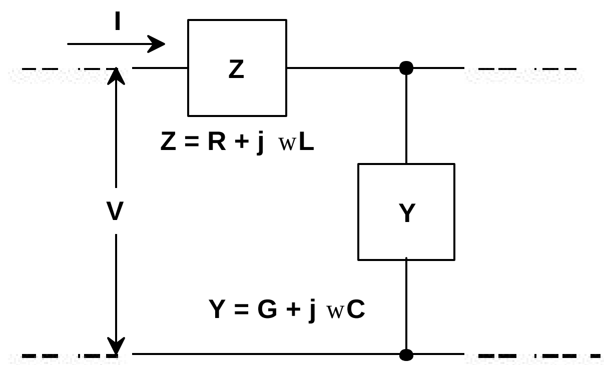
Waves and Impedances on Transmission Lines

Transmission Line Circuit Model[1]

Consider a transmission line consisting of iterated incremental elements as shown here:



Z and Y are the impedance and admittance per unit length ?z.

Z = R + jwL and **Y = G + jwC**, where

R is the series resistance per unit length ?z, Ω/m
L is the series inductance per unit length ?z, H/m
G is the shunt conductance per unit length ?z, S/m
C is the shunt capacitance per unit length ?z, F/m

The equations for V and I are

$\frac{\mathbf{dV}}{\mathbf{dz}} = \mathbf{ZI}$ and $\frac{\mathbf{dI}}{\mathbf{dz}} = \mathbf{YV}$, simultaneous solution of which yields

$\frac{\mathbf{d^2V}}{\mathbf{dz^2}} = \mathbf{ZYV}$ and $\frac{\mathbf{d^2I}}{\mathbf{dz^2}} = \mathbf{ZYI}$; z here represents distance along the transmission line.

The solution of these equations is in the form of waves in the +z and -z direction, which for sinusoidal excitation take the form

$\mathbf{V(z) = V_+e^{wt-jgz} + V_-e^{wt+jgz}}$ and $\mathbf{I(z) = I_+e^{(wt-jgz)} + I_+e^{(wt+jgz)}}$

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[1] This particular derivation is from Terman, *Electronic and Radio Engineering*, 4th Ed., McGraw-Hill, 1955, Ch. 4

The propagation constant γ is given by

$\gamma = \alpha + \mathbf{j}\beta = \sqrt{\mathbf{ZY}}$. For ωL>>R and ωC>>G (low or zero loss case),

$$\beta = \omega\sqrt{\mathbf{LC}}$$

The voltage and current functions represent waves in each direction such that successive peaks and troughs move at a velocity

$$\mathbf{v} = \frac{\omega}{\beta} = \mathbf{f}\lambda \text{, so } \beta = \frac{2\pi}{\lambda}$$

To distinguish it from the free-space wavelength nomenclature λ or λ_o, the wavelength on a waveguide or coaxial transmission line is often referred to as the *guide wavelength* λ_g.

For a single wave solution in one direction, the ratio V(z)/I(z) is the same everywhere on the line, and is defined as the *characteristic impedance* Z_o, which for a lossless line is a real number

$$\mathbf{Z_o} = \frac{\mathbf{V_+}}{\mathbf{I_+}} = \sqrt{\frac{\mathbf{Z}}{\mathbf{Y}}} = \sqrt{\frac{\mathbf{L}}{\mathbf{C}}}$$, where L and C are the inductance and capacitance per unit length.

Thus we can rewrite the current equation as

$$\mathbf{I(z)} = \mathbf{I_+ e^{(\omega t - j\beta z)}} + \mathbf{I_+ e^{(\omega t + j\beta z)}} = \frac{\mathbf{V_+}}{\mathbf{Z_o}}\mathbf{e^{j(\omega t - \beta z)}} - \frac{\mathbf{V_-}}{\mathbf{Z_o}}\mathbf{e^{j(\omega t + \beta z)}}$$

where the minus sign reflects the fact that the magnetic field, and hence the current, of the negative-going propagation is reversed compared to that of a positive-going wave. If both waves exist, the instantaneous voltage or current as function of location is the sum of voltages or currents of both waves. The characteristic impedance Z_o is the ratio of voltage to current of either wave independently, but <u>not necessarily their sum</u>.

<u>Transmission Line Parameters</u>

If we consider an infinite lossless transmission line, we can determine the inductance L and capacitance C per unit length from geometric field considerations. The three physical embodiments that are of interest are the two-wire transmission line, the coaxial transmission and the microstrip transmission line (a simple parallel-plate approximation).

Parameter	Two-wire	Coaxial	Microstrip
L	$\frac{\mu}{p}$ ln (D/a)	$\frac{\mu}{2p}$ ln(b/a)	μT/W
C	$\frac{p\varepsilon}{\ln(D/a)}$	$\frac{2p\varepsilon}{\ln(b/a)}$	εW/T

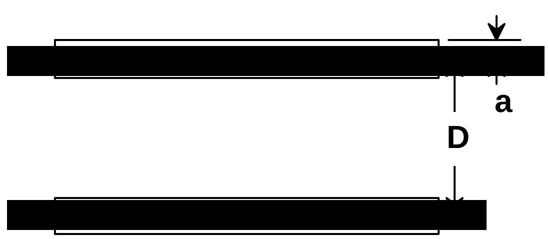
In this table, D and a are the center-to-center spacing and wire radius of the two-wire line, b and a are the outer and inner radius of the coaxial line and T and W are the dielectric thickness and conductor width of the microstrip line. For two-wire line, the expressions include the approximation $\cosh^{-1}(D/2a) \approx \ln(D/a)$ for D/2a >> 1.

If we solve for Z_o of coaxial and microstrip line, we have

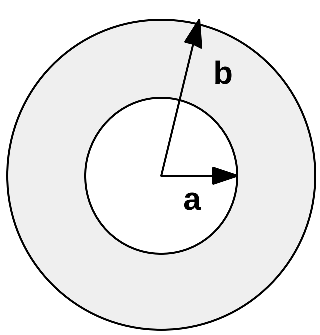
$$\mathbf{Z_o = \frac{377}{2p\sqrt{e_r}} \ln(b/a)}$$ for coaxial line (note use of ln and $\log_{10}$ in different references), and

$$\mathbf{Z_o \approx \frac{377}{\sqrt{e_r}} T/W}$$ for microstrip line, ignoring fringing fields.

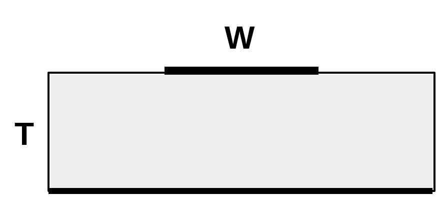
For microstrip, more accurate approximations available in the literature, and there is also a simple Macintosh program used in EE344 Lab that calculates Z_o given ε_r, T and W.



Parallel Wire Line



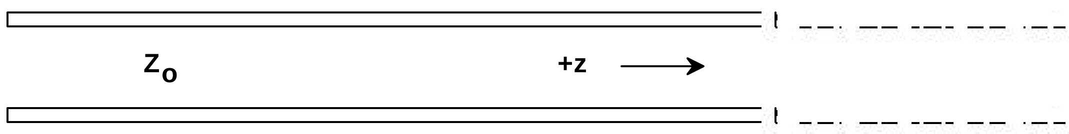
Coaxial Line



Microstrip

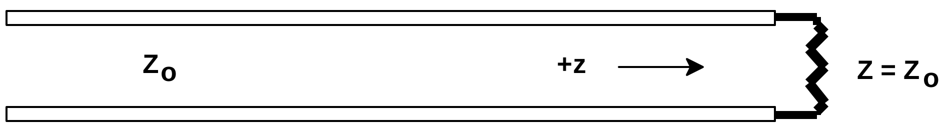
Matched Load

If the transmission is uniform and infinite, the wave in the +z direction will continue indefinitely and never return in the -z direction.



Infinite Transmission Line

If the uniform transmission line is truncated and connected instead to a lumped resistive load $R_L = Z_o$, the entire +z wave is dissipated in the load, which has the same effect as if an infinite line of characteristic impedance Z_o were attached at the same point. This *matched* impedance condition is a unique situation in which all the power of the +z wave is delivered to the load just as if it were an infinite transmission line, with no reflected waves generated in the -z direction.

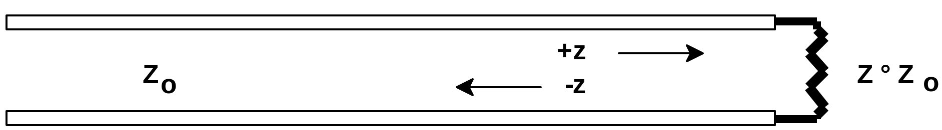


Matched Termination Same as Infinite Line

Boundary conditions at a matched load are the same as for the infinite transmission line.

