Maxwell's Equation

\[ \nabla^2 E + \omega^2 \mu \varepsilon E = 0 \quad (A) \]

\[ \nabla^2 H + \omega^2 \mu \varepsilon H = 0 \quad (B) \]

For a waveguide with arbitrary cross section as shown in the above figure, we assume a plane wave solution and as a first trial, we set \( E_z = 0 \). This defines the TE modes.

From \( \nabla \times E = -\mu \frac{\partial H}{\partial t} \), we have

\[ \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = -\mu \frac{\partial H_x}{\partial t} \Rightarrow +j \beta_z E_y = -j \omega \mu H_x \quad (1) \]

\[ \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = -\mu \frac{\partial H_y}{\partial t} \Rightarrow -j \beta_z E_x = -j \omega \mu H_y \quad (2) \]

\[ \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -\mu \frac{\partial H_z}{\partial t} \Rightarrow \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -j \omega \mu H_z \quad (3) \]

From \( \nabla \times E = -j \omega \mu H \), we get

\[ j \omega \varepsilon E = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ H_x & H_y & H_z \end{vmatrix} \]

\[ \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} = j \omega \varepsilon E_x \Rightarrow \frac{\partial H_z}{\partial y} + j \beta_z H_y = j \omega \varepsilon E_x \quad (4) \]

\[ \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} = j \omega \varepsilon E_y \Rightarrow -j \beta_z H_x - \frac{\partial H_z}{\partial x} = j \omega \varepsilon E_y \quad (5) \]
\[
\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = 0 \quad (6)
\]

We want to express all quantities in terms of \( H_z \).

From (2), we have

\[
H_y = \frac{\beta z E_x}{\omega \mu} \quad (7)
\]

in (4)

\[
\frac{\partial H_z}{\partial y} + j\beta z^2 E_x \omega \mu = j\omega \epsilon E_x \quad (8)
\]

Solving for \( E_x \)

\[
E_x = \frac{j\omega \mu}{\beta z^2 - \omega^2 \mu \epsilon} \frac{\partial H_z}{\partial y} \quad (9)
\]

From (1)

\[
H_x = \frac{-\beta z E_y}{\omega \mu} \quad (10)
\]

in (5)

\[
+j\beta z^2 E_y \omega \mu - \frac{\partial H_z}{\partial x} = j\omega \epsilon E_y \quad (11)
\]

so that

\[
E_y = \frac{-j\omega \mu}{\beta z^2 - \omega^2 \mu \epsilon} \frac{\partial H_z}{\partial x} \quad (12)
\]

\[
H_x = \frac{j\beta z}{\beta z^2 - \omega^2 \mu \epsilon} \frac{\partial H_z}{\partial x} \quad (13)
\]

\[
H_y = \frac{j\beta z}{\beta z^2 - \omega^2 \mu \epsilon} \frac{\partial H_z}{\partial y} \quad (14)
\]

\[
E_z = 0 \quad (15)
\]
Combining solutions for $E_x$ and $E_y$ into (3) gives
\[
\frac{\partial^2 H_z}{\partial x^2} + \frac{\partial^2 H_z}{\partial y^2} = [\beta z^2 - \omega^2 \mu \varepsilon] H_z
\]  
(16)

**RECTANGULAR WAVEGUIDES**

If the cross section of the waveguide is a rectangle, we have a rectangular waveguide and the boundary conditions are such that the tangential electric field is zero on all the PEC walls.

**TE Modes**

The general solution for TE modes with $E_z=0$ is obtained from (16)

\[
H_z = e^{-j\beta_z z} \left[ A e^{-j\beta_x x} + B e^{j\beta_x x} \right] \left[ C e^{-j\beta_y y} + D e^{j\beta_y y} \right]
\]  
(17)

\[
E_y = \frac{\beta_x \omega \mu}{\beta_z^2 - \omega^2 \mu \varepsilon} e^{-j\beta_z z} \left[ -A e^{-j\beta_x x} + B e^{j\beta_x x} \right] \left[ C e^{-j\beta_y y} + D e^{j\beta_y y} \right]
\]  
(18)

\[
E_x = \frac{j\beta_y \omega \mu}{\beta_z^2 - \omega^2 \mu \varepsilon} e^{-j\beta_z z} \left[ A e^{-j\beta_x x} + B e^{j\beta_x x} \right] \left[ -C e^{-j\beta_y y} + D e^{j\beta_y y} \right]
\]  
(19)

