Plane Waves, Polarization and the Poynting Vector

$$\mathbf{H}_y = \frac{\mathbf{E}_{x+}}{\eta} - \frac{\mathbf{E}_{x-}}{\eta}$$

Uniform Plane Wave in Free Space

We have previously established the following properties of plane waves in free space:

- Electric and magnetic field components in propagation direction are zero.
- Electric and magnetic fields ($\vec{E}_x, \vec{B}_y$ or $\vec{E}_y, \vec{B}_x$) are related. Each is the "source" of the other.
- A set of three second order differential equations apply, one for each field component in rectangular coordinates. For example:

$$\frac{\partial^2 \vec{E}_x}{\partial z^2} = \frac{\mu_o \varepsilon_o \partial^2 \vec{E}_x}{\partial t^2} \ ,$$

or for harmonic fields, in phasor notation:

$$\frac{\partial^2 \vec{\mathbf{E}}_x}{\partial z^2} = -\omega^2 \mu_o \varepsilon_o \vec{\mathbf{E}}_x$$

A general solution to the Helmholz equation is written:

$$\vec{\mathbf{E}}_x = \vec{\mathbf{C}}_1 e^{-j\beta_o z} + \vec{\mathbf{C}}_2 e^{j\beta_o z} \ (\vec{\mathbf{C}}_i \text{ can be complex}),$$

where $\beta_o = \omega\sqrt{\mu_o \varepsilon_o} = \omega / c$, $c =$ speed of light.

Propagation of Magnetic Field

Suppose we have an electric field wave travelling in the positive z direction. Recall that **curl** $\vec{\mathbf{E}} = -j\omega \vec{\mathbf{B}}$, or for our travelling wave,

$$\vec{\mathbf{E}}_x = \vec{\mathbf{E}}_{+xo} e^{-j\beta_o z} \qquad \frac{\partial \vec{\mathbf{E}}_x}{\partial z} = -j\omega \vec{\mathbf{B}}_y$$

$$\frac{\partial\left(\vec{\mathbf{E}}_{+xo} e^{-j\beta_o z}\right)}{\partial z} = -j\omega \vec{\mathbf{B}}_y$$

$$-j\beta_o \vec{\mathbf{E}}_{+xo} e^{-j\beta_o z} = -j\omega \vec{\mathbf{B}}_y \Rightarrow \vec{\mathbf{B}}_y = \frac{\beta_o}{\omega} \mathbf{E}_{+xo} e^{-j\beta_o z}$$

$$= \frac{\omega}{c\omega} \vec{\mathbf{E}}_{+xo} e^{-j\beta_o z}$$

- Electric and magnetic ($\vec{B}$) field are orthogonal (perpendicular, Right Hand Rule), in-phase, and the ratio of the field magnitudes is the impedance.

It is very useful to express this ratio of electric and magnetic fields in terms of $\vec{\mathbf{H}}$, rather than $\vec{\mathbf{B}}$.

$$\mu_o \vec{\mathbf{H}}_y = \frac{\vec{\mathbf{E}}_x}{c} = \vec{\mathbf{E}}_x \sqrt{\mu_o \varepsilon_o} \ ,$$

or, after rearranging: $\dfrac{\vec{\mathbf{E}}_x}{\vec{\mathbf{H}}_y} = \mu_o c = \dfrac{\mu_o}{\sqrt{\varepsilon_o \mu_o}} = \sqrt{\dfrac{\mu_o}{\varepsilon_o}}$

Dimensional analysis: $\sqrt{\dfrac{Henry/m}{Farad/m}} = \sqrt{\dfrac{Joule/amp^2}{Joule/volt^2}} = \dfrac{volt}{amp} = Ohm$!!!

or:

Electric fields are volts/meter and $\vec{H}$ fields are amps/meter.

$\dfrac{\vec{\mathbf{E}}_x}{\vec{\mathbf{H}}_y} = \sqrt{\dfrac{\mu_o}{\varepsilon_o}} \equiv \eta_o \ Ohms$, $\eta_o \approx 120\pi = 377 \ Ohms$ (impedance of freespace)

For a wave travelling in the $-\hat{e}_z$ direction, $\dfrac{\vec{\mathbf{E}}_x}{\vec{\mathbf{H}}_y} = -\eta_o$. This just means that $\vec{\mathbf{E}}_x$ is oriented along the $-\hat{e}_x$ direction (or equivalently with $\vec{\mathbf{H}}_y$).

Summary:

- $\vec{\mathbf{E}}$ and $\vec{\mathbf{H}}$ are perpendicular to each other and propagation direction. Right Hand Rule gives direction.
- Ratio of $\vec{\mathbf{H}}$ to $\vec{\mathbf{E}}$ is the *intrinsic wave impedance*, η_o. The wavelength is the distance that the wave travels so that the phase changes by 2π radians.

$$\beta_o \lambda = 2\pi \ , \text{ or } \ \lambda = \frac{2\pi}{\beta_o} = \frac{2\pi}{\omega/c} = \frac{2\pi c}{\omega} = \frac{c}{f}$$

- Picture:

- Phase velocity, v_p

To understand the phase velocity, we must return to the real representation of the field with both space and time dependence; that is,

$$E = E_{+xo}\cos(\omega t - \beta_o z)$$

Consider an observer moving along in z at the same point, say the peak, of the oscillating field. This means mathematically that:

$\omega t - \beta_o z = 0$ or any constant. Thus, $\omega dt = \beta_o dz$,

or $\dfrac{dz}{dt} = \dfrac{\omega}{\beta_o} \equiv v_p$, the phase velocity.