Transmission Line Discontinuities and Load Impedances

If the wave on a transmission line of characteristic impedance Z_o arrives at a boundary with different Z_o, or at a discontinuity, lumped load or termination of Z ? Z_o, the single wave moving in the +z direction cannot simultaneously satisfy the boundary conditions relating V(z) to I(z) on both sides of the boundary. On one side of the boundary $V(z)/I(z) = V_+/I_+ = Z_o$ and on the other side $V(z)/I(z) = (V_+ + V_-)/(I_+ - I_-) = Z_L$. As in the case for a plane



Mismatched Load Creates Reflected Wave

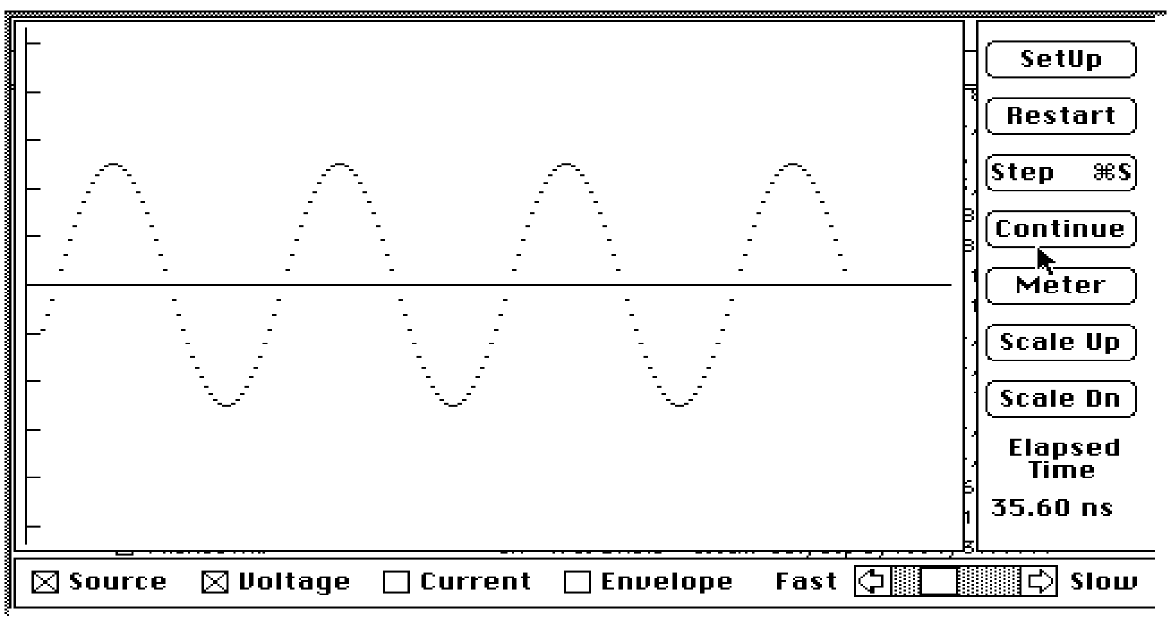
wave reflecting from a dielectric or conducting boundary, transmitted and reflected waves are required to satisfy all the boundary conditions[2].

Waves can exist traveling independently in either direction on a linear transmission line. If a wave in the -z direction is formed by a complete or partial reflection of the +z wave by some discontinuity such as a lumped load of Z? Z_o, the two waves are by definition coherent and an interference pattern will exist.

Even though the waves are traveling in opposite directions, the interference pattern will be stationary with respect to the point of reflection, and will thus be a *standing wave* such as may be found on the strings of musical instrument (of course, these are also defined by a wave equation). The standing wave interference pattern is present both in the resulting V(z) and I(z).

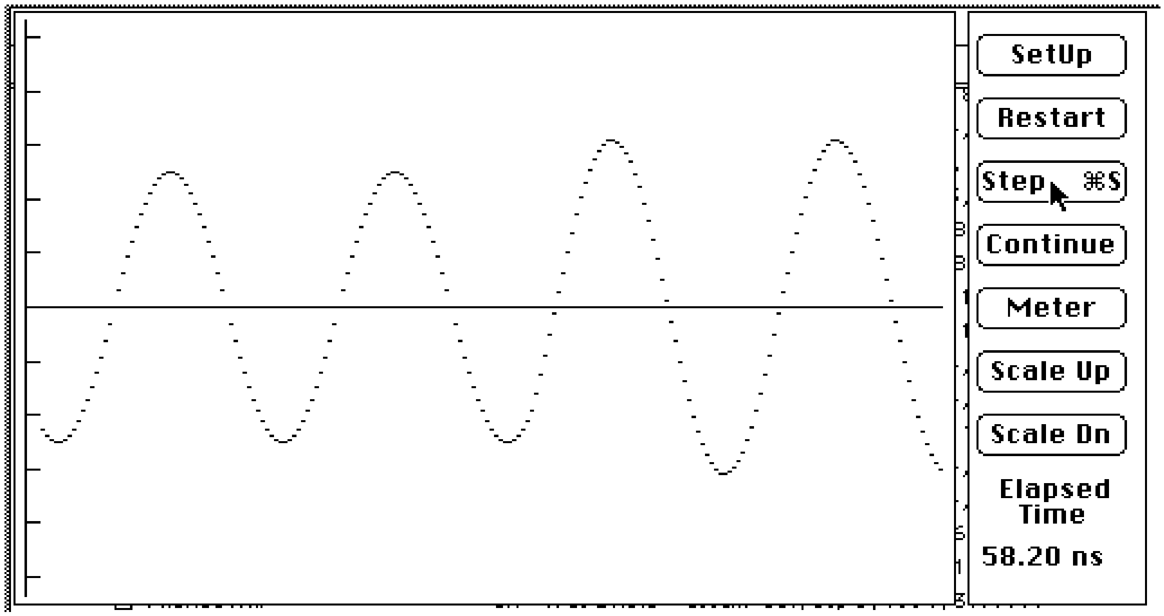
Visualization of Standing Waves

The following set of graphs show the development of the reflected wave, beginning with an initially advancing incident wave moving to the reader's right, which is just about to reach the load point of reflection. For these graphs, Z_o=50? and Z_L=100? .

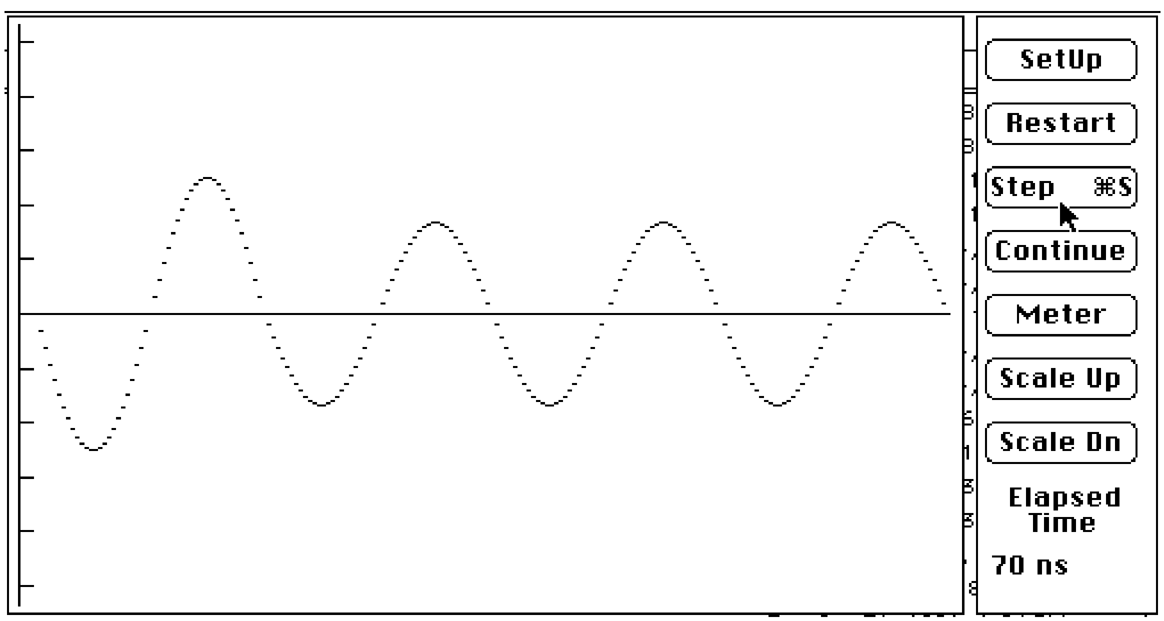


In the next graph, the incident wave has reached the point of reflection, and the reflected wave can be seen to be moving back to the reader's left. In this picture, the waveforms add to a greater magnitude.

