter-clockwise direction.

A wave is classified as *left hand elliptically (circularly) polarized* when the wave is approaching the viewer. A counterclockwise rotation is classified as *right hand elliptically (circularly) polarized*.

When $\phi \neq \pm 90^\circ$, the tip of the vector \mathbf{E} traces out a tilted ellipse. We can show this by expanding E_y in (5).

$$\begin{aligned} E_y &= E_2 \cos \omega t \cos \phi - E_2 \sin \omega t \sin \phi \\ &= \frac{E_2}{E_1} E_x \cos \phi - E_2 \left[1 - \left(\frac{E_x}{E_1} \right)^2 \right]^{\frac{1}{2}} \sin \phi. \end{aligned} \quad (11)$$

Rearranging terms, we get

$$aE_x^2 - bE_xE_y + cE_y^2 = 1, \quad (12)$$

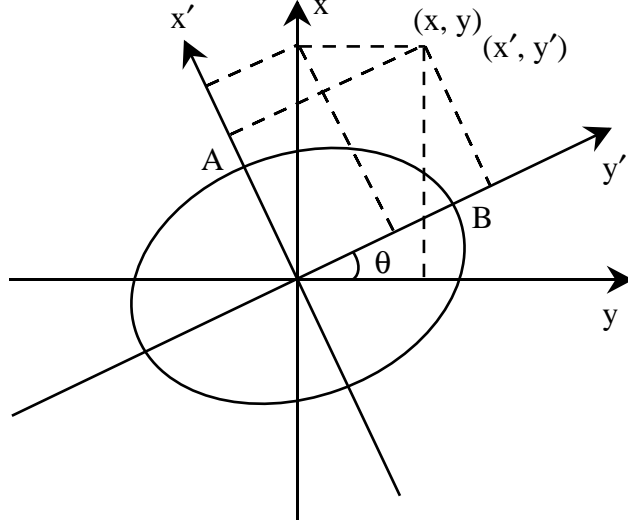
where

$$a = \frac{1}{E_1^2 \sin^2 \phi}, \quad b = \frac{2 \cos \phi}{E_1 E_2 \sin^2 \phi}, \quad c = \frac{1}{E_2^2 \sin^2 \phi}. \quad (13)$$

Equation (12) is of the form

$$ax^2 - bxy + cy^2 = 1, \quad (14)$$

which is the equation of a tilted ellipse.



The equation of an ellipse in its self coordinate is

$$\left(\frac{x'}{A}\right)^2 + \left(\frac{y'}{B}\right)^2 = 1, \quad (15)$$

where A and B are the semi-axes of the ellipse. However,

$$x' = x \cos \theta - y \sin \theta, \quad (16)$$

$$y' = x \sin \theta + y \cos \theta, \quad (17)$$

we have

$$x^2 \left(\frac{\cos^2 \theta}{A^2} + \frac{\sin^2 \theta}{B^2} \right) - xy \sin 2\theta \left(\frac{1}{A^2} - \frac{1}{B^2} \right) + y^2 \left(\frac{\sin^2 \theta}{A^2} + \frac{\cos^2 \theta}{B^2} \right) = 1. \quad (18)$$

Equating (14) and (18), we can deduce that

$$\theta = \frac{1}{2} \tan^{-1} \left(\frac{2 \cos \phi E_1 E_2}{E_2^2 - E_1^2} \right), \quad (19)$$

$$AR = \left(\frac{1 + \Delta}{1 - \Delta} \right)^{\frac{1}{2}}, \quad (20)$$

where

$$\Delta = \left[1 - \frac{4E_1^2 E_2^2 \sin^2 \phi}{E_1^2 + E_2^2} \right]^{\frac{1}{2}}. \quad (21)$$

AR is the axial ratio which is the ratio of the two axes of the ellipse. It is defined to be larger than one always.