Polarization of Place Waves

Consider the propagation characteristics of a plane wave in which the electric field has components in both the x and y directions:

$$\vec{\mathbf{E}} = \left(\vec{\mathbf{E}}_x \hat{e}_x + \vec{\mathbf{E}}_y \hat{e}_y\right) e^{-j\beta_o z},$$

where the field components may be complex. That is,

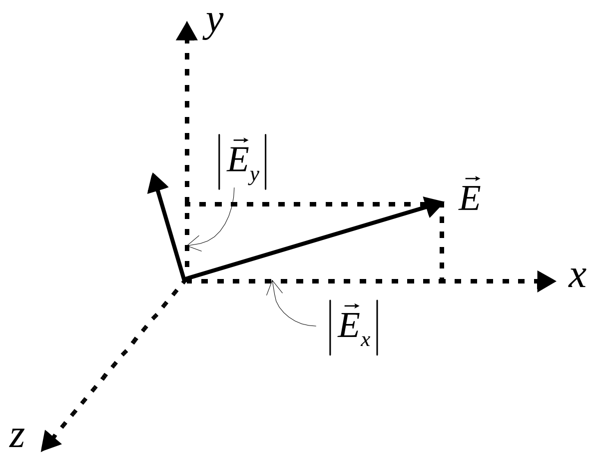
$$\vec{\mathbf{E}}_x = \left|\vec{\mathbf{E}}_x\right| e^{ja} \quad \text{and} \quad \vec{\mathbf{E}}_y = \left|\vec{\mathbf{E}}_y\right| e^{jb}$$

In phase $a = b$,

$$\vec{\mathbf{E}} = \left(\left|\vec{\mathbf{E}}_x\right| \hat{e}_x + \left|\vec{\mathbf{E}}_y\right| \hat{e}_y\right) e^{-j(\beta_o z - a)}$$

or in real form:

$$\vec{E} = \left(\left|\vec{E}_x\right|e_x + \left|\vec{E}_y\right|e_y\right)\cos\left(\omega t - \beta_o z - a\right)$$



This is *linear polarization*.

Now we consider a more general case.

Elliptical Polarization

In this case we allow arbitrary phase relationships a and b:

$$\mathbf{E} = \left(\varepsilon_x\left|\mathbf{E}_x\right|e^{ja} + \varepsilon_y\left|\mathbf{E}_y\right|e^{jb}\right)e^{-j\beta_o z}$$

It is easier to see what this means if we write each component out in real form:

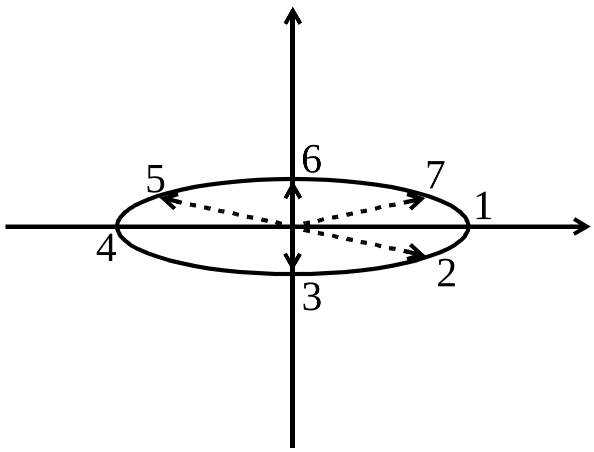
$$\vec{E}_x = \left|\vec{E}_x\right|\cos\left(\omega t + a - \beta z\right)$$
$$\vec{E}_y = \left|\vec{E}_y\right|\cos\left(\omega t + b - \beta z\right)$$

Suppose we let $a = 0$ and $b = \pi/2$. Then:

$$\vec{E}_x = \left|\vec{E}_x\right|\cos\left(\omega t - \beta z\right)$$
$$\vec{E}_y = -\left|\vec{E}_y\right|\sin\left(\omega t - \beta z\right)$$

What can we say about $\vec{E}(z,t) = \vec{E}_x\,\varepsilon_x + \vec{E}_y\,\varepsilon_y$?

Make a plot in the *x*-*y* plane with *t* as a parameter and $\left|\vec{E}_x\right| = 1,\ \left|\vec{E}_y\right| = \frac{1}{2}$.



ωt		$\vec{E}_x$	$\vec{E}_y$
0	1	1	0
$\pi/4$	2		$-.707/2$
$\pi/2$	3	0	$-1/2$
π	4	-1	0
$5\pi/4$	5	$-.707$	$.707/2$
$3\pi/2$	6	0	$1/2$
$7\pi/4$	7		$.707/2$

This clockwise rotation describes an ellipse, with major axes parallel to the *x* axis. As the wave propagates along *z*, the ellipse spreads out to an elliptical helix. This is called elliptical polarization. For the special case of $\left|\vec{E}_x\right| = \left|\vec{E}_y\right|$, we have circular polarization. If we had chosen $b = -\pi/2$, we would have found that an identical ellipse would be formed except that the rotation would be counter-clockwise.