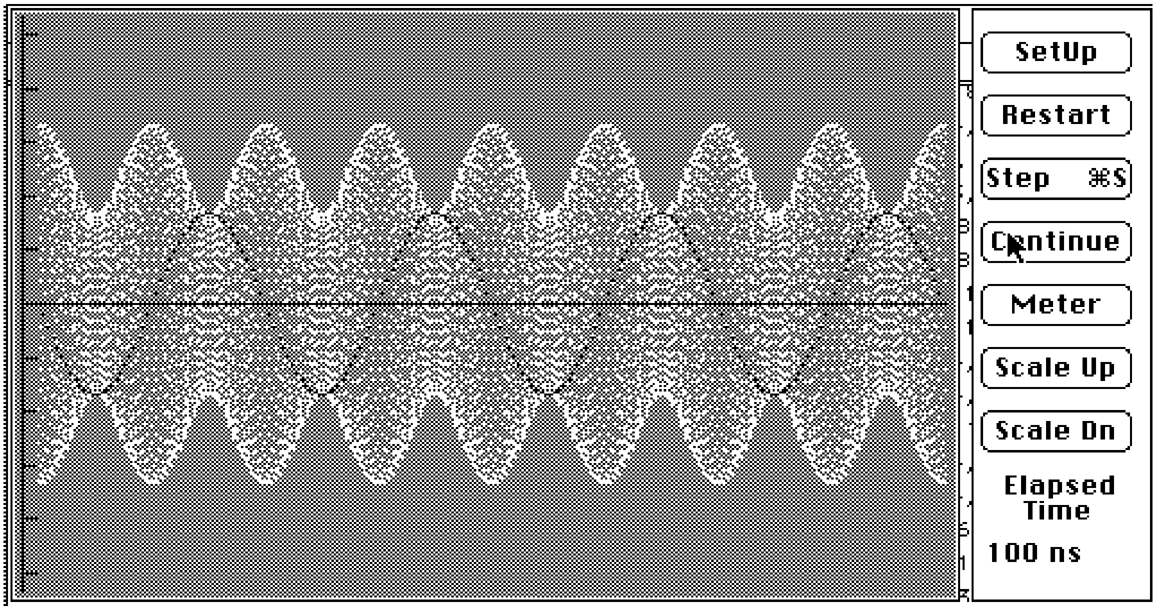
2 Pozar, D., *Microwave Engineering*, 2nd Ed., J. Wiley, 1998, Ch 2



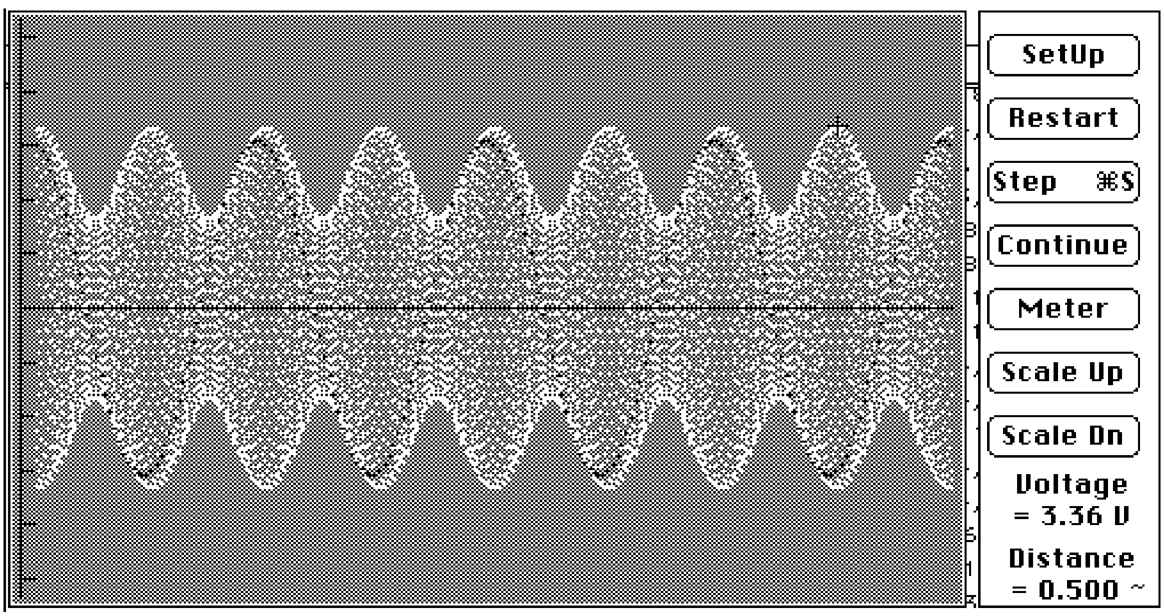
The reflected wave advances further to the left. In this picture, the waves are subtracting.



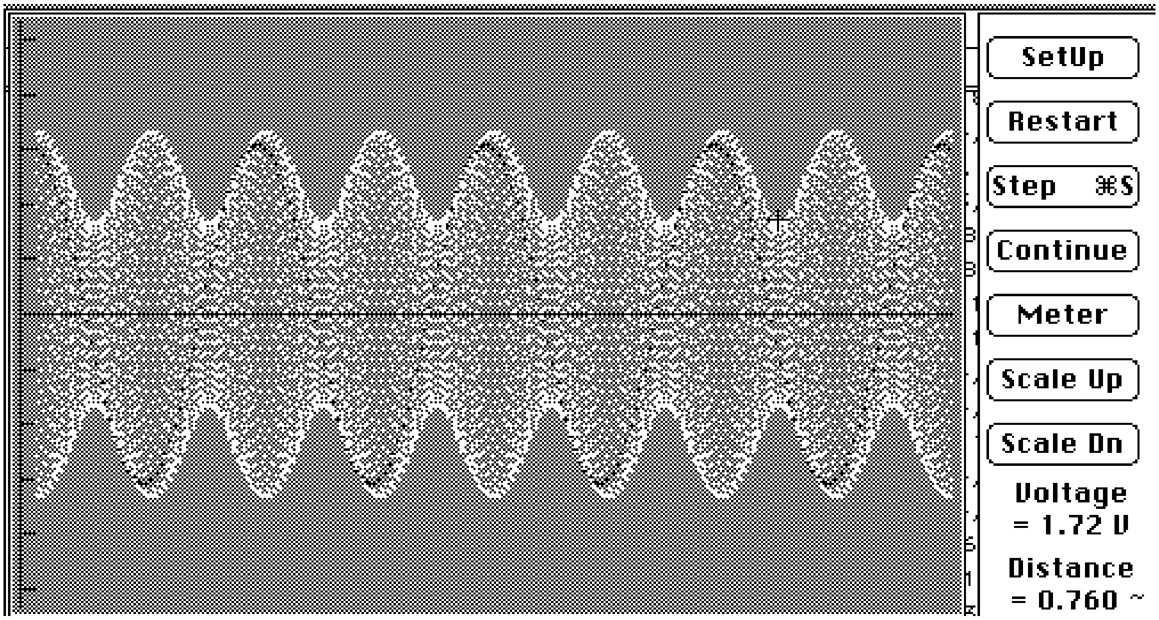
Once the reflected wave has reached steady state and moved off the left of the picture, we can look at the envelope of the combined waveforms. We see that, being coherent, they interfere and form a standing wave, with the voltage maximum at the point of the mismatched load Z_L=200? .



Using the voltmeter function of the software, we can measure the peak as 3.36 V, and in the next picture...



...the minimum as 1.72 V. Hence the ratio of peak to minimum is 3.36/1.72˜2, which is the SWR, the standing wave ratio. This is real number, and does not vary with location on the line.



The Complex Reflection Coefficient Γ

We can express the total voltage and current resulting from waves traveling in both directions:

$\mathbf{V(z) = V_{+}e^{j(\omega t-\beta z)}+V_{-}e^{j(\omega t+\beta z)}}$, and

$$\mathbf{I(z) = \frac{V_{+}}{Z_o}e^{j(\omega t-\beta z)} - \frac{V_{-}}{Z_o}e^{j(\omega t+\beta z)}}.$$

A mismatched load may be either a lumped impedance or an infinite transmission line of a different Z_o. If we consider a complex load impedance Z_L terminating a transmission line Z_o, the magnitude of the -z wave is related to that of the +z wave at the termination by a complex quantity defined as the *reflection coefficient* Γ_L, defined such that

$\mathbf{V_{-}} = \Gamma_{\mathbf{L}}\mathbf{V_{+}}$, where

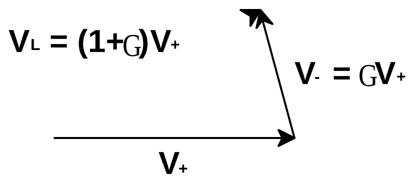
$$\Gamma_{\mathbf{L}} = \frac{\mathbf{V_{-}}}{\mathbf{V_{+}}} = \frac{\mathbf{-I_{-}}}{\mathbf{I_{+}}} = |\Gamma_{\mathbf{L}}|\mathbf{e}^{j\theta} = \rho\ \mathbf{e}^{j\theta} = \rho\,\underline{/\theta}$$

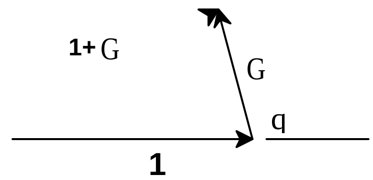
The relationship between the incident wave, the reflected wave and the transmitted wave arising from such a discontinuity such as a lumped load is expressed in terms of the reflection coefficient, so that the reflected wave voltage phasor at the point of reflection is

$\mathbf{V_{-}} = \Gamma_{\mathbf{L}}\mathbf{V_{+}}$ and

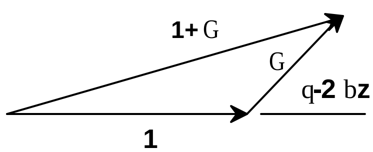
$\mathbf{V_L = V_{+}+V_{-} = V_{+}(1+}\Gamma_{\mathbf{L}}\mathbf{)}$. This is shown in vector form here below left.

