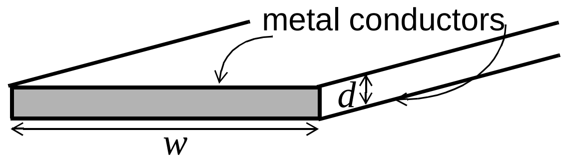
Transmission Lines

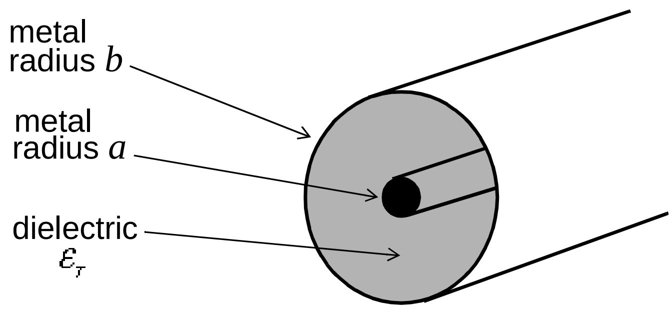
$$Z_{in}(-l) = Z_o \frac{Z_L \cos\beta l + j Z_o \sin\beta l}{Z_o \cos\beta l + j Z_L \sin\beta l} = Z_o \frac{Z_L + j Z_o \tan\beta l}{Z_o + j Z_L \tan\beta l}$$

Examples of Transmission Lines:

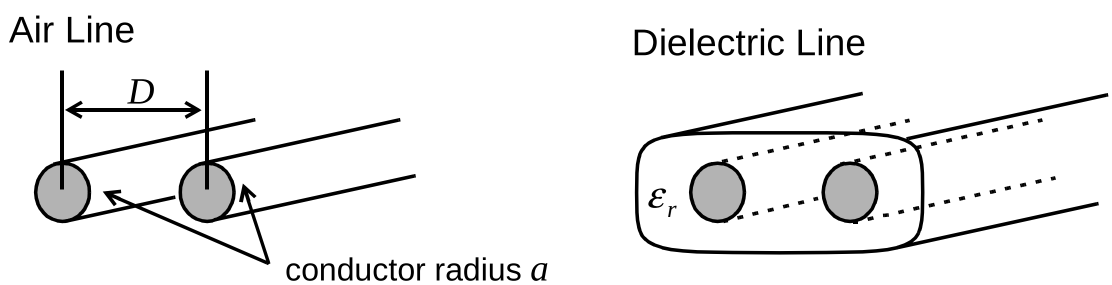
Parallel Strip Line:



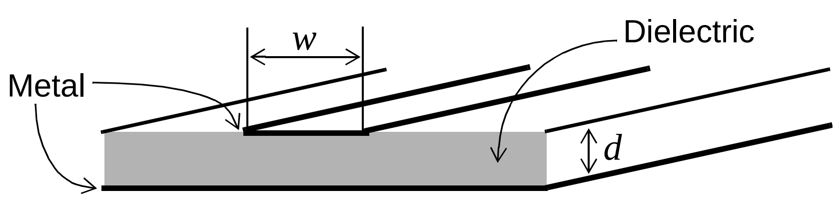
Coaxial Line:



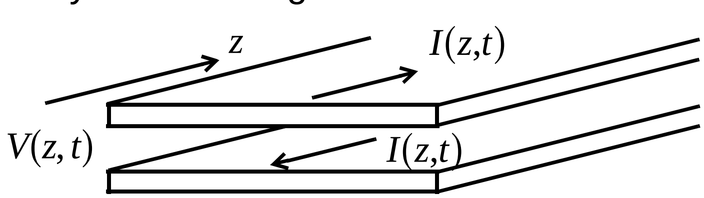
Parallel Wire Line:



Microstrip Line:

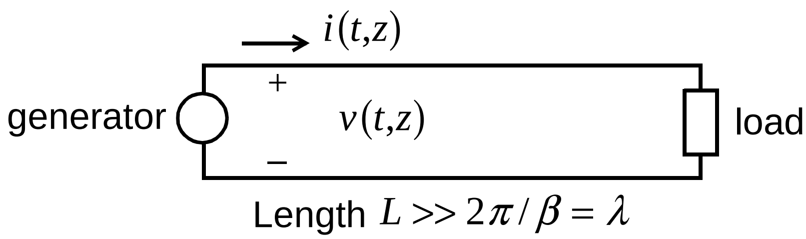


There is a simple way to view the guided wave on a transmission line.



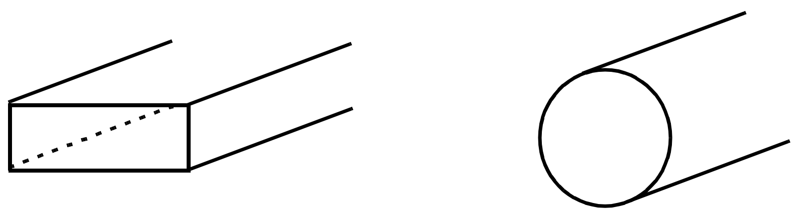
The potential difference (voltage) between the metal conductors with equal and opposite current flowing in them are circuit concepts, except they depend not only on time, but also on the distance z.

So we describe the wave as voltage and current waves.



Other guiding structures:

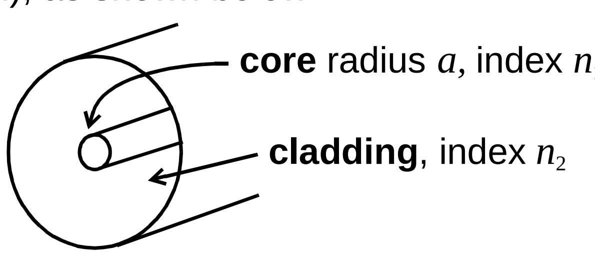
1. **Waveguides** -- consist of a **single** hollow metal tube of various cross-sectional geometry. An EM wave propagates longitudinally **inside** the hollow structure.



The wave propagation in waveguides is **not transverse** (not TEM). That is, it has longitudinal field component(s). Transverse spatial dependence is fairly complicated.

The propagation constant β of a waveguide wave is **not** equal to that of plane waves, and velocity of propagation thus is **not** the same as light.

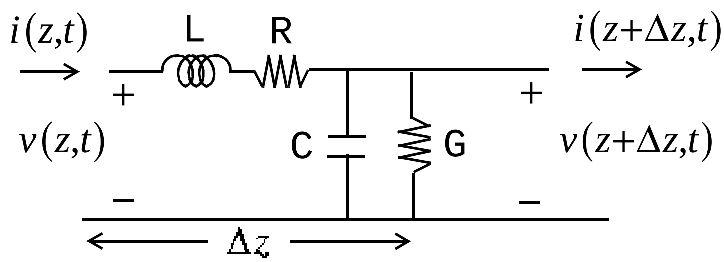
2. **Optical Fibers** -- are used at optical frequencies, at infrared, and at visible wavelengths. An optical fiber consists of a very thin (50-300 μm) dielectric circular cross section cylinder. The material is usually glass or plastic. Inner and outer portions have different dielectric constants (index of refraction), as shown below.



Optical fibers do not support TEM waves, like the hollow metallic guides. Their propagation constant and mode structures are even more complicated than for hollow metallic guides.

