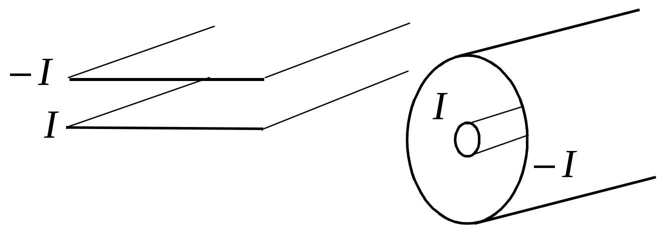
$$\sqrt{\frac{\mu}{\varepsilon}} = \sqrt{\frac{L}{C}}\,\frac{2\pi}{\ln\left(b/a\right)} \Rightarrow Z = \sqrt{\frac{L}{C}} = \sqrt{\frac{\mu}{\varepsilon}}\,\frac{\ln\left(b/a\right)}{2\pi}$$

We know now, from the circuit point of view, that waves exist on a transmission line. Now let's look from the elctromagnetic point of view.

A transmission line consists of 2 parallel metal pieces (unchanged geometry with z!). That is,



Later, it will be shown that, at high frequencies, the currents are surface currents, even if the conductors are thick. However, for now, assume for simplicity that the two conductors are thin strips of metal. Then,

$$I = I_o e^{j(\omega t - \beta z)}$$

We have on one conductor $\vec{J}_s = \hat{e}_z J_{so} e^{j(\omega t - \beta z)}$, where $\vec{J}_s = I_o / l$ (l is the width, or the perimeter). J_s is uniformly distributed in the tranverse plane for these two. From the continuity equation, we have

$$\nabla \cdot \vec{J} = -\frac{\partial \rho}{\partial t} \cdot$$

But $\vec{J}$ is purely axial and is a surface density. Therefore, we have:

$$\frac{\partial}{\partial z} J_s(z) = -\frac{\partial \rho_s(z)}{\partial t}$$

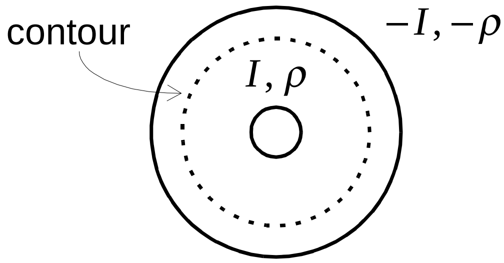
$$J_{so}(-j\beta)\, e^{j(\omega t - \beta z)} = -\frac{d\rho_s}{dt}$$

Therefore, integrating, we get:

$$\rho_s = \frac{\beta}{\omega} J_{so}\, e^{j(\omega t - \beta z)} = \rho_o\, e^{j(\omega t - \beta z)} = \rho_s(z)$$

Thus, we have also a surface charge distribution **equal and opposite sign** on the two conductors. This is also a wave.

Now let's look from the electromagnetic point of view.



I and ρ both have $e^{j(\omega t - \beta z)}$ dependence

Let's consider Maxwell's integral equations.

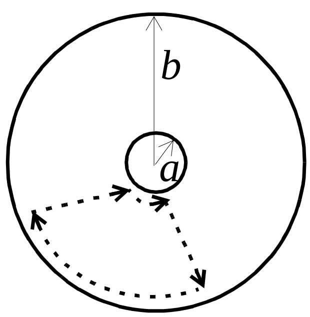
(a) $\oint_c \vec{H} \cdot d\vec{l} = I_{incl} + \iint_s \frac{\partial \vec{D}}{\partial t} \cdot d\vec{s}$

The surface s is in the cross sectional plane. It is obvious that $\vec{E}$ (and $\vec{D}$) is in the plane also and, therefore, the surface integral is **zero**.

$$\therefore \oint \vec{H} \cdot d\vec{l} = I_{incl}$$

Therefore, for a given value of z, we have **same** as the magnetostatic result (which has no z variation).

(b)



$$\oint \vec{E} \cdot d\vec{l} = -\frac{d}{dt} \int_s \vec{B} \cdot d\vec{s}$$

It is obvious that $\vec{B}$ is in the plane. Thus, the surface integral is zero. Therefore, $\oint \vec{E} \cdot d\vec{l} = 0$. That is, for a given value of z, we have the same as the electrostatic result.

We can thus use the concept of the electrostatic potential (for a given value of z)

$$V = -\int_1^2 \vec{E} \cdot d\vec{l}$$

So we can calculate that (i) we can use static E and H (of ∞ long conductors), (ii) add $e^{j(\omega t - \beta z)}$ variation, and (iii):

$$V = -\int_1^2 \vec{E} \cdot d\vec{l} \qquad I = \oint_c \vec{H} \cdot d\vec{l}$$

arbitrary path from conductor 1 to 2 — arbitrary path around one conductor

For the coaxial line, then, we have, by Ampere's Law:

$$I = I_o e^{j(\omega t - \beta z)} \qquad \boxed{\vec{H} = \hat{e}_\varphi \frac{I_o}{2\pi r} e^{j(\omega t - \beta z)}}$$

We also have $\vec{E} = \hat{e}_r \frac{A}{r} e^{j(\omega t - \beta z)}$.