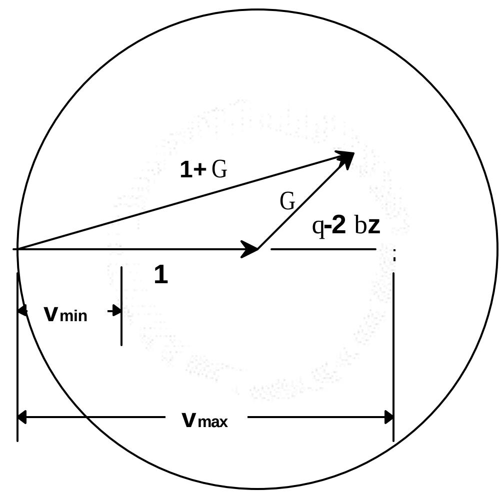
If we normalize to $v_+ = 1$, then $v_- = \Gamma_L$ and $v_L = 1 + \Gamma_L$. This is shown below right.





Now consider what happens if we move a distance -z along the transmission line (in the -z direction, away from the load and toward the generator). At this point we have
$\mathbf{V(d) = 1+\Gamma(z) = 1+\Gamma_L\, e^{-j2\beta z} = 1+|\Gamma|\, e^{j(\theta-2\beta z)}}$, shown in vector form below left.





If we inspect what happens to V(z) as we vary z, we can see from the figure below right that V(z) varies from a maximum of $1+|\Gamma_L|$ to a minimum of $1-|\Gamma_L|$, and that the distance between minima or between maxima is $2\beta z = 2\pi$, which occurs every $\lambda/2$. Also, the distance from a minimum to a maximum is $\lambda/4$.

This particular form of polar presentation has the advantage of containing all possible values of Γ within the circle $|\Gamma| = 1$, and the vector Γ is defined everywhere on a lossless transmission line by the same vector length $|\Gamma|$ or ρ, with the distance to the load in wavelengths identified with the angle of the vector Γ.

Standing Wave Ratio (SWR) and Return Loss

We can identify the maximum and minimum voltages v_{max} and v_{min} (normalized to V_+) by inspection of the figure above.

The ratio of these magnitudes is a real number, the voltage *standing wave ratio* (SWR, or VSWR), given by

$$\mathbf{SWR} = \frac{\mathbf{v_{max}}}{\mathbf{v_{min}}} = \frac{\mathbf{1+|\Gamma_L|}}{\mathbf{1-|\Gamma_L|}} = \frac{\mathbf{1+}\rho}{\mathbf{1-}\rho}$$

. Note that this can be solved for ρ, yielding

$$\rho = \frac{\mathbf{SWR-1}}{\mathbf{SWR+1}}$$

, so if we know SWR we know ρ.

For a matched load ρ=0, SWR =1 and the voltage on the line is just $V(d) = V_+$ for all d; under such a condition the line is termed *flat*.. The ratio of the power in the reflected wave to that in the incident wave, termed the *return loss,* is

$$\frac{\mathbf{P_-}}{\mathbf{P_+}} = \frac{\mathbf{|V_-|^2}}{\mathbf{|V_+|^2}} = \rho^{\mathbf{2}}$$

or, expressed as a loss (a positive number) in dB

$$\mathbf{RL = -10\ log_{10}}\ \rho^{\mathbf{2}} = \mathbf{-20\ log_{10}}\ \rho\,.$$

By measuring the return loss in dB, we can determine

ρ $\mathbf{= 10^{RL/20}}$ and $\mathbf{SWR} = \dfrac{\mathbf{1+}\rho}{\mathbf{1-}\rho}$, which characterizes the degree of impedance match.

In a lossless network, the transmitted power is

$\mathbf{P_t = P_+ - P_- = P_+(1-}\rho^{\mathbf{2}}\mathbf{)}$, and the *transmission loss* TL is

$$\mathbf{TL = -10\ log_{10}\ (1-}\rho^{\mathbf{2}}\mathbf{)\ dB}.$$

At the points of voltage minima and maxima, the impedance is a pure resistance, which makes it possible to evaluate Z at those points in terms of the standing wave ratio SWR.

At a voltage minimum (which is also a current maximum), $Z = R = Z_o$ and

$$\mathbf{Z} = \frac{\mathbf{Z_o}}{\mathbf{SWR}}$$

, a real quantity.

At a voltage maximum (a current minimum), $Z = R = Z_o$ and

$\mathbf{Z = Z_o \times SWR}$, also a real quantity.

<u>The Smith Chart</u>[3]

For a transmission line of characteristic impedance Z_o and load impedance Z_L the reflection coefficient Γ_L is

$$G_{\mathbf{L}} = \frac{\mathbf{Z_L - Z_o}}{\mathbf{Z_L + Z_o}} = \frac{\frac{\mathbf{Z_L}}{\mathbf{Z_o}} - \mathbf{1}}{\frac{\mathbf{Z_L}}{\mathbf{Z_o}} + \mathbf{1}}$$
; normalize Z_L to Z_o by defining z_L such that

$$\mathbf{z_L} = \frac{\mathbf{Z_L}}{\mathbf{Z_o}}.$$ Substituting, we have

$$G_{\mathbf{L}} = \frac{\mathbf{z_L - 1}}{\mathbf{z_L + 1}}$$, and at any distance d from the load we have

$$G\mathbf{(d)} = G_{\mathbf{L}}\mathbf{e}^{\mathbf{-2j}b\,\mathbf{d}} = \frac{\mathbf{z(d)-1}}{\mathbf{z(d)+1}},$$

where z = r+jx, the impedance, resistance and reactance normalized to Z_o.

Solving for z, we have the value of z for any measured Γ at any point d

$$\mathbf{z(d)} = \frac{\mathbf{1}+G\mathbf{(d)}}{\mathbf{1}-G\mathbf{(d)}}.$$ This can be expressed in the very useful form

$$\mathbf{Z_{in}} = \mathbf{Z_o}\frac{\mathbf{Z_L + jZ_o tan}b\,\mathbf{d}}{\mathbf{Z_o + jZ_L tan}b\,\mathbf{d}},$$ the input impedance of a line of length d, Z_o and load Z_L.

If we plot Γ on the polar plot, and overlay the circles of constant r and x, <u>this yields the Smith Chart</u>, on which we can convert from Γ to Z (or the reverse) by inspection.

To see how the Smith Chart works, first consider a matched load, $Z = Z_o$ and $\Gamma = 0$. This point is at the origin of the plot, since $\Gamma = 0 + j0$. This is plotted below left.

Next, consider a transmission line terminated with an open circuit at d=0.

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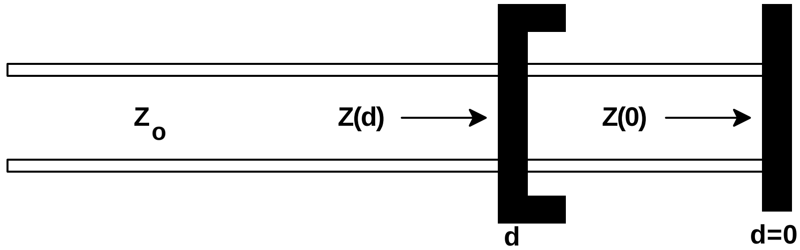
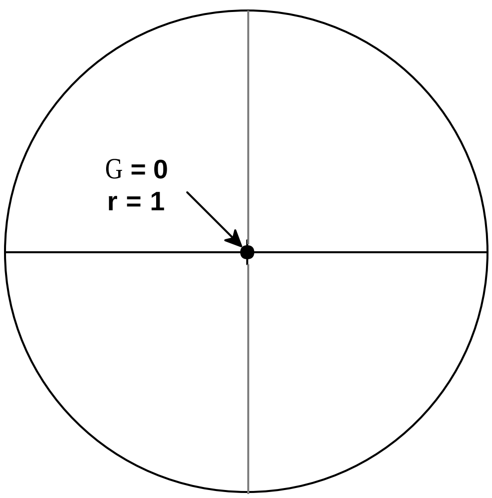
[3] Smith, P. H., "Transmission line calculator", *Electronics*, vol. 12, pg. 29, Jan. 1939 and "An improved transmission-line calculator", *Electronics*, vol. 17, pg. 130, Jan. 1944; for an interesting biography of P. H. Smith see also http://www.noblepub.com/Noble/Smthbiog.html