Derivation of Transmission Line Equations (1-3)

Let us consider a length Δz of a transmission line at location z. The **circuit model** is clearly a series inductance and resistance (since the whole line with a load at the end forms a loop) and a shunt capacitance and shunt leakage conductance (between the good conductors, across the dielectric which also has a small conductance). With the above circuit parameters being $\mathsf{L}, \mathsf{R}, \mathsf{C},$ and G, the model is



A straight forward application of Kirchoff's Loop Law gives

$$v(z+\Delta z,t) - v(z,t) = -\,\mathsf{L}\,\Delta z \frac{\partial i(z,t)}{\partial t} - \mathsf{R}\,\Delta z\, i(z,t)\,,$$

and Kirchoff's Current Law at the upper node gives

$$i(z+\Delta z,t) - i(z,t) = -\,\mathsf{G}\,\Delta z\, v(z+\Delta z,t) - \mathsf{C}\,\Delta z \frac{\partial v(z+\Delta z,t)}{\partial t}\,.$$

Dividing through by Δz and taking the limit $\Delta z \to 0$, we get:

$$\frac{\partial v}{\partial z} = -\,\mathsf{L}\frac{\partial i}{\partial t} - \mathsf{R}\,i \quad \text{(1-5)}$$

$$\frac{\partial i}{\partial z} = -\,\mathsf{C}\frac{\partial v}{\partial t} - \mathsf{G}\,v \quad \text{(1-6)}$$

Sinusoidal Analysis of Transmission Lines (using phasors)

$$\frac{\partial \mathbf{v}}{\partial z} = -(\mathsf{R} + j\omega \mathsf{L})\,\mathbf{i} \qquad (1)$$

$$\frac{\partial \mathbf{i}}{\partial z} = -(\mathsf{G} + j\omega \mathsf{C})\,\mathbf{v} \qquad (2)$$

Differentiate (1) with respect to z:

$$\frac{\partial^2 \mathbf{v}}{\partial z^2} = -(\mathsf{R} + j\omega \mathsf{L})\frac{d\mathbf{i}}{dz}$$

Substitute from (2) $\partial \mathbf{i}/\partial z$:

$$\frac{\partial^2 \mathbf{v}}{\partial z^2} = (\mathsf{G} + j\omega \mathsf{C})(\mathsf{R} + j\omega \mathsf{L})\mathbf{v}$$

Similarly, we get:

$$\frac{\partial^2 \mathbf{i}}{\partial z^2} = (\mathsf{G} + j\omega \mathsf{C})(\mathsf{R} + j\omega \mathsf{L})\mathbf{i}$$

The Lossless Transmission Line

$\mathsf{R} = 0$ (perfect conductor) $\mathsf{G} = 0$ (lossless dielectric)

This is not only instructive and simple, but also a good approximation of many real lines which are made of very good conductors (typically, copper) and nearly lossless dielectrics (e.g., teflon). For example, a commercial coaxial line is a nearly lossless line.

$$\frac{\partial^2 \mathbf{v}}{\partial z^2} = j\omega \mathsf{C} \cdot j\omega \mathsf{L}\mathbf{v}$$

$$\frac{\partial^2 \mathbf{v}}{\partial z^2} = -\left(\omega^2 \mathsf{L}\mathsf{C}\right)\mathbf{v} \qquad \text{Also, } \frac{\partial^2 \mathbf{i}}{\partial z^2} = -\left(\omega^2 \mathsf{L}\mathsf{C}\right)\mathbf{i}$$

$$\frac{\partial^2 \mathbf{v}}{\partial z^2} = -\beta^2 \mathbf{v} \qquad \frac{\partial^2 \mathbf{i}}{\partial z^2} = -\beta^2 \mathbf{i}$$

$$\frac{\partial V}{\partial z} = -\mathsf{L}\frac{\partial I}{\partial t} \qquad \frac{\partial I}{\partial z} = -\mathsf{C}\frac{\partial V}{\partial t}$$

The solutions are travelling waves:

$e^{\pm j\beta z}$ (with $e^{j\omega t}$ already assumed)

$e^{j(\omega t - \beta z)}$ or $e^{j(\omega t + \beta z)}$ or sum of both.

$$\beta = \omega\sqrt{\mathsf{LC}} = \omega\sqrt{\mu\varepsilon}$$

It will be shown later, when we consider transmission lines from the Electromagnetic Field's point of view, that these voltage and current waves correspond to EM waves, also. It will be shown that:

$$\sqrt{\mathsf{LC}} = \sqrt{\mu\varepsilon} = \frac{1}{v},$$

where v is velocity of light in the medium between the conductors.

We can get the relationship of V and I by substituting back into the original transmission line equations.

$$\mathbf{v} = V e^{j(\omega t - \beta z)} \qquad \mathbf{i} = I e^{j(\omega t - \beta z)}$$

(i.e., only $+\mathbf{z}$ going wave)

$$\frac{d\mathbf{v}}{dz} = -j\omega \mathsf{L}\mathbf{i} \qquad \frac{d\mathbf{i}}{dz} = -j\omega \mathsf{C}\mathbf{v}$$

$$-j\beta\mathbf{v} = -j\omega \mathsf{L}\mathbf{i} \qquad -j\beta\mathbf{i} = -j\omega \mathsf{C}\mathbf{v}$$

$$\frac{\mathbf{v}}{\mathbf{i}} = \frac{\omega \mathsf{L}}{\beta} = \frac{\beta}{\omega \mathsf{C}} \qquad \beta = \omega\sqrt{\mathsf{LC}} = \omega\sqrt{\mu\varepsilon},$$

$$\boxed{\frac{\mathbf{v}_+}{\mathbf{i}_+} = \sqrt{\frac{\mathsf{L}}{\mathsf{C}}} = Z_o}$$ This is for $+\mathbf{z}$ going wave only.

Z_o is called the **characteristic** impedance of the line. For this lossless line, Z_o is real. It means that V and I are in phase. For example, power is **propagated**. None is absorbed. (Contrast this with ordinary circuits, where real impedance means power is absorbed and dissipated!)

Power Propagated on a Lossless Line

The power propagated can be calculated either from the electromagnetic wave or from the voltage-current wave. It is usual to use voltage and current for a transmission line. The power, of course, is an oscillating quantity. We are mostly interested, however, in the **average power flow**. If $v = V_o$ (real), then from the **circuit picture**:

$$P_{av} = \textbf{Re}\,\frac{1}{2}\mathbf{v}\mathbf{i}^* = \frac{1}{2}\frac{V_o^2}{Z_o} \quad \text{for lossless line}$$

If we have only a $-\mathbf{z}$ going wave, the same kind of derivation we did for the $+\mathbf{z}$ going wave gives:

$$\boxed{\frac{\mathbf{v}_-}{\mathbf{i}_-} = -Z_o} \qquad (-\mathbf{z} \text{ going wave})$$

Circuits vs. Transmission Lines

1) Q: When does the "regular" circuit (E17) method hold correct; that is, only $e^{j\omega t}$ dependence (no propagation) and when does the transmission line method need to be used?

A: Simple. When the wavelength is large compared to the dimensions, we can neglect transmission line concepts:

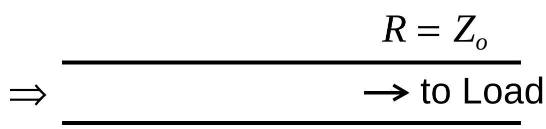
$$\beta z = \frac{2\pi}{\lambda} z \to 0$$

That is, whenever the distance scale (z) is small and/or the wavelength is long; that is, $z << \lambda$.

