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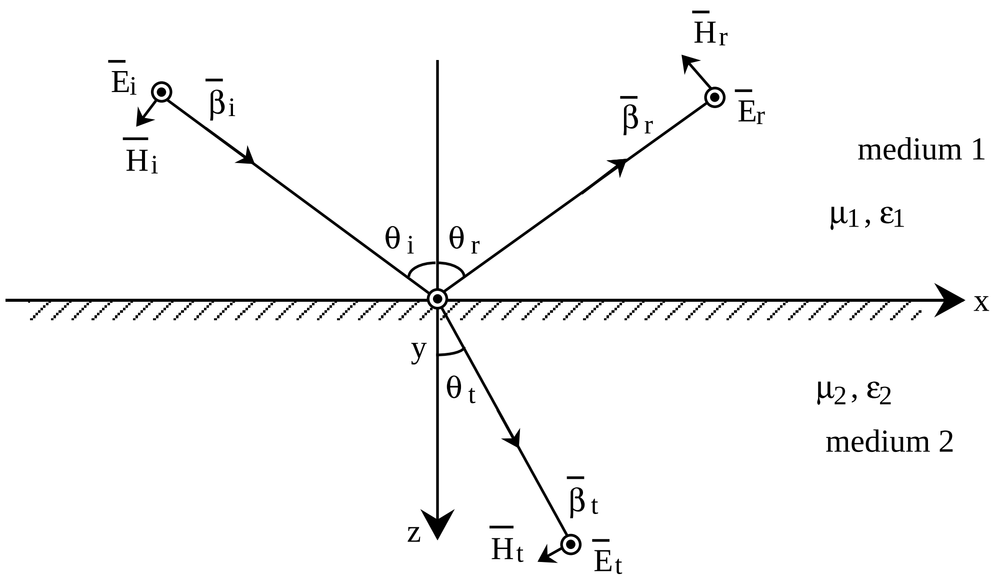
Reflections and Refractions of Plane Waves.



Perpendicular Case (Transverse Electric or TE case)

When an incident wave impinges on a dielectric interface, a reflected wave as well as a transmitted wave is generated. We can express the three waves as

$$\mathbf{E}_i = \hat{y} E_0 e^{-j\boldsymbol{\beta}_i \cdot \mathbf{r}}, \tag{1}$$

$$\mathbf{E}_r = \hat{y} \rho_\perp E_0 e^{-j\boldsymbol{\beta}_r \cdot \mathbf{r}}, \tag{2}$$

$$\mathbf{E}_t = \hat{y} \tau_\perp E_0 e^{-j\boldsymbol{\beta}_t \cdot \mathbf{r}}. \tag{3}$$

The electric field is perpendicular to the xz plane, and $\boldsymbol{\beta}_i$, $\boldsymbol{\beta}_r$, and $\boldsymbol{\beta}_t$ are their respective directions of propagation. The $\boldsymbol{\beta}$'s are also known as **propagation vectors** In particular,

$$\boldsymbol{\beta}_i = \hat{x}\beta_{ix} + \hat{z}\beta_{iz}, \tag{4}$$

$$\boldsymbol{\beta}_r = \hat{x}\beta_{rx} - \hat{z}\beta_{rz}, \tag{5}$$

$$\boldsymbol{\beta}_{tx} = \hat{x}\beta_{tx} + \hat{z}\beta_{tz}. \tag{6}$$

Since $\mathbf{E}_i$ and $\mathbf{E}_r$ are in medium 1, we have

$$\beta_{ix}^2 + \beta_{iz}^2 = \beta_1^2 = \omega^2 \mu_1 \epsilon_1, \tag{7}$$

$$\beta_{rx}^2 + \beta_{rz}^2 = \beta_1^2 = \omega^2 \mu_1 \epsilon_1, \tag{8}$$

and for $\mathbf{E}_t$ in medium 2, we have

$$\beta_{tx}^2 + \beta_{tz}^2 = \beta_2^2 = \omega^2 \mu_2 \epsilon_2. \tag{9}$$

(7), (8), and (9) are known as the **dispersion relations** for the components of the propagation vectors. From the figure, we note that

$$\beta_{ix} = \beta_1 \sin\theta_i, \quad \beta_{iz} = \beta_1 \cos\theta_i, \tag{10}$$

$$\beta_{rx} = \beta_1 \sin\theta_r, \quad \beta_{rz} = \beta_1 \cos\theta_r, \tag{11}$$

$$\beta_{tx} = \beta_2 \sin\theta_t, \quad \beta_{tz} = \beta_2 \cos\theta_t. \tag{12}$$