Poynting Theorem

We know that energy is propagated by waves, in general, and electromagnetic waves, in particular. We need to quantify the associated power flow. We easily obtain a hint of how to calculate power flow by recalling our circuit theory, where $P = VI^*$, or by a dimensional analysis of the fields.

Remember that the units of *H* are *Amp/meter* and the units of *E* are *V/m*. Their product *Amp Volts/*m^2 has units of *Watts/area*, which is a power density, just what we want. The product must clearly have a direction associated with it, and it ought to somehow point in a reasonable direction. A reasonable direction for power flow in lossless, homogeneous, linear media, such as free space, would be in the direction of propagation. We might therefore guess that $\vec{E} \times \vec{H}$ would be a reasonable definition of power density.

We will consider the complex Poynting vector for time harmonic plane electromagnetic waves in phasor notation. This requires some thought because of the nonlinear nature of $\vec{E}\times\vec{H}$.

Let $\vec{E}(z,t) = \text{Re}\left[\vec{\mathbf{E}}(z)e^{j\omega t}\right]\hat{e}_x$ and

$$\vec{H}(z,t) = \text{Re}\left[\vec{\mathbf{H}}(z)e^{j\omega t}\right]\hat{e}_y$$

We will write out the real and imaginary parts of E and H.

$$\vec{E}(z,t) = \text{Re}\left[\vec{\mathbf{E}}_r(z)e^{j\omega t} + j\,\vec{\mathbf{E}}_i(z)e^{j\omega t}\right]\hat{e}_x \quad \text{and}$$

$$\vec{H}(z,t) = \text{Re}\left[\vec{\mathbf{H}}_r(z)e^{j\omega t} + j\,\vec{\mathbf{H}}_i(z)e^{j\omega t}\right]\hat{e}_y$$

Now expand the complex exponential and take the real part:

$$\vec{E}(z,t) = \left[\vec{E}_r(z)\cos\omega t - \vec{E}_i(z)\sin\omega t\right]\hat{e}_x$$

$$\vec{H}(z,t) = \left[\vec{H}_r(z)\cos\omega t - \vec{H}_i(z)\sin\omega t\right]\hat{e}_y$$

The Poynting vector is: $\vec{P}(z,t) = \vec{E}\times\vec{H} =$

$$\left[\vec{E}_r\,\vec{H}_r\cos^2\omega t + \vec{E}_i\,\vec{H}_i\sin^2\omega t - \vec{E}_r\,\vec{H}_i\cos\omega t\sin\omega t - \vec{E}_i\,\vec{H}_r\sin\omega t\cos\omega t\right]\hat{e}_z$$

The time average Poynting vector is:

$$\vec{P}_{av}(z) = \frac{1}{T}\int_0^T \vec{P}(z,t)\,dt = \frac{1}{2}\vec{E}_r\,\vec{H}_r + \frac{1}{2}\vec{E}_i\,\vec{H}_i$$

The very same expression can be obtained directly from the phasors by the following rule:

$$\vec{P}(z,t) = \frac{1}{2}\text{Re}\left(\vec{\mathbf{E}}\times\vec{\mathbf{H}}^*\right) = \frac{1}{2}\text{Re}\left[\left(\vec{\mathbf{E}}_r + j\,\vec{\mathbf{E}}_i\right)\hat{e}_x\times\left(\vec{\mathbf{H}}_r - j\,\vec{\mathbf{H}}_i\right)\hat{e}_y\right]$$

$$= \frac{1}{2}\left(\vec{E}_r\,\vec{H}_r + \vec{E}_i\,\vec{H}_i\right)\hat{e}_z$$

Example

What is the time average Poynting vector for a plane wave propagating in free space with the following (phasor) fields:

$$\vec{\mathbf{E}}(z) = E_{+0}\, e^{-j\beta_o z}\, \hat{e}_x \quad \text{and} \quad \vec{\mathbf{H}}(z) = \frac{E_{+0}}{\eta_o}\, e^{-j\beta_o z}\, \hat{e}_y \text{?}$$

Then $\vec{P}_{av}(z,t) = \frac{1}{2}\mathrm{Re}\left(\vec{\mathbf{E}} \times \vec{\mathbf{H}}^{*}\right) =$

$$\frac{1}{2}\mathrm{Re}\left(E_{+0}\, e^{-j\beta_o z}\, \hat{e}_x \times \frac{E_{+0}}{\eta_o}\, e^{-j\beta_o z}\, \hat{e}_y \right) = \frac{1}{2}\mathrm{Re}\, \frac{E_{+0}^{2}}{\eta_o}\, \hat{e}_z\,,$$

or:

$$\vec{P}_{av}(z,t) = \frac{E_{+0}^{2}}{2\,\eta_o}\, \hat{e}_z$$