2) **For circuits**: If Z is real (that is, $Z = R$), $P_{av} = \frac{1}{2} I^2 R$. This is power **absorbed** in the real impedance (resistance).
For a lossless transmission line: Z_o is real. If we have only + going wave:

$$P_{av} = \frac{1}{2}\frac{|\mathbf{v}_+|^2}{Z_o} = \frac{1}{2}|\mathbf{i}_+|^2 Z_o$$

This is not absorbed power. It is power **propagating** in the $+\mathbf{z}$ direction down the lossless line.



Power is absorbed in the **load** where it does “something.”

If we have – going wave:

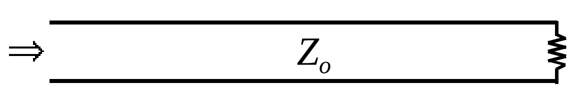
$$P_{av} = \textbf{Re}\,\frac{1}{2}\mathbf{v}_-\mathbf{i}_-^* = -\frac{1}{2}\frac{|\mathbf{v}_-|^2}{Z_o} = -\frac{1}{2}|\mathbf{i}_-|^2 Z_o$$

The negative sign means that power is propagating in the negative direction. Unless otherwise stated, we always will mean average power.

Transmission Line Circuits

Transmission lines are used to carry signals from one location to another. Thus, they usually have a signal generator at one end and a load at another end. Let’s now examine the behavior of a transmission line with load. We use phasors and will assume **lossless** transmission line.

1) **Transmission line terminated in its characteristic impedance.** This is called “matched.”



$Z_L = Z_o$

Let’s assume we launched a $+\mathbf{z}$ going wave on this transmission line of characteristic impedance Z_o. Then, at any point z, we know that $\mathbf{v}_+ / \mathbf{i}_+ = Z_o$. At the load, $\mathbf{v}_L / \mathbf{i}_L = Z_L$. (This is definition of Z_L.) Thus, if

$Z_L = Z_o$, then $\mathbf{v}_L = \mathbf{v}_+$ $(Z=L)$ and $\mathbf{i}_L = \mathbf{i}_+$ $(Z=L)$. Thus, there is **no** reflected wave ($-\mathbf{z}$ going wave). Rather, the wave continues **into the load**, where it gets dissipated. (In practical situations, the load is a whole circuit, for example, a detector, a radio, an oscilloscope. As long as its input impedance is Z_o, the foregoing discussion is true.)

2) **Transmission line terminated in other impedance** (1-7, text)

Assume again we launched a signal $\mathbf{v}_+, \mathbf{i}_+$ at some frequency. Then, $\mathbf{v}_+ / \mathbf{i}_+ = Z_o$. But at the load, $\mathbf{v}_L / \mathbf{i}_L = Z_L$. If $Z_L \neq Z_o$, $\mathbf{v}_L$ cannot be $\mathbf{v}_+$ and $\mathbf{i}_L$ cannot be $\mathbf{i}_+$. The boundary condition at the load can only be satisfied if **there is also a reflected wave** $\mathbf{v}_-, \mathbf{i}_-$. This wave propagates backward in the $-\mathbf{z}$ direction. Then, at any point on the line:

$$\mathbf{v} = V_+ e^{j(\omega t - \beta z)} + V_- e^{j(\omega t + \beta z)} = \mathbf{v}_+ + \mathbf{v}_-$$

$$\mathbf{i} = \frac{V_+}{Z_o} e^{j(\omega t - \beta z)} - \frac{V_-}{Z_o} e^{j(\omega t + \beta z)} = \mathbf{i}_+ + \mathbf{i}_- \qquad \mathbf{i}_- = -\frac{\mathbf{v}_-}{Z_o}$$

Note the $-$ sign on the current of the reflected wave! Now we can satisfy the condition at the load $\mathbf{v}/\mathbf{i} = Z_L$. Only one value of V_- will satisfy.

Let us define:

$$\boxed{\frac{\mathbf{v}_-}{\mathbf{v}_+} = \Gamma, \text{ the reflection coefficient}}$$

Then:

$$\mathbf{v}_+ + \Gamma \mathbf{v}_+ = \mathbf{v} \qquad (1)$$

$$\frac{\mathbf{v}_+}{Z_o} - \Gamma \frac{\mathbf{v}_+}{Z_o} = \frac{\mathbf{v}}{Z_L} \qquad (2)$$

We solve for Γ to get: (Problem: Solve to Γ)

$$\boxed{\Gamma = \frac{Z_L - Z_o}{Z_L + Z_o}}$$

Note: Γ is **complex, in general**, because Z_L may be complex.

$$-1 \leq |\Gamma| \leq 1$$

Fraction of the incident power reflected back:

Incident Power $$P_i = \textbf{Re}\left[\frac{1}{2}\mathbf{v}_+\mathbf{i}_+^*\right] = \textbf{Re}\left[\frac{1}{2}\left|\mathbf{v}_+\right|^2/Z_o\right] = \frac{1}{2}\left|V_+\right|^2/Z_o$$

Reflected Power $$P_r = \textbf{Re}\left[\frac{1}{2}\mathbf{v}_-\mathbf{i}_-^*\right] = \textbf{Re}\left[\frac{1}{2}\Gamma\,\mathbf{v}_-\,\Gamma^*\,\mathbf{i}_+^*\right](-1)$$

$$= \textbf{Re}\left[-\left|\Gamma\right|^2\frac{1}{2}\mathbf{v}_+\mathbf{i}_+^*\right] = -\left|\Gamma\right|^2 P_i$$

Total Power $$P_T = \textbf{Re}\left[\frac{1}{2}\mathbf{v}_T\,\mathbf{i}_T^*\right] = \textbf{Re}\left[\frac{1}{2}(1+\Gamma)\,\mathbf{v}_+\left(1-\Gamma^*\right)\mathbf{i}_+^*\right]$$

$$= \textbf{Re}\left[\frac{1}{2}\left(1+\Gamma-\Gamma^*-\Gamma\Gamma^*\right)\mathbf{v}_+\mathbf{i}_+^*\right]$$

$$= \left[1-\left|\Gamma\right|^2\right]P_i \equiv \text{Transmitted power (into Load)}$$

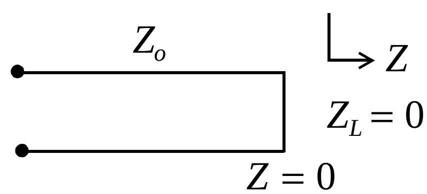
Note: $\frac{1}{2}\left(\Gamma-\Gamma^*\right) = \mathtt{Im}\left[\Gamma\right]$ is purely imaginary.

3) **Special Cases of Termination**

A special case of the general load is $Z_L = 0$, that is, a **short circuited** transmission line. (A transmission line may be shorted deliberately in order to achieve a certain result, or it may be shorted accidentally, something that has to be repaired.)

For this case:

$$\Gamma = -\frac{Z_o}{Z_o} = -1$$



We can say here that $\tilde{\Gamma} = \left|\Gamma\right|e^{j\varphi} = 1\,e^{+j\pi}$ here.