To find the unknown $\rho_\perp$ and $\tau_\perp$, we need to match boundary conditions for the fields at the dielectric interface. The boundary conditions are the equality of the tangential electric and magnetic fields on both sides of the interface. The magnetic fields can be derived via Maxwell's equations.

$$\mathbf{H}_i = \frac{\nabla \times \mathbf{E}_i}{-j\omega\mu_1} = \frac{\boldsymbol{\beta}_i \times \mathbf{E}_i}{\omega\mu_1} = (\hat{z}\beta_{ix} - \hat{x}\beta_{iz}) \frac{E_0}{\omega\mu_1} e^{-j\boldsymbol{\beta}_i \cdot \mathbf{r}}. \tag{13}$$

Similarly,

$$\mathbf{H}_r = (\hat{z}\beta_{rx} + \hat{x}\beta_{rz}) \frac{\rho_\perp E_0}{\omega\mu_1} e^{-j\boldsymbol{\beta}_r \cdot \mathbf{r}}, \tag{14}$$

$$\mathbf{H}_t = (\hat{z}\beta_{tx} - \hat{x}\beta_{tz}) \frac{\tau_\perp E_0}{\omega\mu_2} e^{-j\boldsymbol{\beta}_t \cdot \mathbf{r}}. \tag{15}$$

Continuity of the tangential electric fields across the interface implies

$$E_0 e^{-j\beta_{ix}x} + \rho_\perp E_0 e^{-j\beta_{rx}x} = \tau_\perp E_0 e^{-j\beta_{tx}x}. \tag{16}$$

The above equation is to be satisfied for all x. This is only possible if

$$\beta_{ix} = \beta_{rx} = \beta_{tx} = \beta_x. \tag{17}$$

This condition is known as **phase matching**. From (10), (11), and (12), we know that (17) implies

$$\beta_1 \sin\theta_i = \beta_1 \sin\theta_r = \beta_2 \sin\theta_t. \tag{18}$$

The above implies that $\theta_r = \theta_i$. Furthermore,

$$\sqrt{\mu_1\epsilon_1} \sin\theta_i = \sqrt{\mu_2\epsilon_2} \sin\theta_t. \tag{19a}$$

If we define a refractive index $n_i = \sqrt{\frac{\mu_i\epsilon_i}{\mu_0\epsilon_0}}$, then (19a) becomes

$$n_1 \sin\theta_i = n_2 \sin\theta_t, \tag{19b}$$

which is the well known **Snell's Law**. Consequently, equation (16) becomes

$$1 + \rho_\perp = \tau_\perp. \tag{20}$$

From the continuity of the tangential magnetic fields, we have

$$-\beta_{iz}\frac{E_0}{\omega\mu_1}+\beta_{rz}\frac{\rho_\perp E_0}{\omega\mu_1}=-\beta_{tz}\frac{\tau_\perp E_0}{\omega\mu_2}. \tag{21}$$

Since $\theta_r=\theta_i$, we have $\beta_{iz}=\beta_{rz}$. Therefore, (21) becomes

$$1-\rho_\perp=\frac{\mu_1}{\mu_2}\frac{\beta_{tz}}{\beta_{iz}}\tau_\perp. \tag{22}$$

Solving (20) and (22), we have

$$\rho_\perp=\frac{\mu_2\beta_{iz}-\mu_1\beta_{tz}}{\mu_2\beta_{iz}+\mu_1\beta_{tz}}, \tag{23}$$

$$\tau_\perp=\frac{2\mu_2\beta_{iz}}{\mu_2\beta_{iz}+\mu_1\beta_{tz}}. \tag{24}$$