At $y=0$, $E_x=0$ which leads to $C=D$

At $x=0$ $E_y=0$ which leads to $A=B$

\[
H_z = H_0 e^{-j\beta_z z \cos \beta_x x \cos \beta_y y}
\]  
(20)

\[
E_y = \frac{\beta_x \omega \mu}{\beta_z^2 - \omega^2 \mu \varepsilon} H_0 e^{-j\beta_z z \sin \beta_x x \cos \beta_y y}
\]  
(21)

\[
E_x = \frac{j\beta_y \omega \mu}{\beta_z^2 - \omega^2 \mu \varepsilon} H_0 e^{-j\beta_z z \cos \beta_x x \sin \beta_y y}
\]  
(22)
At $x=a$, $E_y=0$; this leads to $\beta_x = \frac{m\pi}{a}$

At $y=b$, $E_x=0$; this leads to $\beta_y = \frac{n\pi}{b}$

The dispersion relation is obtained by placing (20) in (16)

$$\beta_x^2 + \beta_y^2 + \beta_z^2 = \omega^2\mu\varepsilon$$

(23)

$$\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 + \beta_z^2 = \omega^2\mu\varepsilon$$

(24)

and

$$\beta_z = \sqrt{\omega^2\mu\varepsilon - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}$$

(25)

The guidance condition is

$$\omega^2\mu\varepsilon > \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2$$

(26)

or

$f > f_c$ where $f_c$ is the cutoff frequency of the TE$_{mn}$ mode given by the relation

$$f_c = \frac{1}{2\sqrt{\mu\varepsilon}} \sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}$$

(27)

The TE$_{mn}$ mode will not propagate unless $f$ is greater than $f_c$. Obviously, different modes will have different cutoff frequencies.

**TM Modes**

The transverse magnetic modes for a general waveguide are obtained by assuming $H_z=0$. By duality with the TE modes, we have

$$\frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} = [\beta_z^2 - \omega^2\mu\varepsilon]E_z$$

(28)
with general solution

\[ E_z = e^{-j\beta_z z} \left[ Ae^{-j\beta_x x} + Be^{+j\beta_x x} \right] C e^{-j\beta_y y} + De^{+j\beta_y y} \]  

(29)

The boundary conditions are

At x=0, \( E_z = 0 \) which leads to \( A = -B \)

At y=0, \( E_z = 0 \) which leads to \( C = -D \)

At x=a, \( E_z = 0 \) which leads to \( \beta_x = \frac{m\pi}{a} \)

At y=b, \( E_z = 0 \) which leads to \( \beta_y = \frac{n\pi}{b} \)

so that the generating equation for the TM_{mn} modes is

\[ E_z = E_0 e^{-j\beta_z z} \sin \left( \frac{m\pi x}{a} \right) \sin \left( \frac{n\pi y}{b} \right) \]  

(30)

NOTE: THE DISPERSION RELATION, GUIDANCE CONDITION AND CUTOFF EQUATIONS FOR A RECTANGULAR WAVEGUIDE ARE THE SAME FOR TE AND TM MODES.

For additional information on the field equations see Rao, page 539 Table 9.1.

There is no TE_{00} mode

There are no TM_{m0} or TM_{0n} modes

The first TE mode is the TE_{10} mode

The first TM mode is the TM_{11} mode

Impedance of a Waveguide

For a TE mode, we define the transverse impedance as

\[ \eta_{gTE} = \frac{-E_y}{H_x} = \frac{E_x}{H_y} = \frac{\omega \mu}{\beta_z} \]
From the relationship for $\beta_z$ and using

$$f_c^2 = \frac{1}{4\pi^2 \mu \varepsilon} \left[ \left( \frac{m}{a} \right)^2 + \left( \frac{n}{b} \right)^2 \right]$$

we get

$$\eta_{gTE} = \frac{\eta}{\sqrt{1 - \frac{f_c^2}{f^2}}}$$

where $\eta$ is the intrinsic impedance $\eta = \sqrt{\frac{\mu}{\varepsilon}}$. Analogously, for TM modes, it can be shown that

$$\eta_{gTM} = \eta \sqrt{1 - \frac{f_c^2}{f^2}}$$

**Power Flow in a Rectangular Waveguide (TE\textsubscript{10})**

The time-average Poynting vector for the TE\textsubscript{10} mode in a rectangular waveguide is given by

$$\langle P \rangle = \frac{1}{2} \text{Re}[E \times H^*] = \hat{z} \frac{|E_o|^2}{2} \frac{\beta_z}{\omega \mu} \sin^2 \frac{\pi x}{a}$$

$$\langle \text{Power} \rangle = \int_0^a \int_0^b |E_o|^2 \frac{\beta_z}{\omega \mu} \sin^2 \frac{\pi x}{a} \text{d}x \text{d}y$$

$$\langle \text{Power} \rangle = \frac{|E_o|^2}{4} \frac{\beta_z a b}{\omega \mu} = \frac{|E_o|^2 a b}{4\eta_{gTE,10}}$$

Therefore the time-average power flow in a waveguide is proportional to its cross-section area.