If $\mathbf{v}_+ = Ae^{j(\omega t-\beta z)}$,

$$\mathbf{v}(z) = Ae^{-j\beta z} - Ae^{j\beta z} \quad \left(e^{j\omega t}\text{ understood}\right)$$

$$\mathbf{i}(z) = \frac{1}{Z_o}\left[Ae^{-j\beta z} + Ae^{j\beta z}\right]$$

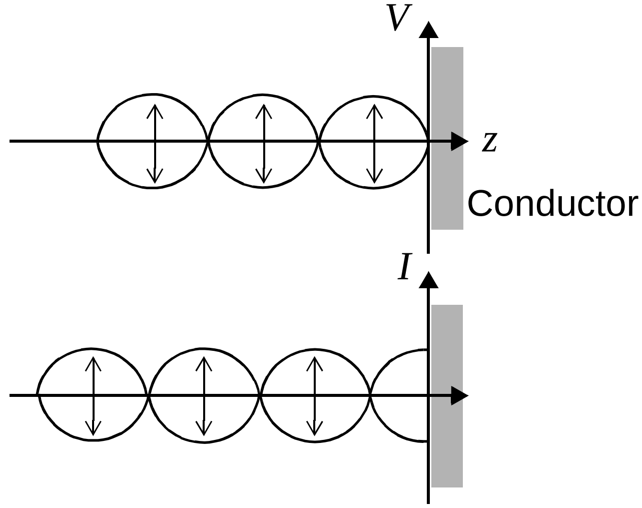
Note that this time, the z dependence has two different parts. Thus, the z dependence cannot be lumped together with $e^{j\omega t}$ dependence, as it can be for a wave traveling in one direction only. Now, the entire V or I, including z dependence, is the phasor:

$$\mathbf{v}(z) = -2jA\sin\beta z$$
$$\mathbf{i}(z) = \frac{2A}{Z_o}\cos\beta z \qquad (e^{j\omega t} \text{ understood})$$

The **real voltage** and **current** are (for $A = |A|e^{j\theta}$)

$$v(z,t) = \mathbf{Re}\left\{-2j|A|e^{j\theta}\sin\beta z\, e^{j\omega t}\right\}$$
$$= 2|A|\sin\beta z\sin(\omega t+\theta)$$

$$i(z,t) = \mathbf{Re}\left\{\frac{2|A|e^{j\theta}}{Z_o}\cos\beta z\, e^{j\omega t}\right\}$$
$$= \frac{2|A|}{Z_o}\cos\beta z\cos(\omega t+\theta)$$



Both voltage and current oscillate sinusoidally in time with different maximum values (amplitudes) at different locations. This is known as a standing wave--"standing," because the amplitude remains the same at each location and the oscillating pattern is standing. Contrast this with "travelling" wave, in which a given point on the wave form progresses in distance with time. The amplitude of a travelling wave, however, is a constant value regardless of location.

The *instantaneous* power flow is (must be calculated from **real** V, I):

$$P(z,t) = \mathbf{v}(z,t)\mathbf{i}(z,t)$$
$$= \frac{4|A|^2}{Z_o}\sin\beta z\cos\beta z\sin(\omega t+\theta)\cos(\omega t+\theta)$$
$$= \frac{|A|^2}{Z_o}\sin 2\beta z\sin 2(\omega t+\theta)$$

The *average* power flow (we calculate it here direction from phasors) is:

$$\textbf{Re}\left[\frac{1}{2}V\,I^*\right] = \textbf{Re}\left[-\frac{1}{2}\,2\,jA\sin\beta z\cdot\frac{2\,A^*}{Z_o}\cos\beta z\right]$$

$$= \textbf{Re}\left[2j\,|A|^2\sin\beta z\,\cos\beta z\right] = 0$$

(purely imaginary)

This makes physical sense, since no power flows into the short circuit. Thus, all the power must be reflected back, giving **net** power flow equal to zero (average). The same result, of course, could be derived from the field point of view. The short circuit is a perfectly conducting plate closing off the line. **No** fields penetrate into it. Thus,

$$\boldsymbol{P}_{av} = e_z\,P_+ - e_z\,P_- = 0$$

The Concept of Input Impedance (Z_{in})

Characteristic impedance Z_o is the ratio of voltage to current of **one** wave direction.

$$Z_o = \frac{\mathbf{v}_+}{\mathbf{i}_+} = \frac{|\mathbf{v}_-|}{|\mathbf{i}_-|}$$

Since transmission lines are used in conjunction with lumped circuits, we must be able to treat them as regular circuits as well. The input impedance of any circuit is $Z_{in} = V/I$ at the terminals. Thus, for a transmission line, **at any point** on it:

$$Z_{in} = \frac{\mathbf{v}_T}{\mathbf{i}_T},$$

where $\mathbf{v}_T$ and $\mathbf{i}_T$ are the **total** phasor fields at that point.

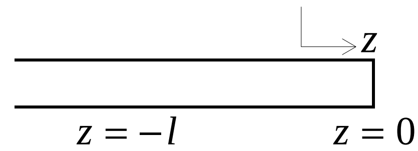
Important: Since $Z_{in} = \mathbf{v}_T/\mathbf{i}_T$, this is now the same as a regular circuit concept. So we can always replace a transmission line circuit with its input impedance!

If there is only a single wave propagating in one direction (transmission line terminated in $Z_L = Z_o$), then, of course, $Z_{in} = Z_o$. But, **in general**, $Z_{in} \neq Z_o$, and Z_{in} is a function of both the load and of the position z on the line.

Let's measure Z_{in} as follows: $Z = 0$ at the load. Therefore, z is negative on the line $z = -l$.

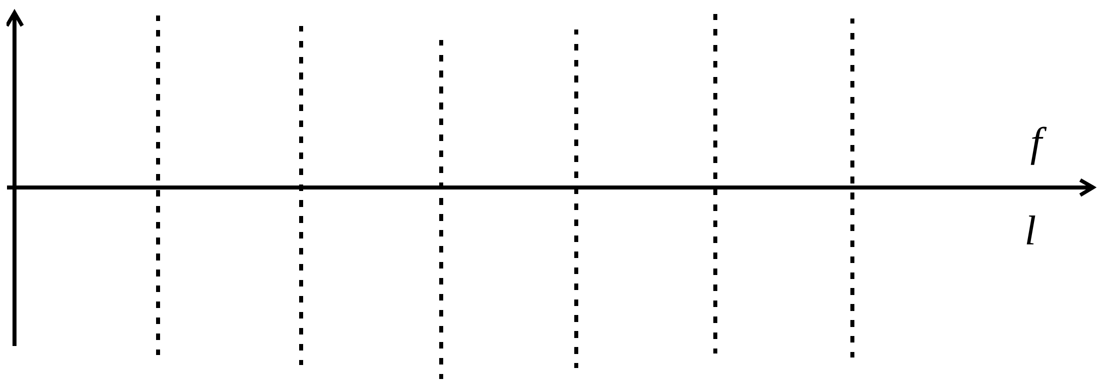
For the short circuited lossless transmission line:

$$Z_{in}(-l) = \frac{\mathbf{v}(-l)}{\mathbf{i}(-l)} = \frac{-2jA\sin\beta(-l)}{\frac{2A}{Z_o}\cos\beta(-l)}$$



$$Z_{in}(-l) = jZ_o\tan\beta l = jZ_o\tan\frac{2\pi}{\lambda}l = jZ_o\tan\frac{2\pi f}{v}l \quad \text{(for a shorted line)}$$

The input impedance of a short circuited lossless line is purely imaginary (as it should be, since it consists of distributed capacitance and inductance.) The values of the input impedance vary with the distance if the frequency is fixed or varies with frequency if the length is constant.