Integrate from a to b:

$$V = -\int \vec{E} \cdot d\vec{r} = A \ln\left(b/a\right) e^{j(\omega t - \beta z)} = V_o e^{j(\omega t - \beta z)}$$

$$\therefore A = \frac{V_o}{\ln\left(b/a\right)}$$

$$\boxed{\vec{E} = \hat{e}_r \frac{V_o}{\ln\left(b/a\right)} \frac{1}{r} e^{j(\omega t - \beta z)}}$$

Note that, for the coaxial line, both E_r and H_φ are functions of r.

But $\vec{E}$ and $\vec{H}$ obey the dynamic Maxwell equations (when counting z and ω dependence). In phasor form:

$$\nabla \times \vec{\mathbf{E}} = -j\omega\mu \vec{\mathbf{H}} \qquad \nabla \times \vec{\mathbf{H}} = \vec{\mathbf{J}} + j\omega\varepsilon \vec{\mathbf{E}}$$

The only components are $\mathbf{E}_r$ and $\mathbf{H}_\varphi$ and j_s. Therefore, the above curl equations simplify to:

$$\frac{\partial \mathbf{E}_r}{\partial z} = -j\omega\mu \mathbf{H}_\varphi \quad (1) \qquad -\frac{\partial \mathbf{H}_\varphi}{\partial z} = j\omega\varepsilon \mathbf{E}_r \quad (2)$$

In full-time domain form, these are:

$$\frac{\partial \mathbf{E}_r}{\partial z} = -\mu \frac{\partial \mathbf{H}_\varphi}{\partial t} \quad (3) \qquad -\frac{\partial \mathbf{H}_\varphi}{\partial z} = \varepsilon \frac{\partial \mathbf{E}_r}{\partial t} \quad (4)$$

Notice the exact analogy to the transmission line equations (!)

$$\frac{\partial V}{\partial z} = -\mathsf{L}\frac{\partial I}{\partial t} \qquad \frac{\partial I}{\partial z} = -\mathsf{C}\frac{\partial V}{\partial t}$$

$$E \Leftrightarrow V \qquad H \Leftrightarrow I \qquad \mu \Leftrightarrow \mathsf{L} \qquad \varepsilon \Leftrightarrow \mathsf{C}$$

constants: $\mathsf{L} =$ inductance / unit length
$\mathsf{C} =$ capacitance / unit length

Combining (3) and (4), we get:

$$\frac{\partial^2 \mathbf{E}_r}{\partial z^2} = +\mu\varepsilon \frac{\partial^2 \mathbf{E}_r}{\partial t^2}, \text{ or } \frac{\partial^2 \mathbf{H}_\varphi}{\partial z^2} = +\mu\varepsilon \frac{\partial^2 \mathbf{H}_\varphi}{\partial t^2}$$

The wave equation.

Or, in phasor form, combining (1) and (2), we get:

$$\frac{\partial^2 \mathbf{E}_r}{\partial z^2} = -\omega^2 \mu\varepsilon \mathbf{E}_r \qquad \omega^2\mu\varepsilon \equiv \beta^2$$

$$\frac{\partial^2 \mathbf{E}_r}{\partial z^2} = -\beta^2 \mathbf{E}_r \qquad \frac{\partial^2 \mathbf{H}_\varphi}{\partial z^2} = -\beta^2 \mathbf{H}_\varphi$$

The Helmholtz equation with solution:

$$\mathbf{E}_r(r,z) = E(r)e^{j(\omega t-\beta z)} = \frac{V_o}{\ln\left(b/a\right)}\frac{1}{r}e^{j(\omega t-\beta z)} \qquad \mathbf{H}_\varphi(r,z) = H(r)e^{j(\omega t-\beta z)} = \frac{I_o}{2\pi}\frac{1}{r}e^{j(\omega t-\beta z)}$$

Substituting back into the curl equation:

$$-j\beta\mathbf{E}_r = -j\omega\mu\mathbf{H}_\varphi \qquad -j\beta\mathbf{H}_\varphi = -j\omega\varepsilon\mathbf{E}_r$$

$$\frac{\mathbf{E}_r}{\mathbf{H}_\varphi} = \frac{\omega\mu}{\beta} = \sqrt{\frac{\mu}{\varepsilon}} = \eta \qquad \frac{\mathbf{E}_r}{\mathbf{H}_\varphi} = \frac{\beta}{\omega\varepsilon} = \sqrt{\frac{\mu}{\varepsilon}} = \eta$$