Using (10), (11), and (12), we can rewrite the above as

$$\rho_\perp=\frac{\eta_2\cos\theta_i-\eta_1\cos\theta_t}{\eta_2\cos\theta_i+\eta_1\cos\theta_t}, \tag{25}$$

$$\tau_\perp=\frac{2\eta_2\cos\theta_i}{\eta_2\cos\theta_i+\eta_1\cos\theta_t}. \tag{26}$$

If the media are non-magnetic so that $\mu_1=\mu_2=\mu_0$, we can use (19) to rewrite (25) as

$$\rho_\perp=\frac{\eta_2\cos\theta_i-\eta_1\sqrt{1-\frac{\epsilon_1}{\epsilon_2}\sin^2\theta_i}}{\eta_2\cos\theta_i+\eta_1\sqrt{1-\frac{\epsilon_1}{\epsilon_2}\sin^2\theta_i}}. \tag{27}$$

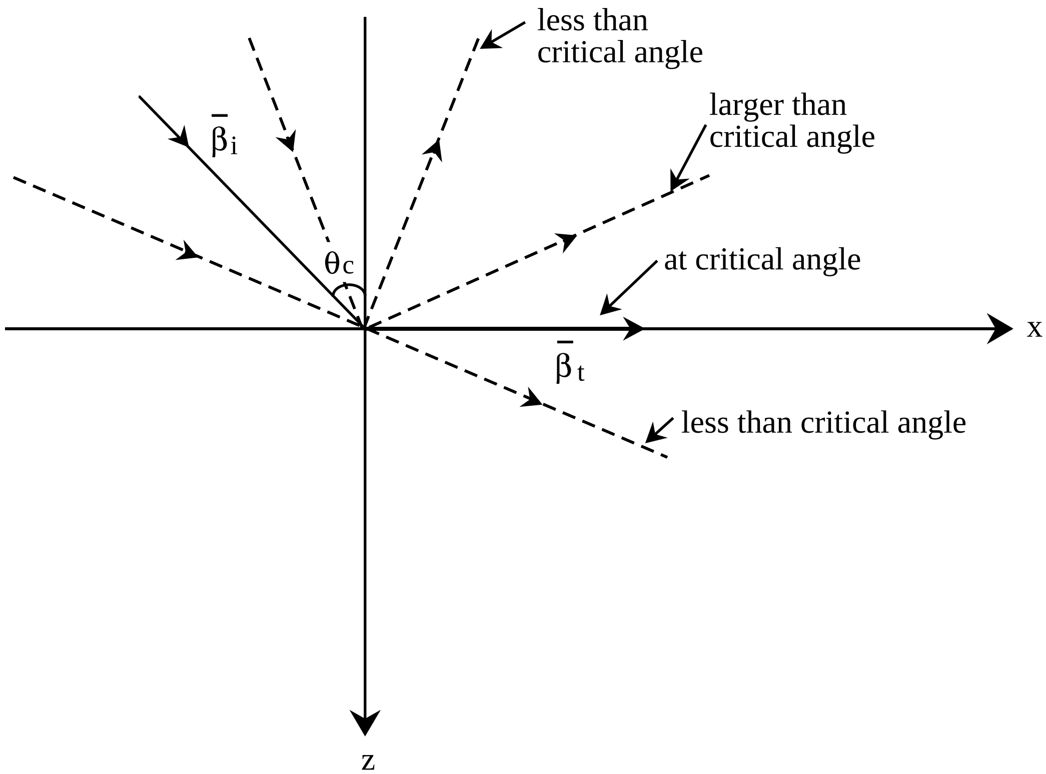
If $\sqrt{\frac{\epsilon_1}{\epsilon_2}}\sin\theta_i>1$, which is possible if $\frac{\epsilon_1}{\epsilon_2}>1$, when $\theta_i<\frac{\pi}{2}$, then $\rho_\perp$ is of the form

$$\rho_\perp=\frac{A-jB}{A+jB}, \tag{28}$$

which always has a magnitude of 1. In this case, all energy will be reflected. This is known as a **total internal reflection**. This occurs when $\theta_i>\theta_c$ where $\sqrt{\frac{\epsilon_1}{\epsilon_2}}\sin\theta_c=1$. or

$$\theta_c=\sin^{-1}\sqrt{\frac{\epsilon_2}{\epsilon_1}},\quad \epsilon_2<\epsilon_1. \tag{29}$$

When $\theta_i = \theta_c$, $\theta_t = 90^\circ$ from (19). The figure below denotes the phenomenon.