Reactance can be either capacitive or inductive, and its absolute value varies from zero to infinity. This property can be utilized in a number of ways.

(a) Easily adjustable values of reactance.
(b) Location of an accidental short on a line. (Problem)
(c) Matching of a load not equal to Z_o (will study later)

Open Circuited Lossless Line

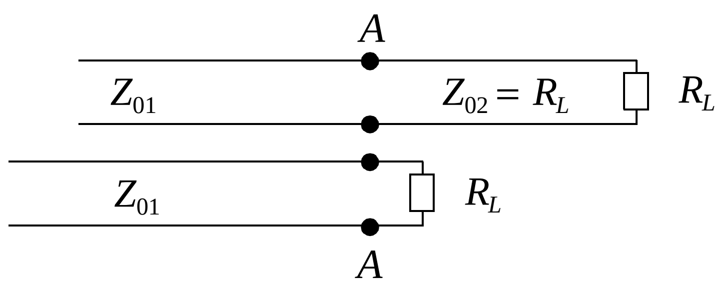
In practice, we try to avoid open circuiting a transmission line. **Reason:** Theoretically, open circuit (i.e., open ends) means infinite load impedance ($R_L = \infty$). This would mean that:

$$\Gamma = \frac{\infty - Z_o}{\infty + Z_o} = +1$$

That is, all of the wave is reflected, resulting in standing waves.

However, in practice, this is not really true. The open end is not a truly infinite impedance. While no current flows beyond the open end, some of the electromagnetic wave goes out the open end and becomes a free space wave. Only a small fraction does, since the open circuited transmission line is “poorly matched” to space (i.e., it is a poor antenna). Yet, if not all the power is reflected, we obviously do not have infinite impedance.

Lossless transmission line terminated in a real load $R_L \neq Z_{01}$



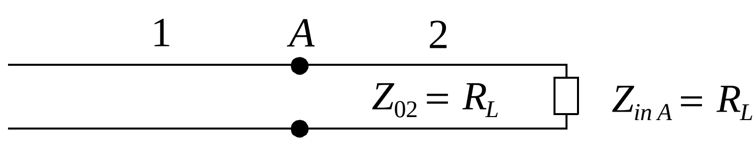
The above two cases are identical as far as T.L. 1 is concerned, since the impedance at point A is R_L for both.

> We can always replace any transmission line by its equivalent input impedance. The fields up to that input point will be unchanged.

For the above, since $Z_{02} \neq Z_{01}$:

$$\Gamma = \frac{R_L - Z_{01}}{R_L + Z_{01}}$$

Γ is real and its value varies from -1 to $+1$.



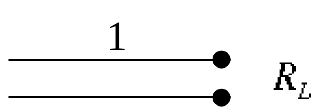
Line 1 has both $+\mathbf{z}$ and $-\mathbf{z}$ going wave.

Line 2 has only $+\mathbf{z}$ going wave, since it is properly terminated.

Just to the left of junction A: $\mathbf{v} = \mathbf{v}_{1+} + \mathbf{v}_{1-}$

Just to the right of junction A: $\mathbf{v} = \mathbf{v}_{2+}$

These are obviously **equal**. So we replace at A by R_L:



$$\tau = \frac{\mathbf{v}_{2+}}{\mathbf{v}_{1+}} = \frac{\mathbf{v}_{1+} + \mathbf{v}_{1-}}{\mathbf{v}_{1+}} = 1 + \frac{\mathbf{v}_{1-}}{\mathbf{v}_{1+}} = 1 + \Gamma$$

τ is called the transmission coefficient. Thus:

$$\mathbf{v}_{2+} = \tau \mathbf{v}_{1+} \qquad \tau = 1 + \Gamma = \frac{2 Z_L}{Z_L + Z_o}$$

τ can be greater than unity, but that does **not** mean more than 100% of power transmitted.

Power balance: $P_r + P_t = P_i$ $\qquad 0 \le |\tau| \le 2$

$$P_r = \frac{\mathbf{v}_{1-}^2}{2 Z_o} = \Gamma^2 \frac{\mathbf{v}_{1+}^2}{2 Z_o} = \Gamma^2 P_i$$

$$P_t = \frac{\mathbf{v}_{2+}^2}{2 Z_L} = \frac{\tau^2 \mathbf{v}_{1+}^2}{2 Z_L} = \tau^2 \frac{\mathbf{v}_{1+}^2}{2 Z_o} \cdot \frac{Z_o}{Z_L}$$

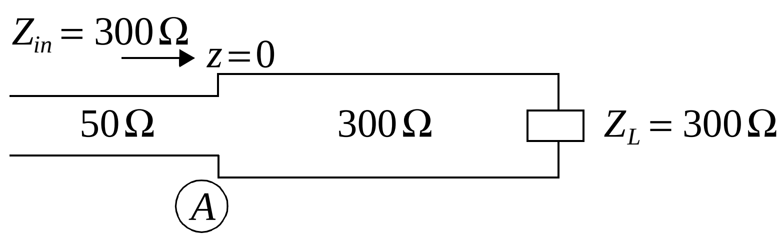
Substituting for Γ and τ and writing the sum of the reflected and transmitted power = incident:

$$\frac{\mathbf{v}_{1+}^2}{Z_o}\left[\left(\frac{Z_L - Z_o}{Z_L + Z_o}\right)^2 + \frac{4 Z_L^2}{(Z_L + Z_o)^2}\left(\frac{Z_o}{Z_L}\right)\right] = \frac{\mathbf{v}_{1+}^2}{Z_o}$$

$$\frac{Z_L^2 + Z_o^2 - 2 Z_L Z_o + 4 Z_L Z_o}{(Z_L + Z_o)^2} = 1$$

$$1 = 1 \quad \text{Q.E.D.}$$

Example:



Commercial coaxial line (Radio Shack) is $50\,\Omega$, flat antenna line is $300\,\Omega$. The above situation would happen if you connect them together without a matching transformer. (We shall learn later what a matching transformer is.)

At the junction $A(z=0)$:

$$\Gamma = \frac{Z_{in} - Z_o}{Z_{in} + Z_o} = \frac{300 - 50}{300 + 50} = 0.714$$

$$\tau = 1 + \Gamma = 1.714$$

The fraction of power reflected is $\Gamma^2 = 0.51$.

The fraction of power transmitted to $300\,\Omega$ line and on to the load is $1 - 0.51 = 0.49$.

General Transmission Line

Now let's return to a general case and find the total voltage and current on the line as a function of distance:

$$\mathbf{v}_1 = V_{1+}\, e^{-j\beta z} + \Gamma\, V_{1+}\, e^{+j\beta z} = V_o\, e^{-j\beta z}\left(1 + \Gamma\, e^{2j\beta z}\right) \qquad \text{(Let } V_{1+} = V_o\text{)}$$

$$\mathbf{i}_1 = \frac{V_{1+}}{Z_o} e^{-j\beta z} - \Gamma \frac{V_{1+}}{Z_o} e^{+j\beta z} = \frac{V_o}{Z_o} e^{-j\beta z}\left(1 - \Gamma\, e^{2j\beta z}\right)$$

$e^{j\omega t}$ variation understood! We have dropped $e^{j\omega t}$.