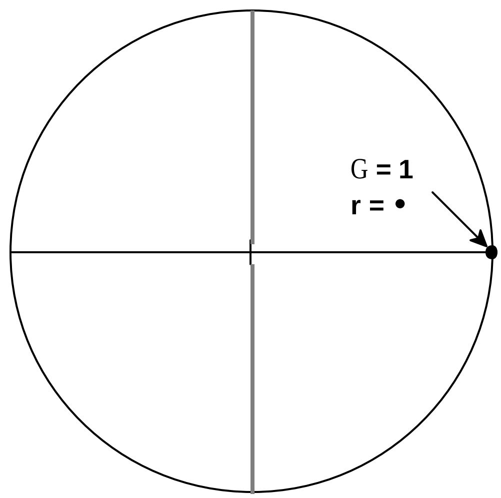


At d=0, the plane of the open, the current is constrained to be zero, so the reflected wave current must equal the incident wave current and be out of phase (i.e., $I_- = - I_+$, so that $V_- = V_+$). The impedance Z(0) at this point is 8, and the reflection coefficient Γ is

$G = \frac{z-1}{z+1}$ **= 1/0**. This value of Γ is plotted on the polar chart below right.

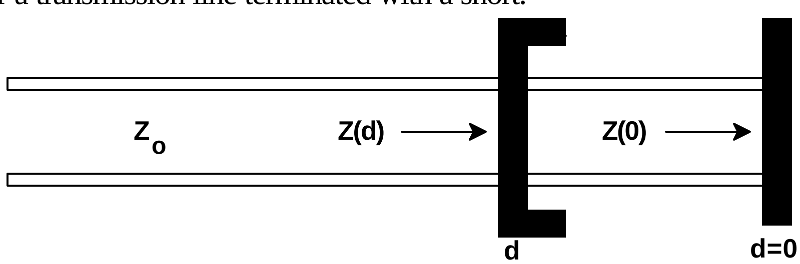


Matched load (G=0)



Open circuit load (G=+1)

Now consider a transmission line terminated with a short.



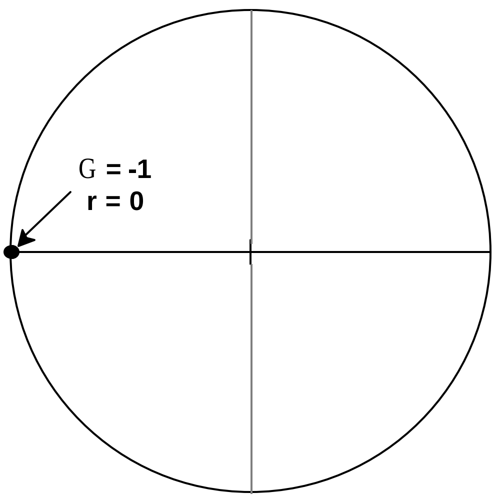
At d=0, the plane of the short, the voltage is zero, so the reflected wave voltage must equal the incident wave voltage and be out of phase (i.e., $V_- = - V_+$). This is the same as Γ = -1/0 which is Γ = 1/p. This value of Γ is plotted on the polar chart below left.

$$z(0) = \frac{1+G(0)}{1-G(0)} = \frac{1+(-1)}{1-(-1)} = 0 \text{ at this point.}$$

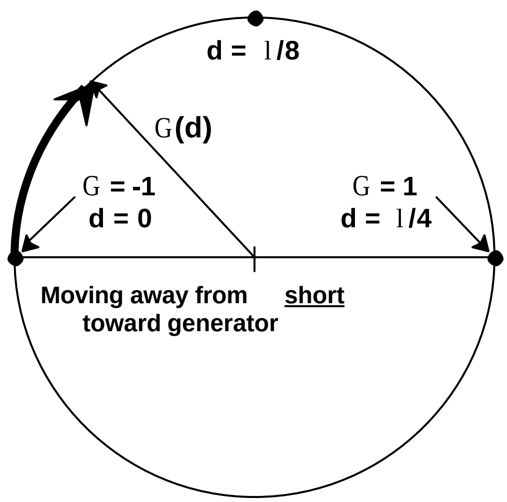
Now consider what happens as we move back along the transmission line away from the short circuit load toward the generator. In the shorted case, Γ rotates from $\Gamma = 1\underline{/p}$ toward $\Gamma = 1\underline{/0}$, while $|\Gamma| = 1$. A very short transmission line terminated with a short is inductive and has positive reactance $Z = j\omega L$, and in fact the entire upper hemisphere of the Γ plot is inductive (positive reactance). If we consider the case of $Z = jX = jZ_o$, or $z = j$, Γ is

$$G = \frac{\mathbf{z-1}}{\mathbf{z+1}} = \frac{\mathbf{j-1}}{\mathbf{j+1}} \mathbf{= j = 1\underline{/p/2}}.$$

Since the phase angle of Γ is $-2\beta d$, this point corresponds to moving $\beta d = p/4$, or $d/\lambda = 1/8$. When we have moved $d = \lambda/4$, the phase angle of Γ reaches 0. This is plotted below right. Because the polar plot will become crowded with information, the vertical (j) axis of the preceding polar plots will not be repeated from this point of the derivation.



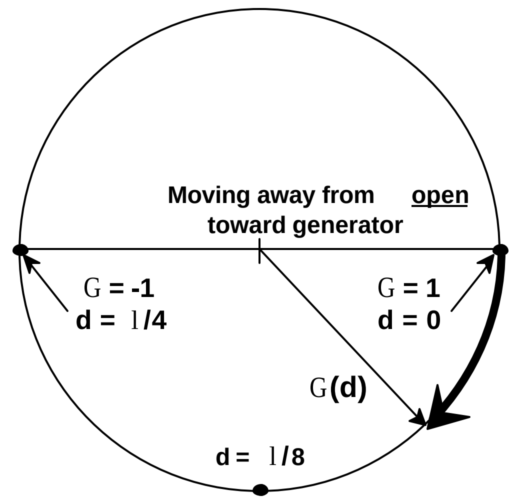
Short circuit load (G=-1)



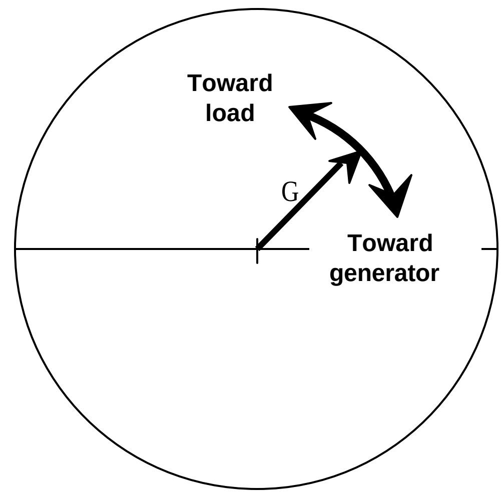
Short through arbitrary line length

By the same reasoning, if we move back on the open-terminated transmission line away from the open toward the generator, Γ rotates from $\Gamma = 1\underline{/0}$ toward $\Gamma = 1\underline{/-p}$, while $|\Gamma| = 1$. A very short transmission line terminated with a open is capacitive and has negative reactance $Z = 1/j\omega C$, and in fact this entire hemisphere of the Γ plot is capacitive (negative reactance). When we have moved $\lambda/8$, the phase angle of Γ reaches $-p/2$, and $z = -j$. When we have moved $\lambda/4$, the phase angle of Γ reaches $-p$, and $z = 0$. This is plotted below left.

But the fact is that any arbitrary impedance z or reflection coefficient Γ will have the same behavior if we move along the transmission from the point it is measured toward the generator. And if the impedance is measured at a point on the transmission line other than at the termination, we can move toward the load as well. It is this variation only of the phase angle, and not the magnitude, of Γ that is plotted below right.