When $\theta_i > \theta_c$, $\beta_{tz} = \sqrt{\beta_2^2 - \beta_1^2 \sin^2\theta_i}$, or

$$\beta_{tz} = \omega\sqrt{\mu_0\epsilon_2}\left(1 - \frac{\epsilon_1}{\epsilon_2}\sin^2\theta_i\right)^{\frac{1}{2}}. \tag{30}$$

The quantity in the parenthesis is purely negative, so that

$$\beta_{tz} = -j\alpha_{tz}, \tag{31}$$

a pure imaginary number. In this case, the electric field in medium 2 is

$$\mathbf{E}_t = \hat{y}\tau_\perp E_0 e^{-j\beta_x x - \alpha_{tz} z}. \tag{32}$$

The field is exponentially decaying in the positive z direction. We call such a wave an **evanescent wave**, or an **inhomogeneous wave** as opposed to **uniform plane wave**. The magnitude of a uniform plane wave is a **constant** of space while the magnitude of an evanescent wave or an inhomogeneous wave is **not a constant** of space. The corresponding magnetic field is

$$\mathbf{H}_t = (\hat{z}\beta_x + \hat{x}j\alpha_{tz})\frac{\tau_\perp E_0}{\omega\mu_2}e^{-j\beta_x x - \alpha_{tz} z}. \tag{33}$$

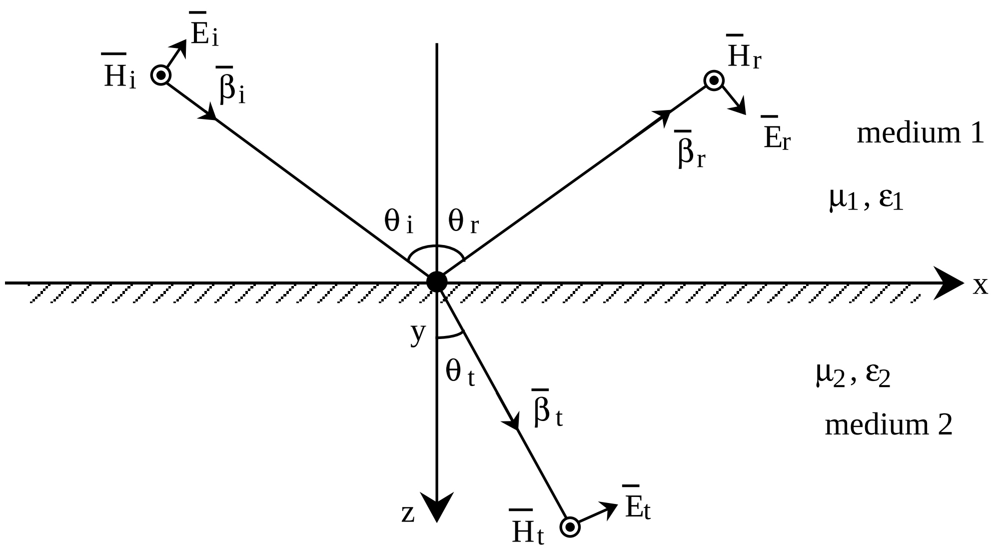
The complex power in the transmitted wave is

$$\underline{\mathbf{S}} = \mathbf{E}_t \times \mathbf{H}_t^* = (\hat{x}\beta_x + \hat{z}j\alpha_{tz})\frac{|\tau_\perp|^2 |E_0|^2}{\omega\mu_2} e^{-2\alpha_{tz}z}. \tag{34}$$

We note that $\underline{S}_x$ is pure real implying the presence of net time average power flowing in the $\hat{x}$-direction. However, $\underline{S}_z$ is pure imaginary implying that the power that is flowing in the $\hat{z}$-direction is purely reactive. Hence, no net time average power is flowing in the $\hat{z}$-direction.