The real fields, of course, vary sinusoidally in time. $\mathbf{v}_1$ and $\mathbf{i}_1$ are the complex phasor amplitudes. Since, for a complex voltage $\mathbf{v} = Ae^{j\omega t}$, the maximum value (or sinusoidal amplitude) is $|A|$, and the sinusoidal amplitude of V_1 along the line is:

$$|\mathbf{v}_1| = V_o \left|e^{-j\beta z}\right| \left|1 + \Gamma\, e^{2j\beta z}\right| = V_o \left|1 + \Gamma\left(\cos 2\beta z + j \sin 2\beta z\right)\right| \qquad \text{(for } \Gamma \text{ real)}$$

$$= V_o \sqrt{(1 + \Gamma \cos 2\beta z)^2 + \Gamma^2 \sin^2 2\beta z} = V_o \sqrt{1 + \Gamma^2 + 2\Gamma \cos 2\beta z}$$

Remember that Γ may be positive or negative.

$$2\beta z = \frac{4\pi}{\lambda} z$$

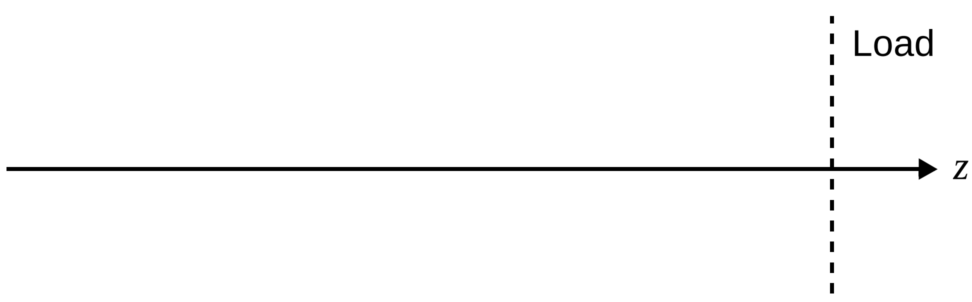
The minimum value of this is (if Γ is positive):

$$|\mathbf{v}_1|_{min} = V_o \sqrt{1 + \Gamma^2 - 2\Gamma} = V_o (1 - \Gamma)$$

For a general Γ: $|\mathbf{v}_1|_{min} = V_o\left(1 - |\Gamma|\right)$

Similarly: $|\mathbf{v}_1|_{max} = V_o \sqrt{1 + \Gamma^2 + 2\Gamma} = V_o\left(1 + |\Gamma|\right)$

The location of consecutive minima and maxima are $\lambda/2$ apart. At the load, we have either a maximum or a minimum (max. for Γ positive). The shape of the function vs. z is not a pure sinusoid. The minima are not infinitely sharp (as they are for a short circuit).



An A.C. voltmeter, of course, does not measure the time variation, but only the amplitude (top solid line). (Actually the RMS value)
This is called a partial standing wave. An important measurable quantity is the Standing Wave Ratio (SWR) = S.

$$\boxed{S = \frac{V_{max}}{V_{min}} = \frac{1+|\Gamma|}{1-|\Gamma|}}$$

S can range from 1 to ∞.

Example: For the previous example,

$$S = \frac{1.714}{0.286} = 6$$

We note also that for Γ positive ($Z_L > Z_o$, Z_L real),

$$S = \frac{1+\dfrac{Z_L - Z_o}{Z_L + Z_o}}{1-\dfrac{Z_L - Z_o}{Z_L + Z_o}} = \frac{2Z_L}{2Z_o} = \frac{Z_L}{Z_o}$$

and for Γ negative ($Z_L < Z_o$, Z_L real),

$$S = \frac{1+\dfrac{Z_o - Z_L}{Z_o + Z_L}}{1-\dfrac{Z_o - Z_L}{Z_o + Z_L}} = \frac{2Z_o}{2Z_L} = \frac{Z_o}{Z_L}$$

This also agrees with our example: $6 = 300/50$

If a transmission line is connected to an unknown load, the SWR can be measured and, from this, the load value found. But to fully understand how, we must consider a more general load.

Lossless Transmission Line Terminated in a General Complex Load Z_L

Most practical circuits/devices (whether the device be a transmitting antenna, a TV, a receiver, a microwave amplifier, an oscilloscope, etc.) have complex input impedance. Thus, if such a device is fed by a transmission line, the transmission line sees a complex load. Without something special being done, the line is not matched to the load. There is a reflected wave.

$$\Gamma = \frac{Z_L - Z_o}{Z_L + Z_o} \qquad \text{(complex, in general)}$$

$$\mathbf{v}_T = V_+ e^{-j\beta z} + \Gamma V_+ e^{+j\beta z}$$

$$\mathbf{i}_T = \frac{V_+}{Z_o} e^{-j\beta z} - \Gamma \frac{V_+}{Z_o} e^{+j\beta z}$$

Assume for simplicity that $\mathbf{v}_+ = V_o \angle 0°$ and that the load is at $Z = 0$. Therefore, on the transmission line, $z = -l$. Then,

$$\mathbf{v}_T = V_o e^{j\beta l} + \Gamma V_o e^{-j\beta l}$$

$$\mathbf{i}_T = \frac{V_o}{Z_o} e^{j\beta l} - \Gamma \frac{V_o}{Z_o} e^{-j\beta l},$$

where ***l* is the distance measured backwards from the load**.

The input impedance at a point l distance back from the load is:

$$Z_{in} = Z_o \frac{e^{j\beta l} + \Gamma e^{-j\beta l}}{e^{j\beta l} - \Gamma e^{-j\beta l}} = Z_o \frac{1 + \Gamma e^{-2j\beta l}}{1 - \Gamma e^{-2j\beta l}}$$

Substituting the value of Γ and rearranging, we get the alternate forms:

$$Z_{in}(-l) = Z_o \frac{Z_L \cos\beta l + j Z_o \sin\beta l}{Z_o \cos\beta l + j Z_L \sin\beta l} = Z_o \frac{Z_L + j Z_o \tan\beta l}{Z_o + j Z_L \tan\beta l}$$

When $Z_L = Z_o$, we get $Z_{in} = Z_o$.

Now returning to the voltage and current:

$$\mathbf{v} = V_o e^{j\beta l}\left[1 + \Gamma e^{-2j\beta l}\right] \qquad Z_{in} = Z_o \frac{1 + \Gamma e^{-2j\beta l}}{1 - \Gamma e^{-2j\beta l}}$$

$$\mathbf{i} = \frac{V_o}{Z_o} e^{j\beta l}\left[1 - \Gamma e^{-2j\beta l}\right] \qquad \textit{another form}$$

This is the same expression as before ($z = -l$), except now Γ is complex $\Gamma = \Gamma \angle \theta_r$. It is easiest to see how this changes the standing wave by considering phasor diagrams. The amplitude of the quantity in square brackets determines the amplitude of the standing wave, since $\left|e^{j\beta l}\right| = 1$.