So, the behavior of the fields is like a uniform plane wave. **But**, they are not uniform (there is transverse dependence and they are of finite transverse extent!). However, they **are plane** (phase fonts are constant z planes). The ratio of $\mathbf{E}_r / \mathbf{H}_\varphi$ does not depend on r, but is constant (η), just like for uniform plane waves. We already know from the transmission line equations that $\mathbf{V}/\mathbf{I} = \sqrt{\mathsf{L}/\mathsf{C}} = Z$. Therefore, for the coaxial transmission line, we have:

$$\eta = \frac{\mathbf{E}_r(r,z)}{\mathbf{H}_\varphi(r,z)} = \frac{E(r)}{H(r)} = \frac{V_o}{\ln\left(b/a\right)} \frac{1}{\not r} \frac{1}{I_o} \not r\, 2\pi = \frac{V_o}{I_o} \frac{2\pi}{\ln\left(b/a\right)}$$

$$\sqrt{\frac{\mu}{\varepsilon}} = \sqrt{\frac{\mathsf{L}}{\mathsf{C}}} \frac{2\pi}{\ln\left(b/a\right)} \Rightarrow Z = \sqrt{\frac{\mathsf{L}}{\mathsf{C}}} = \sqrt{\frac{\mu}{\varepsilon}} \frac{\ln\left(b/a\right)}{2\pi}$$

This is correct with our known values of L,C, calculated from 2-dimensional statics.

$$\mathsf{L} = \frac{\mu}{2\pi} \ln\left(b/a\right) \qquad \mathsf{C} = 2\pi\varepsilon \frac{1}{\ln\left(b/a\right)}$$

For a general transmission line:

$\mathsf{C} = \varepsilon \frac{1}{g} \quad \mathsf{L} = \mu g$ (g is a geometrically determined constant.)

Therefore, $\sqrt{\frac{\mathsf{L}}{\mathsf{C}}} = \sqrt{\frac{\mu}{\varepsilon}}\, g$

$\beta = \omega\sqrt{\mathsf{LC}} = \omega\sqrt{\mu\varepsilon}$ (same as plane wave propagation constant)

It is also instructive to calculate the power flow from the electromagnetic point of view. The Poynting vector $\vec{P}_{av}$ is the complex Poynting vector:

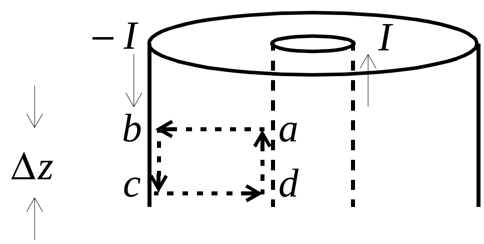
$$\vec{P}_{av} = \frac{1}{2}\,\vec{\mathbf{E}}\times\vec{\mathbf{H}}^{*} = \hat{e}_z\,\frac{1}{2}\mathbf{E}_r\,\mathbf{H}_\varphi = \hat{e}_z\,\frac{V_o}{\ln\left(b/a\right)}\,\frac{1}{r}\,\frac{I_o^{*}}{2\pi}\,\frac{1}{r}$$

$$\vec{P}_{av} = \hat{e}_z\,\frac{1}{2}\,\frac{V_o\,I_o^{*}}{2\pi\,\ln\left(b/a\right)}\,\frac{1}{r^2}$$

$$\vec{P}_{av} = \hat{e}_z\int_0^{2\pi}\int_a^b P_{av}\,r\,dr\,d\varphi = \frac{1}{2}V_o\,I_o^{*}\,\frac{1}{2\pi}\,\frac{\hat{e}_z}{\ln\left(b/a\right)}\,2\pi\int_a^b\frac{dr}{r}$$

$$\vec{P}_{av} = \hat{e}_z\,\frac{1}{2}\,V_o\,I_o^{*}$$

Another look: The transmission line equations derived from Maxwell's equations.



which is just another form of

$$\nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$

$$\oint\vec{E}\cdot d\vec{l} = -\frac{d}{dt}\,\Phi$$

Take line integral around path indicated:

$$\int_a^b \mathbf{E}_r(r,z)\,dr + \underbrace{\int_b^c \mathbf{E}_r(r,z)\,dr}_{=0} + \int_c^d \mathbf{E}_r(r,z+\Delta z)\,dr + \underbrace{\int_d^a \mathbf{E}_r(r,z+\Delta z)\,dr}_{=0} = -\frac{d}{dt}\,\Phi$$

Since the definition of $\mathbf{L} = \dfrac{\Phi}{I}$,

$$V(z) - V(z+\Delta z) = -\,\mathbf{L}\,\frac{\partial I}{\partial t} = -\,\mathrm{L}\,\Delta z\,\frac{\partial I}{\partial t}$$

where L' is the inductance per unit length,
$\mathbf{L}$ is the total inductance, and $\mathbf{L} = \mathrm{L}'\Delta z$.

$$\therefore \frac{V(z+\Delta z)-V(z)}{\Delta z} = -\mathsf{L}\frac{\partial I}{\partial t}$$

$$\frac{\partial V}{\partial z} = -\mathsf{L}\frac{\partial I}{\partial t}$$