Parallel case (Transverse Magnetic or TM case)

In this case, the electric field is parallel to the xz plane that contains the plane of incidence.



The magnetic field is polarized in the y direction, and they can be written as

$$\mathbf{H}_i = \hat{y}\frac{E_0}{\eta_1}e^{-j\boldsymbol{\beta}_i\cdot\mathbf{r}}, \tag{35}$$

$$\mathbf{H}_r = -\hat{y}\rho_\parallel\frac{E_0}{\eta_1}e^{-j\boldsymbol{\beta}_r\cdot\mathbf{r}}, \tag{36}$$

$$\mathbf{H}_t = \hat{y}\tau_\parallel\frac{E_0}{\eta_2}e^{-j\boldsymbol{\beta}_t\cdot\mathbf{r}}. \tag{37}$$

We put a negative sign in the definition for $\rho_\parallel$ to follow the convention of transmission line theory, where reflection coefficients are defined for voltages, and hence has a negative sign when used for currents. The magnetic field is the analogue of a current in transmission theory.

In this case, the electric field has to be orthogonal to $\boldsymbol{\beta}$ and $\hat{y}$, and they can be derived using

$$\mathbf{E}_i = -\frac{\boldsymbol{\beta}_i \times \mathbf{H}_i}{\omega \epsilon_1}$$

to be

$$\mathbf{E}_i = \frac{\hat{y} \times \boldsymbol{\beta}_i}{\beta} E_0 e^{-j\boldsymbol{\beta}_i \cdot \mathbf{r}} = (\hat{x}\beta_{iz} - \hat{z}\beta_{ix}) \frac{E_0}{\beta_1} e^{-j\boldsymbol{\beta}_i \cdot \mathbf{r}}, \tag{38}$$

$$\mathbf{E}_r = (\hat{x}\beta_{rz} + \hat{z}\beta_{rx}) \frac{\rho_{\|} E_0}{\beta_1} e^{-j\boldsymbol{\beta}_r \cdot \mathbf{r}}, \tag{39}$$

$$\mathbf{E}_t = (\hat{x}\beta_{tz} - \hat{z}\beta_{tx}) \frac{\tau_{\|} E_0}{\beta_2} e^{-j\boldsymbol{\beta}_t \cdot \mathbf{r}}. \tag{40}$$

Imposing the boundary conditions as before, we have

$$1 + \rho_{\|} = \frac{\beta_{tz}}{\beta_2} \frac{\beta_1}{\beta_{iz}} \tau_{\|}, \tag{41}$$

$$1 - \rho_{\|} = \frac{\eta_1}{\eta_2} \tau_{\|}. \tag{42}$$

The above can be solved to give

$$\rho_{\|} = \frac{\epsilon_1 \beta_{tz} - \epsilon_2 \beta_{iz}}{\epsilon_2 \beta_{iz} + \epsilon_1 \beta_{tz}} = \frac{\eta_2 \cos\theta_t - \eta_1 \cos\theta_i}{\eta_2 \cos\theta_t + \eta_1 \cos\theta_i}, \tag{43}$$

and

$$\tau_{\|} = \frac{2\epsilon_2 \beta_{iz}}{\epsilon_2 \beta_{iz} + \epsilon_1 \beta_{tz}} \frac{\eta_2}{\eta_1} = \frac{2\eta_2 \cos\theta_i}{\eta_2 \cos\theta_t + \eta_1 \cos\theta_i}. \tag{44}$$

In (43), $\rho_{\|}$ will be zero if

$$\eta_2^2 \cos^2\theta_t = \eta_1^2 \cos^2\theta_i. \tag{45}$$

Using Snell's Law, or (19), $\cos^2\theta_t = 1 - \frac{\mu_1\epsilon_1}{\mu_2\epsilon_2}\sin^2\theta_i$, and (45) becomes

$$1 - \frac{\mu_1\epsilon_1}{\mu_2\epsilon_2}\sin^2\theta_i = \frac{\mu_1\epsilon_2}{\mu_2\epsilon_1}\cos^2\theta_i. \tag{46}$$

