

## Review of Vector Analysis

A vector  $\mathbf{A}$  can be written as

$$\mathbf{A} = \hat{x}A_x + \hat{y}A_y + \hat{z}A_z. \quad (1)$$

Similarly, a vector  $\mathbf{B}$  can be written as

$$\mathbf{B} = \hat{x}B_x + \hat{y}B_y + \hat{z}B_z. \quad (2)$$

In the above,  $\hat{x}, \hat{y}, \hat{z}$  are unit vectors pointing in the  $x, y, z$  directions respectively.  $A_x, A_y$  and  $A_z$  are the components of the vector  $\mathbf{A}$  in the  $x, y, z$  directions respectively. The same statement applies to  $B_x, B_y$ , and  $B_z$ .

### Addition

$$\mathbf{A} + \mathbf{B} = \hat{x}(A_x + B_x) + \hat{y}(A_y + B_y) + \hat{z}(A_z + B_z). \quad (3)$$

### Multiplication

(a) Dot Product (scalar product)

$$\mathbf{A} \cdot \mathbf{B} = A_xB_x + A_yB_y + A_zB_z, \quad (4)$$

$$\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A}, \quad \text{commutative property} \quad (5)$$

$$\mathbf{A} \cdot (\mathbf{B} + \mathbf{C}) = \mathbf{A} \cdot \mathbf{B} + \mathbf{A} \cdot \mathbf{C}, \quad \text{distributive property} \quad (6)$$

$$\mathbf{A} \cdot \mathbf{B} = |\mathbf{A}| |\mathbf{B}| \cos \theta. \quad (7)$$

In (7),  $\theta$  is the angle between vectors  $\mathbf{A}$  and  $\mathbf{B}$ .

(b) Cross Product (vector product)

$$\begin{aligned} \mathbf{A} \times \mathbf{B} &= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = \hat{x}(A_yB_z - A_zB_y) + \hat{y}(A_zB_x - A_xB_z) \\ &\quad + \hat{z}(A_xB_y - A_yB_x), \end{aligned} \quad (8)$$

$$\mathbf{A} \times \mathbf{B} = \hat{u} |\mathbf{A}| |\mathbf{B}| \sin \theta, \quad (9)$$

where  $\hat{u}$  is a unit vector obtained from  $\mathbf{A}$  and  $\mathbf{B}$  via the right hand rule.

$$\mathbf{A} \times (\mathbf{B} + \mathbf{C}) = \mathbf{A} \times \mathbf{B} + \mathbf{A} \times \mathbf{C}, \quad \text{distributive property} \quad (10)$$

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) \neq (\mathbf{A} \times \mathbf{B}) \times \mathbf{C}, \quad \text{non-associative property} \quad (11)$$

$$\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A}, \quad \text{anti-commutative property} \quad (12)$$

## Vector Derivatives

$$\text{Del } \nabla = \hat{x} \frac{\partial}{\partial x} + \hat{y} \frac{\partial}{\partial y} + \hat{z} \frac{\partial}{\partial z}, \quad (13)$$

$$\text{Gradient } \nabla \phi = \hat{x} \frac{\partial}{\partial x} \phi + \hat{y} \frac{\partial}{\partial y} \phi + \hat{z} \frac{\partial}{\partial z} \phi, \quad (14)$$

$$\text{Divergent } \nabla \cdot \mathbf{A} = \frac{\partial}{\partial x} A_x + \frac{\partial}{\partial y} A_y + \frac{\partial}{\partial z} A_z, \quad (15)$$

$$\begin{aligned} \text{Curl } \nabla \times \mathbf{A} &= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} \\ &= \hat{x} \left( \frac{\partial}{\partial y} A_z - \frac{\partial}{\partial z} A_y \right) + \hat{y} \left( \frac{\partial}{\partial z} A_x - \frac{\partial}{\partial x} A_z \right) \\ &\quad + \hat{z} \left( \frac{\partial}{\partial x} A_y - \frac{\partial}{\partial y} A_x \right). \end{aligned} \quad (16)$$

## Divergence Theorem

$$\oint_V \nabla \cdot \mathbf{A} dV = \oint_S \mathbf{A} \cdot \hat{n} dS. \quad (17)$$

## Stokes Theorem

$$\oint_S (\nabla \times \mathbf{A}) \cdot \hat{n} dS = \oint_C \mathbf{A} \cdot d\mathbf{l}. \quad (18)$$

### Some Useful Vector Identities

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}), \quad (19)$$

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b}), \quad (20)$$

$$\mathbf{a} \times \mathbf{a} = \mathbf{0}, \quad (21)$$

$$\mathbf{a} \cdot (\mathbf{a} \times \mathbf{b}) = 0, \quad (22)$$

$$\nabla \times (\nabla \phi) = \mathbf{0}, \quad (23)$$

$$\nabla \cdot (\nabla \times \mathbf{A}) = 0, \quad (24)$$

$$\nabla \cdot (\psi \mathbf{A}) = \mathbf{A} \cdot \nabla \psi + \psi \nabla \cdot \mathbf{A}, \quad (25)$$

$$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot \nabla \times \mathbf{A} - \mathbf{A} \cdot \nabla \times \mathbf{B}, \quad (26)$$

$$\nabla \times \nabla \times \mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - \nabla \cdot \nabla \mathbf{A}, \quad (27)$$

$$\nabla^2 = \nabla \cdot \nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}. \quad (28)$$