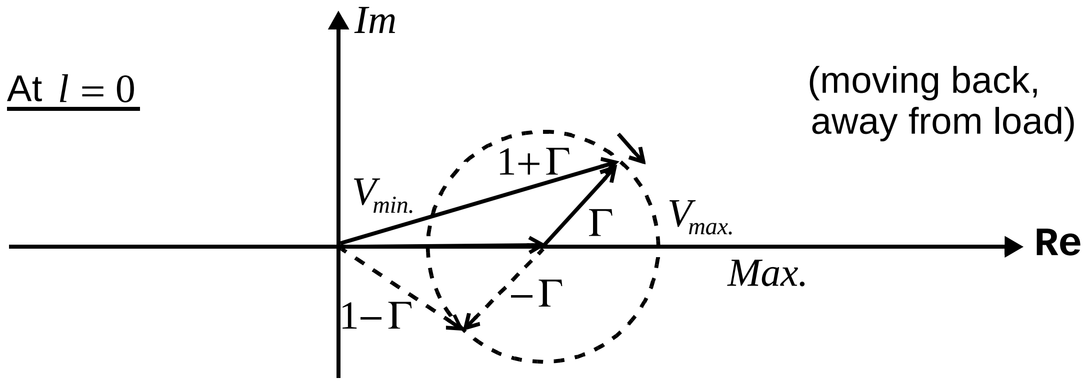
Definition of Generalized Reflection Coefficient

$$\Gamma(l) = \frac{V_-(l)}{V_+(l)} = \frac{V_-(0)e^{-j\beta l}}{V_+(0)e^{+j\beta l}} = \Gamma(0)e^{-2j\beta l} \qquad z = -l$$

In my notation $\Gamma = \Gamma(0)$, reflection coefficient at the load (complex quantity in general) is equivalent to the book's $\tilde{\Gamma}_r$.

Note:

1) Γ is real if load is real and line is lossless. Γ is complex if load is complex.
2) $\Gamma(l)$ is always complex, except at certain points. These points have special significance!
3) $|\Gamma(l)| \equiv |\Gamma|$, that is, moving about on the line only changes the phase angle of $\Gamma(l)$.



It is obvious from this phasor diagram that the cases of real load and complex load are not fundamentally different. $\theta_r = 0$ or $180°$ for a real load, but can be anything for a complex load. The phasor rotates as l increases, making a complete cycle for $2\beta l = 2\pi$; that is, $4\pi l/\lambda = 2\pi$, or $l = \lambda/2$.

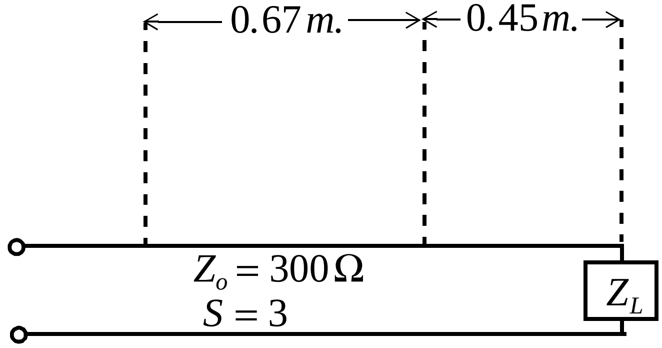
V_{max} is still $1+\Gamma$ and $V_{min} = 1-\Gamma$, where $\Gamma = |\Gamma|$. Obviously, the shape of the standing wave is also the same as for a real load, except the maximum or minimum no longer occurs at the load.

Location of maxima and minima: Maximum occurs at:

$$\theta_r - 2\beta l_{max} = 0, \pm 2\pi, \pm 4\pi, \text{ etc.}$$

Similarly, $\theta_r - 2\beta l_{min} = \pm\pi, \pm 3\pi,$ etc.

Example: An RF signal is sent on a $300\,\Omega$ parallel wire transmission line, which has $\varepsilon_r = 5$, to a receiver. A VSWR of $S = 3$ is measured on the line. The distance between two minima is $0.67\,m.$ and the nearest minimum to the load minimum is $0.45\,m.$ away. Find the frequency of the signal and the The'vnenin equivalent circuit of the receiver input.



Solution:

$$\lambda_m/2 = 0.67 \qquad \lambda_m = 1.34$$

$$\lambda = \lambda_m\sqrt{\varepsilon_r} = 1.34 \cdot \sqrt{5} = 3\,m.$$

$$f = \frac{c}{\lambda} = \frac{3\times10^8}{3} = 10^8 = 100\,MHz$$

$$\beta = \frac{2\pi}{\lambda_m} = \frac{2\pi}{1.34} = 1.49\pi\,m^{-1}$$

$$-2\beta l_{min} + \theta_r = \pi \qquad \therefore 2.98\pi l_m + \pi = \theta_r$$

$$2.34\pi = \theta_r = 0.34\pi = 61.2^\circ$$

$$S = \frac{1+|\Gamma|}{1-|\Gamma|} \qquad \therefore |\Gamma| = \frac{S-1}{S+1} = \frac{2}{4} = 0.5$$

$$\Gamma = 0.5\angle 61.2^\circ = 0.24 + j0.44$$

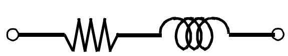
$$\Gamma = \frac{Z_L - Z_o}{Z_L + Z_o} = \frac{Z_l - 300}{Z_L + 300}$$

$$Z_L(1-\Gamma) = 300(1+\Gamma)$$

$$Z_L(0.76 - j0.44) = 300(1.24 + j0.44)$$

$$Z_L = 300\frac{1.24 + j0.44}{0.76 - j0.44} = 300\frac{1.32\angle 19.5^\circ}{0.88\angle -30^\circ} = 450\angle 49.5^\circ$$

$$= 292 + j342 \qquad \Omega = R + j\omega L \qquad L = \frac{342}{2\pi\times10^8} = 0.54\times10^{-6}h$$



Load is $R = 292\,\Omega \quad L = 0.54\,\mu h \qquad ANS.$

Looking at the definition of Z_{in} and at the rotating phasor diagram of the previous page, we also see that:

At the location of a voltage maximum:

$$Z_{in} = Z_o \frac{1+|\Gamma|}{1-|\Gamma|} = Z_o S$$

At the location of a voltage minimum:

$$Z_{in} = Z_o \frac{1-|\Gamma|}{1+|\Gamma|} = \frac{Z_o}{S}$$

$Z_{in} = Z_o S$	at voltage maximum
$Z_{in} = \frac{Z_o}{S}$	at voltage minimum

This is an important observation. It will be used soon for matching.

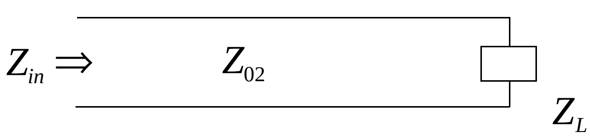
Matching a transmission line to an unmatched load

Reflections on a transmission line are undesirable for a number of reasons. For example, digital signals from PCM reflected pulses can bounce back again from the sending end if that end is also unmatched. Then false signals can be detected.

Q: How do we match an unmatched load?
A: There are several methods. We now study the first one.

I. Quarter Wave Matching ($\lambda/4$ "Transformer")

Suppose we have a transmission line terminated in a **real** load $Z_L (\equiv R_L)$



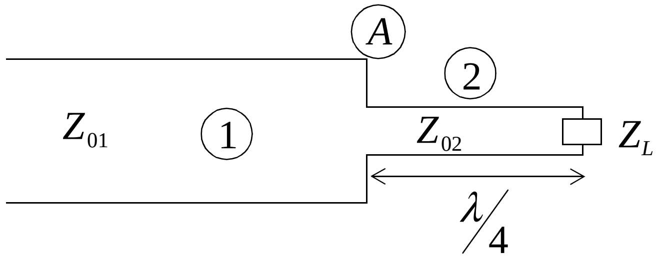
Z_{02} is intrinsic impedance.