Solving the above, we get

$$\sin\theta_i = \left(\frac{1 - \frac{\mu_1\epsilon_2}{\mu_2\epsilon_1}}{\frac{\mu_1\epsilon_1}{\mu_2\epsilon_2} - \frac{\mu_1\epsilon_2}{\mu_2\epsilon_1}} \right)^{\frac{1}{2}}. \tag{47}$$

Most materials are non-magnetic in this world so that $\mu = \mu_0$, then

$$\sin\theta_i = \sqrt{\frac{\epsilon_2}{\epsilon_2 + \epsilon_1}}. \tag{48}$$

The angle for θ_i at which $\rho_\parallel = 0$ is known as the **Brewster angle**. It is given by

$$\theta_{ib} = \sin^{-1}\sqrt{\frac{\epsilon_2}{\epsilon_2+\epsilon_1}} = \tan^{-1}\sqrt{\frac{\epsilon_2}{\epsilon_1}}. \tag{49}$$

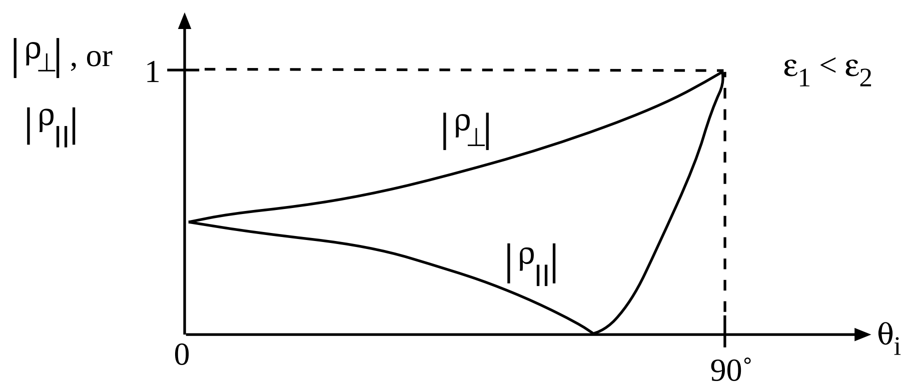
At this angle of incident, the wave will not be reflected but totally transmitted. Furthermore, we can show that

$$\sin^2\theta_{ib} + \sin^2\theta_{tb} = 1, \tag{50}$$

implying that

$$\theta_{ib} + \theta_{tb} = \frac{\pi}{2}. \tag{51}$$

On the contrary, $\rho_\perp$ can never be zero for $\mu = \mu_0$ or non-magnetic materials. Hence, a plot of $\left|\rho_\parallel\right|$ as a function of θ_i goes through a zero while the plot of $|\rho_\perp|$ is always larger than zero for non-magnetic materials.



At normal incidence, i.e., $\theta_i = 0, \rho_\perp = \rho_\parallel$ since we cannot distinguish between perpendicular and parallel polarizations. When $\theta_i = 90^\circ$, $|\rho_\perp| = \left|\rho_\parallel\right| = 1$. On the whole, $|\rho_\perp| \geq \left|\rho_\parallel\right|$ for non-magnetic materials.

The above equations are defined for lossless media. However, for lossy media, if we define a complex permittivity $\underline{\epsilon} = \epsilon - j\frac{\sigma}{\omega}$, Maxwell's equations remain unchanged. Hence, the expressions for $\rho_\perp$, $\tau_\perp$, $\rho_\parallel$, and $\tau_\parallel$ remain the same, except that we replace real permittivities with complex permittivities. For example, if medium 2 is metallic so that $\sigma \to \infty$, then, $\eta_2 = \sqrt{\frac{\mu_2}{\underline{\epsilon}_2}} \to 0$, and $\rho_\perp = -1$, and $\tau_\perp = 0$. Similarly, $\rho_\parallel = -1$ and $\tau_\parallel = 0$.