Open through arbitrary line length

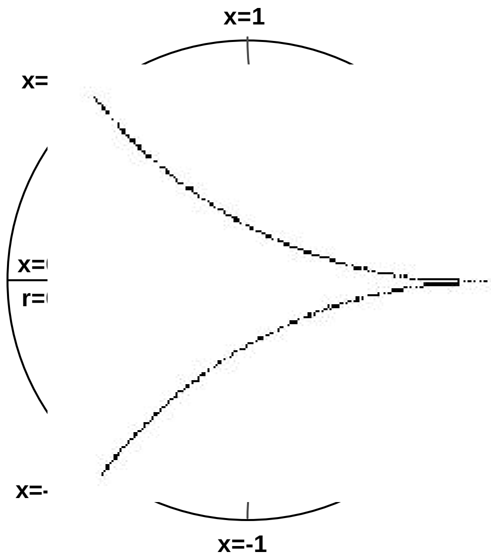


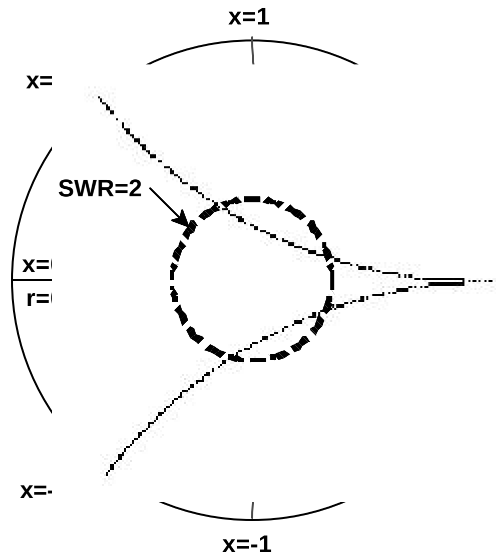
Only the angle of G changes

Because 2βd, the phase angle of Γ, is twice the electrical length βd of the motion along the line, Γ and its corresponding impedance z repeat every λ/2. It is the fact that Γ and z contain exactly the same information, but that only Γ varies gracefully as we move along a transmission line, that makes the polar plot of Γ, the Smith Chart, so useful.

Now if we overlay circles of constant r and constant x, we can enter either g or z, can convert between them by inspection, and can account for changes in line length by angle only. This is the Smith Chart, plotted below left.

We can also add, for example, the circle for SWR = 2, for which |Γ| = 1/3, as shown on the right. No matter what load impedance results in this SWR, as we move along the line at some point we pass through r = 0.5 and r = 2, repeating both every λ/2. Recalling the earlier figure showing V_{max} and V_{min}, this construction demonstrates that at the voltage maxima and minima the impedance is real and is either z = SWR or z =1/SWR.





The Smith Chart has at least four benefits:

1. All possible values of Γ, hence all possible values of Z, lie within the unit circle.

2. For a given termination, the variation of Γ with transmission line position is simply a rotation on the chart with no change in magnitude |Γ|, and hence, no change in SWR.

3. Lines of constant R and X are uniquely defined circles on the chart, so we can input data in Γ format and read the result in Z format by inspection.

4. Data from a slotted line can be entered directly in terms of SWR and distance between minima.

The Smith Chart is a mapping onto the complex Γ plane from the complex z plane. We let

G $\mathbf{= u + jv}$ and $\mathbf{z = r + jx}$

$$G = \mathbf{u + jv} = \frac{\mathbf{z-1}}{\mathbf{z+1}} = \frac{\mathbf{r+jx-1}}{\mathbf{r+jx+1}} = \frac{\mathbf{(r-1)+jx}}{\mathbf{(r+1)+jx}} = \frac{\mathbf{r^2-1+j2x}}{\mathbf{(r+1)^2+x^2}} = \frac{\mathbf{r^2-1}}{\mathbf{(r+1)^2+x^2}} + \mathbf{j}\frac{\mathbf{2x}}{\mathbf{(r+1)^2+x^2}}$$

Circles of constant Γ are concentric with the center point of the chart, which represent z = 1 + j0, hence Γ = 0 + j0. Lines of constant r and variable x transform into circles passing through the point labeled r on the axis running from r = 0 to r = 8. Lines of constant x and variable r transform into circles passing through specific points on the outer circle |Γ| = 1. The Smith Chart expresses the polar form of Γ directly by inspection, the polar angle being the angle from the resistance axis.

The benefit of the Smith Chart in connection with the use of computers for transmission-line calculations lies in its ability to assist in visualizing transmission-line problems, even though the accuracy available from computation is much greater. The Smith Chart is used so widely it could be considered the logo of microwave engineering. It is found on the cover of almost every book on the subject.

Transmission Lines as Reactance and Resonators

On the Smith Chart, if we rotate around a full circle toward the generator (away from the load) to return to the same impedance, we have gone λ/2 on the transmission line. Notice that if we choose a real impedance of z?1, by rotating one half circle (λ/4) we transform z to 1/z. This property of a λ/4 transmission line can be used to match an impedance Z to a transmission line Z_o by interposing between the line and the load an impedance matching λ/4 line whose characteristic impedance $Z_m = Z_o^2/Z$. This is known as a quarter-wave transformer. Any odd-multiple of λ/4 can be used at a single frequency, but the frequency sensitivity of the resulting match will be greater.

For the λ/4 case

$$\mathbf{z_{in} = 1/z_L} \text{ or } \mathbf{Z = \frac{Z_o^2}{Z_L}}$$

For the case of λ/2 we see that the impedance is the same as the load impedance. This repeats every half wavelength along the line.

For the case of shorted load ($Z_L = 0$),

$\mathbf{Z = jZ_o \tan(2pd/l)}$. For the case of a short line (d/λ<<1), this can be expressed as

$\mathbf{Z \,\tilde{}\; jZ_o(2pd/l)}$, which is an inductive reactance.

Since fλ=v, the velocity of propagation, we can say

$\mathbf{Z \,\tilde{}\; jZ_o 2pfd/v.}$

Since Z = jX = 2pfL we can write the inductance L of a short shorted line:

$$\mathbf{L \,\tilde{}\; \frac{Z_o d}{v}}$$

For a shorted line of length d = λ/4 or odd multiples thereof, the impedance will be high and will vary with frequency in exactly the same manner as a lumped parallel resonant circuit. For lengths that are multiples of λ/2, the impedance will be low and will vary in exactly manner as a lumped series resonant circuit.

For open-circuited load (Z_L = 8) we have

$$\mathbf{Z = \frac{-jZ_o}{tan(2pd/l)}}$$

For the case of a short line (d/λ<<1), this can be expressed as

$$\mathbf{Z = \frac{-jZ_o}{(2pd/l)}}$$

, which is an inductive reactance. As above, we can express

$$\mathbf{C \sim \frac{v}{Z_o d}}$$

For an open line of length d = λ/4 or odd multiples thereof, the impedance will be low and will vary with frequency in exactly the same manner as a lumped series resonant circuit. For lengths that are multiples of λ/2, the impedance will be high and will vary in exactly manner as a lumped parallel resonant circuit. The Q of such a resonator is determined by the line losses.

In the more general case, the Smith Chart is quite useful to determine impedance matching networks using various configurations of transmission lines and lumped elements. It is also possible to use multiple quarter-wave sections to provide broader band matching. These will be covered in detail in the future.

Here's an interesting extra problem: Using the Smith Chart, find the line length for x = 1 (X = Z_o)

The SWR of a line can be measured by use of a slotted transmission line, arranged to probe the voltage as a function of position. This method is instructive, but has been replaced by modern instruments that generally measure impedance directly and display the result on a Smith Chart display. Direct measurement of impedance can be very precise, but the accuracy is determined by the accuracy with which the line length is determined from the measuring device to the point of measurement.

Also, because the currents of the incident and reflected waves are of opposite signs, it is common to make a directional coupler that adds the voltage through a small capacitance (nondirectional) to the voltage developed on a small magnetic coupling loop (directional) to form a directional detector that can measure the incident and reflected wave amplitudes separately, permitting the direct measurement of SWR.

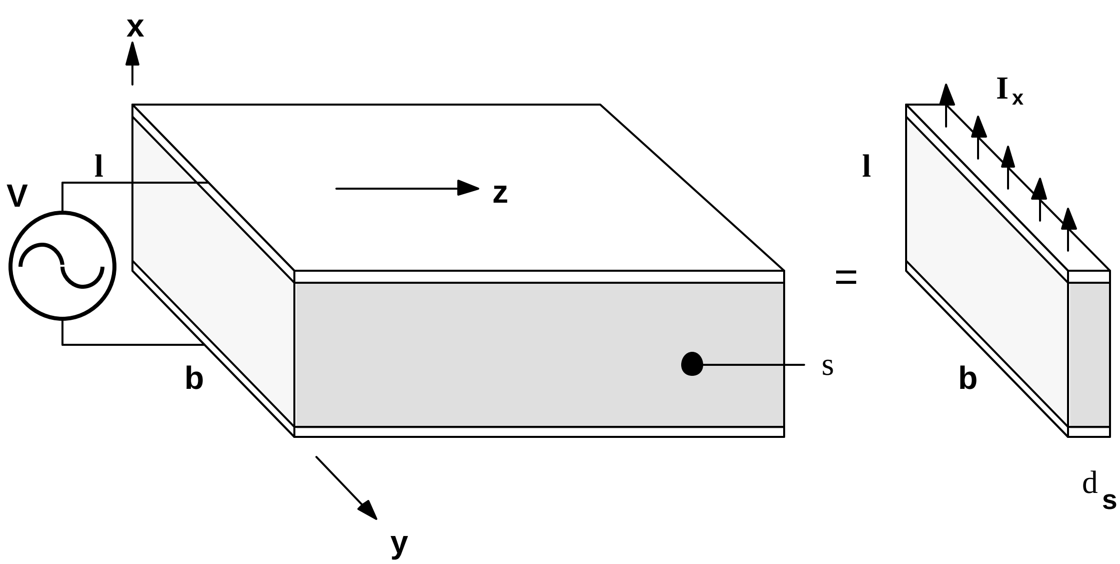
Modern coaxial transmission lines range from miniature semirigid cables of 0.085" diameter up to cables that are as large as several inches in diameter. Connectors with low reflection are an important element of the application of transmission lines, and time-domain reflectometry is used to locate and remove discontinuities that can cause reflections at certain frequencies. Active devices are composed of semiconductor elements in connection with directional couplers, power splitters and other specialized microwave components such as filters.