If the length of this line is exactly $l = \lambda/4$, then $Z_{in} = \frac{Z_{02}^2}{Z_L}$, since

$$\sin \beta l = \sin \frac{2\pi}{\lambda} \cdot \frac{\lambda}{4} = \sin \frac{\pi}{2} = 1$$

and $\cos \beta l = 0$

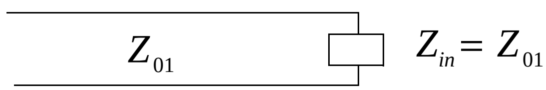
This then offers a method to match this load to **any** lossless transmission line Z_{01}, intrinsic impendance $\overline{\underline{Z_0}}$ to be matched to $\boxed{Z_L}$. Z_L real (R_L)



We insert a transmission line of $\lambda/4$, length (2), and having $Z_{02} = \sqrt{Z_{01} \cdot Z_L}$.
Then, at (A):

$$Z_{in} = \frac{Z_{02}^2}{Z_L} = \frac{Z_{01} Z_L}{Z_L} = Z_{01}!$$

Since $Z_{inA} = Z_{01}$, we are matched. No reflection, since at point A, we could replace the whole combination by Z_{inA}.

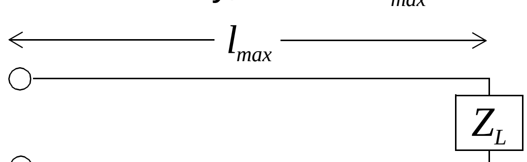


How is this possible physically? We won't go into a detailed analysis of what really happens to the right of point A. But briefly, we do have (in actual physical fact) reflection of waves at point A and at Z_L. The two reflections, however, cancel each other out on the line 1.

Q: This obviously works if the load is real, since Z_{01} is also real, resulting in real Z_{02}. But if the load is complex, then what to do?

A: Recall that the input impedance at a voltage minimum or maximum is real $Z_o S$ and Z_o/S. Thus, these are the steps for matching a complex load Z_L to a line Z_{01}.

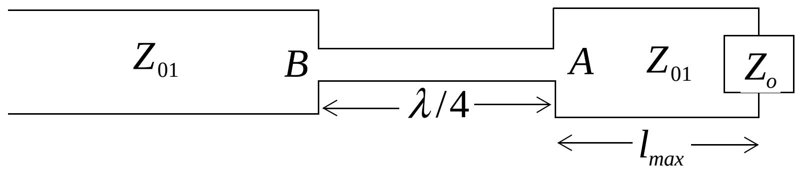
1) Add a section of Z_{01} line to the load. The length should be l_{max} or l_{min} of the standing wave that results. Say, we use l_{max}:



This is equivalent to a load of $Z_L = S Z_{01}$.

2) Now use $\lambda/4$, matching on **this** equivalent load. For example, insert a line:

$$Z_{02} = \sqrt{Z_L Z_{01}} = \sqrt{S\,Z_{01} Z_{01}} = \sqrt{S}\,Z_{01},\ \text{length } \lambda/4$$



The impedance at point B then is:

$$Z_{inB} = \frac{S\,Z_{01}^2}{Z_L^1} = \frac{S\,Z_{01}^2}{SZ_{01}} = Z_{01}$$

Thus we are **matched**. **No reflection** going back on line 1. Same comment applies as before. Actually, there **are** reflections at points A and B, but they are 180° out of phase and cancel out.

The $50\Omega - 300\Omega$ "transformer" supplied for TV and video components is a lumped circuit simulation of a $\lambda/4$ transformer!

Important Note: $l = \lambda/4$ means $\beta l = \frac{2\pi}{\lambda}\cdot\frac{\lambda}{4} = \frac{\pi}{2}$

The matching is only valid at one frequency! Since $\beta = \frac{2\pi f}{v} = \frac{2\pi}{\lambda}$, if f is changed, the wavelength is changed and $\beta l \neq \pi/2$ for the same l.

Example: (a) Match a 50 coaxial cable to a 300Ω real input impedance device at a frequency of $180\ MHz$. Assume all transmission lines used have $\varepsilon_r = 6$. (b) Find the reflection coefficient of the matched circuit at 90 MHz.

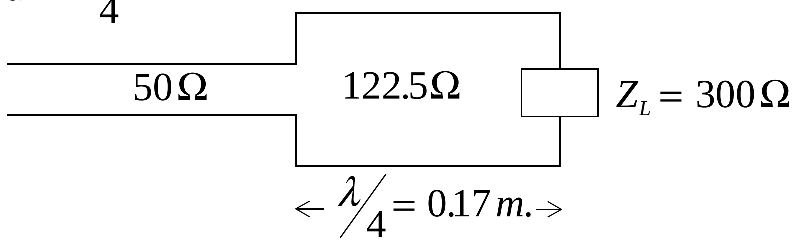
(a) At $180\ MHz$:

$$\lambda_o = \frac{c}{f} = \frac{3\times10^8}{1.8\times10^8} = 1.67\,m.$$

$$\lambda \text{ on the T.L., } \lambda_m = \frac{\lambda_o}{\sqrt{6}} = 0.68\,m.$$

$$\text{T.L. impedance needed } Z_{02} = \sqrt{50\cdot300} = 122.5\Omega \quad ans.$$

$$\text{length needed} = \frac{0.68}{4} = 0.17\,m. \quad ans.$$

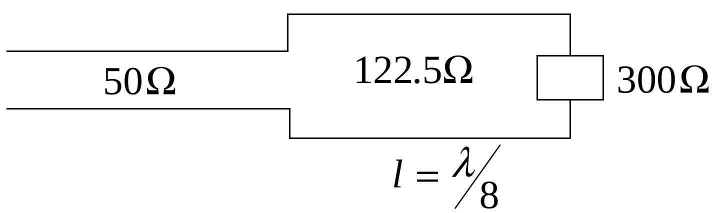


We cannot readily buy a 122.5Ω transmission line. However, any transmission line at these R.F. frequencies can be simulated by a proper lumped circuit network. (In this class, we won't study how.)

(b) If the frequency is changed to $90\ mHz$:

$$\lambda_m = 0.68\times2 = 1.36 \qquad \beta l = \frac{2\pi}{1.36}\times0.17 = \frac{\pi}{4}$$

We have:



$$\beta l = \frac{\pi}{4}$$

$$\tan\beta l = 1$$

$$Z_{inA} = 122.5\frac{300+j122.5\times1}{122.5+j300\times1} = 122.5\frac{324\angle22.2^\circ}{324\angle90-22}$$

$$= 122.5\angle-45.6^\circ = 85.8-j87.5$$

Reflection coefficient at A is:

$$\Gamma = \frac{85.8-j87.5-50}{85.8-j87.5+50} = \frac{33.5-j87.5}{135.8-j87.5}$$

$$= \frac{93.7\angle-69^\circ}{161.5\angle-32.8^\circ} = 0.58\angle-36.2^\circ \quad ans.$$

Fraction of power reflected back at A is $\Gamma^2 = 0.34$.