Present microwave circuit practice is primarily based on microstrip, which is a planar conductor suspended on a dielectric above a conducting plate. Because this configuration can be built with photolithographic techniques, it has been widely studied and applied. The fields are not confined to the dielectric, and thus there are two dielectric regimes, with the result that microstrip is dispersive and is more complex than coaxial structures to analyze and synthesize, but all the considerations of waves on transmission lines apply.

Transmission Line Losses from Resistance

It is of interest to calculate the losses due to surface resistivity, in order to calculate the losses in a transmission line composed of good, but not perfect, conductors.

Consider a bar of conducting material shown in Fig. 1, extending indefinitely in the z direction (into the material). An AC source V is connected at z = 0 to the two conducting planes at x = 0 and x = l. Since this will give rise to only E_x, we can assume that the fields are uniform in the x-y plane, and waves will move in the +z direction, the depth into the conductor.



This figure, and the derivation that follows, are based on Rizzi[4].

The E-field at z = 0 will be determined by

$\mathbf{E_o}^+ = \dfrac{\mathbf{V}}{\mathbf{l}}$ and $E^+(z)$ will move toward +z in the form

[4] Rizzi, P., *Microwave Engineering*, Prentice Hall, 1988, pg. 44ff; see also Pozar, *Microwave Engineering*, and Moreno, *Microwave Transmission Design Data*, Ch 4

$\mathbf{E}^+ = \mathbf{E_o}^+ \mathbf{e}^{-\gamma \mathbf{z}}$, where

$$\gamma = \alpha + \mathbf{j}\beta = \sqrt{\pi \mathbf{f} \mu \sigma} + \mathbf{j}\sqrt{\pi \mathbf{f} \mu \sigma}$$

We have defined the skin-depth δ_S to be

$$\delta_{\mathbf{s}} = \frac{\mathbf{1}}{\sqrt{\pi \mathbf{f} \mu \sigma}}$$

To get to the loss resistance, we need to calculate the current I_X in the direction of the voltage V, which is

$\mathbf{I_x} = \int_{\mathbf{z=0}}^{\infty} \int_{\mathbf{y=0}}^{\mathbf{b}} \mathbf{J_x\, dydz}$. We know that

$$\mathbf{J_x(z)} = \sigma \mathbf{E_x} = \sigma \mathbf{E_o}^+ \mathbf{e}^{-\gamma \mathbf{z}}$$

Integrating with respect to y simply results in the dimension b, so we are left to integrate with respect to z to get

$$\mathbf{I_x} = \frac{\sigma \mathbf{b E_o}^+}{\gamma}$$

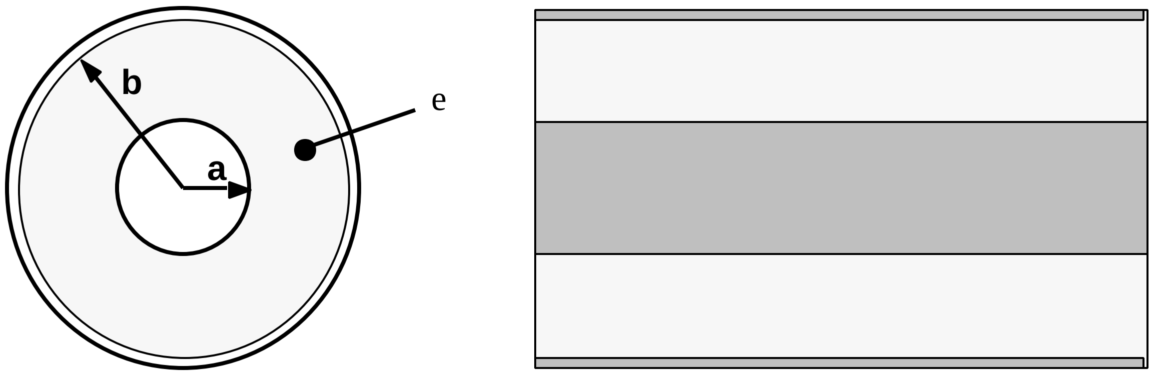
To determine Z, we take the ratio of voltage to current

$\mathbf{Z} = \frac{\mathbf{V}}{\mathbf{I_x}}$ to yield

$\mathbf{Z} = \frac{\mathbf{l}}{\sigma \mathbf{b} \delta_{\mathbf{s}}}\mathbf{(1 + j)}$. We're only interested in the real part, so we have

$\mathbf{R_s} = \frac{\mathbf{l}}{\sigma \mathbf{b} \delta_{\mathbf{s}}}$ ohms. Note the phase angle of Z is 45°.

If we consider the case of a round conductor of radius a, we have the width b = 2πa, so the series resistance is



$$\mathbf{R_s} = \frac{\mathbf{l}}{\mathrm{s}\,\mathbf{(2pa)}\mathrm{d}_\mathbf{s}} \text{ ohms.}$$

Recall that the skin depth is

$$\mathrm{d}_\mathbf{s} = \frac{\mathbf{1}}{\sqrt{\mathrm{p}\,\mathbf{f}\mathrm{m}\,\mathrm{s}}}\,.$$

For copper, $\mu_r = 1$, so $\mu = 4\mathrm{p} \times 10^{-7}$ and $\sigma = 5.8 \times 10\text{-}7$ s/m and the skin depth reduces to

$$\mathrm{d}_\mathbf{cu} = \frac{\mathbf{6.6 \times 10^{-2}}}{\sqrt{\mathbf{f}}} \text{ meters}$$

At $f = 10^9$ Hz, skin depth is 2.1×10^{-3} mm, or about 80 millionths of an inch, so you can see that only the surface of the conductor has much effect. Often copper or silver are plated onto a lower-conductivity material for microwave use. If inadequate plating is used, the losses of the underlying material can have a drastic effect on transmission-line losses.

Surface roughness also affects losses, since it increases the effective surface resistive path. An approximation for R_s in the case of surface roughness is

$\mathbf{R_s' = R_s[1 + 2/p\ tan^{-1}\ 1.4(?/\mathrm{d}_s)2]}$, where ? is the rms surface roughness.

<u>Losses in Coaxial Cables</u>

For coaxial cable, there are two cylinders that must be considered. The outer is often specified to have radius b (not to be confused with the dummy width variable above), while the inner has radius a. For such a cable, the loss resistance is the sum of the resistances of both the inner and outer conductors. Physical cables are often specified in terms of diameter (a measurable) rather than radius, and dimensions are often stated in inches.

The magnitude of a wave on a low loss transmission line can be expressed as

$$\mathbf{V(d) = V_{+}e^{-\alpha d}}$$

If we compare V(d) to V(0), we see that

$$\frac{\mathbf{V(d)}}{\mathbf{V(0)}} = \mathbf{e^{-\alpha d}}$$, and the power ratio is

$$\frac{\mathbf{P(d)}}{\mathbf{P(0)}} = \mathbf{e^{-2\alpha d}}$$

To express this in decibel form, the line *loss* in dB is the positive number

$$\mathbf{Loss = 10\log(e^{-2\alpha d}) = \frac{10}{2.3}\ln(e^{-2\alpha d}) = 4.343(-2\alpha d) = 8.686\alpha d} \text{ dB per length } d$$

For the lumped-constant equivalent circuits, the parameters for a coaxial transmission line are as follows:

$$\mathbf{Z_o = 60}\sqrt{\frac{\mu_r}{\varepsilon_r}}\ \mathbf{\ln\left(\frac{b}{a}\right)}$$

$$\mathbf{G} = \omega\frac{2\pi\varepsilon_o\varepsilon_r}{\ln\left(\frac{b}{a}\right)}\ \mathbf{\tan}\,\delta, \text{ where here } \mathbf{\tan}\,\delta = \frac{\omega\varepsilon'' + \sigma_d}{\omega\varepsilon'}$$

$$\mathbf{R} = \frac{1}{2\pi a\delta_s\sigma} + \frac{1}{2\pi b\delta_s\sigma} = \frac{1}{2\pi\delta_s\sigma}\left(\frac{1}{a} + \frac{1}{b}\right)$$

Loss per unit length consists of two components[5], conductive and dielectric losses

$$\alpha = \frac{R}{2Z_o} + \frac{GZ_o}{2}$$

$$\alpha = \alpha_c + \alpha_d = \frac{a+b}{4\pi Z_o ab\delta_s\sigma} + 120\pi^2\varepsilon_o(\varepsilon_r)^{3/2} f.$$ But since

$$\delta_s = \frac{1}{\sqrt{f}\sqrt{\pi\mu_r\mu_o\sigma}}$$ we can see that α will be of the form $\alpha = k_1\sqrt{f} + k_2 f$

For 1/4" semirigid cable, Z_o=50Ω, b=7x10^{-3} m., ε_r=2.1 and tanδ = 0.00015 (Teflon[6]),

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[5] Rizzi, P., *Microwave Engineering*, Prentice Hall, 1988, pg. 185

$$a \;\mathbf{= 2.5x10^{-6}\sqrt{f} + 2.0x10^{-11}\, f}\ \text{dB/meter}$$

For RG214 cable, Z_o=50? , b=1x10^{-2} m., ε_r=2.3 and tanδ = 0.00031 (polyethylene),

$$a \;\mathbf{= 1.5x10^{-6}\sqrt{f} + 4.2x10^{-11}\, f}\ \text{dB/meter}$$

Some more modern RG-8 type cables use a foam dielectric, which has a lower dielectric constant and loss tangent than polyethylene. Also, the larger center conductor diameter required to maintain Z_o = 50? with foam dielectric results in lower resistive loss.

In order to see the two components of loss, we plot α(f) vs. f on a *log-log plot*, which is the equivalent of plotting log α vs. log f.

Please note the discussion of Nepers and decibels in Pozar[7]. Most exponential or natural logarithmic ratios result naturally in answers in Nepers, which can be converted to the more commonly-used decibels by the conversion factor **1 Np = 8.686 dB**. Of course, keep in mind that power ratios are the square of voltage ratios, which gives rise to the use of dB = 20 log (V_1/V_2) for voltage ratios and dB = 10 log (P_1/P_2) for power ratios.

<u>Coaxial Line Impedance for Minimum Attenuation and Maximum Power Capacity</u>

The expression for conductive loss α_c in a coaxial line, assuming equal resistivity of inner and outer conductors, is of the form

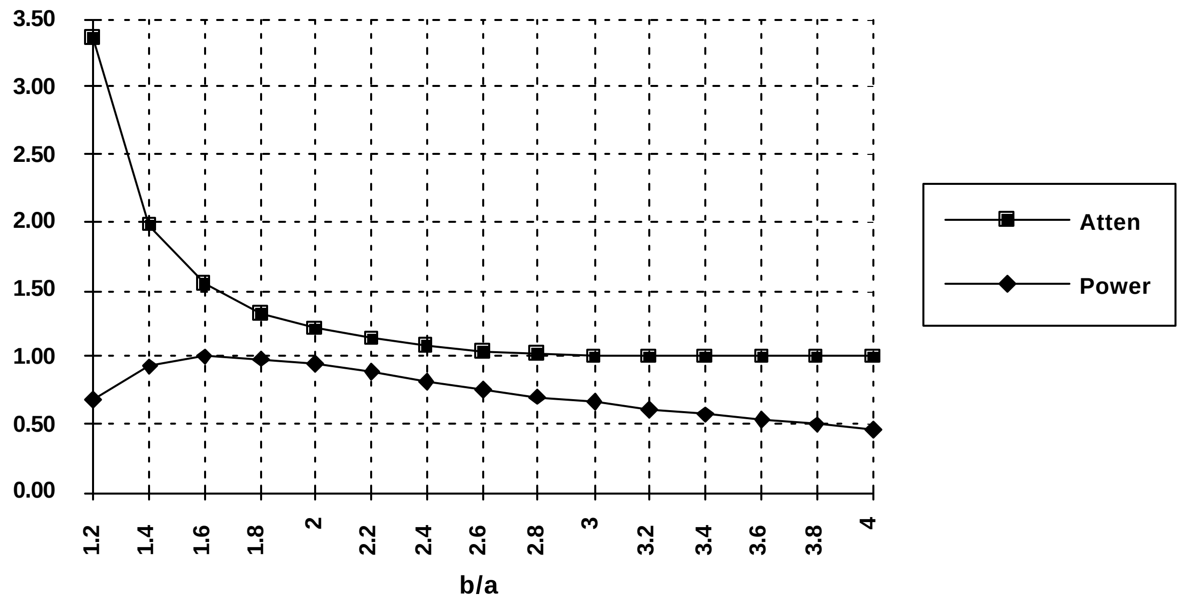
$$a_{\mathbf{c}} = \mathbf{k}_a \frac{\mathbf{(1+x)}}{\mathbf{\ln x}}$$, where x=b/a, the ratio of outer to inner radius.

If we plot this expression, normalized against its minimum value, as a function of b/a, we can see there is a broad minimum around x=3.6, which corresponds to Z_o=77 ? for air line and Z_o=51 ? for polyethylene dielectric (ε_r=2.3).

The expression for breakdown voltage is of the form $\mathbf{V_{max} = k_v \frac{\ln x}{x}}$, but since $\mathbf{P_{max}=V_{max}^2/Z_o}$, we need to consider instead the maximum power capacity, of the form

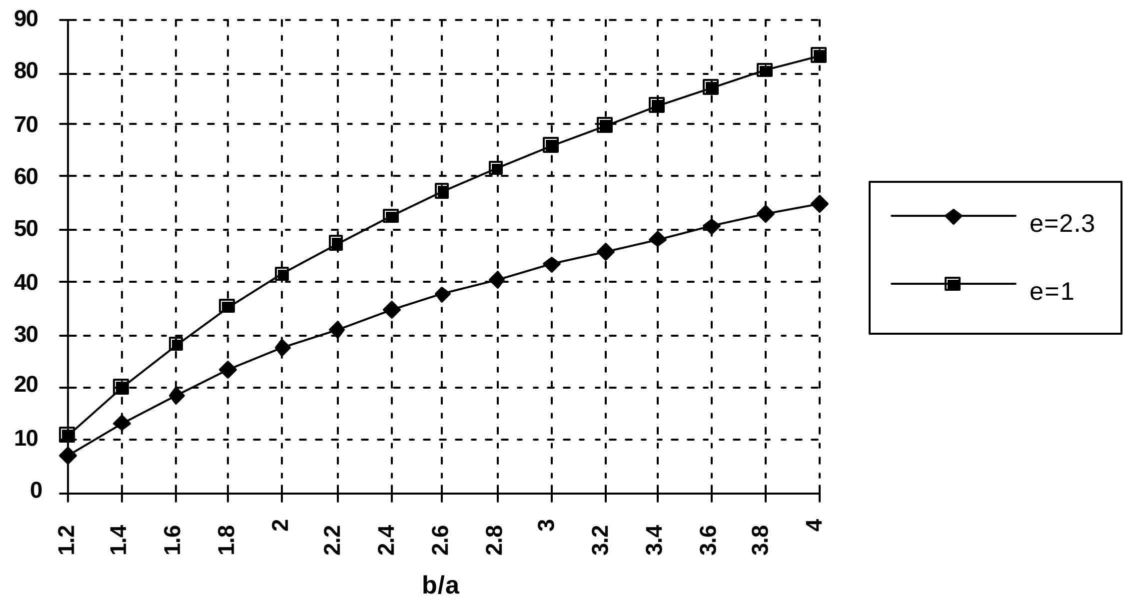
6 *Reference Data for Radio Engineers*, Fifth Edition, Howard W. Sams, 1968, pg. 4-28

7 Pozar, D. M., *Microwave Engineering*, 2nd Ed., J. Wiley, 1998, pg. 72-73



$$P_{max}=k_p\frac{\ln x}{x^2}\,.$$

If we plot this expression, normalized against its maximum value, as a function of b/a, we can see there is a maximum around x=1.6, which corresponds to Z_o=28 ? for air line and Z_o=19 ? for polyethylene dielectric (ε_r=2.3).



These curves appear in Moreno[8], and interestingly, are calculated incorrectly in Rizzi[9].

Appendix I: Reference Summary of Relationships Between Complex Z and Γ

The input wave is $\mathbf{V^+}$ and $\mathbf{I^+}$, and the characteristic impedance Z_o is defined as $\mathbf{Z_o} = \dfrac{\mathbf{V^+}}{\mathbf{I^+}}$.

$\mathbf{V^-} = G\mathbf{V^+}$ and $\mathbf{I^-} = -G\mathbf{I^+}$. The total voltage V is

$\mathbf{V} = \mathbf{V^+} + \mathbf{V^-} = \mathbf{V^+} + G\mathbf{V^+} = \mathbf{V^+}(\mathbf{1}+G)$, and the total current I is

$\mathbf{I} = \mathbf{I^+} + \mathbf{I^-} = \mathbf{I^+} - G\mathbf{I^+} = \mathbf{I^+}(\mathbf{1}-G)$. The ratio of V/I is the impedance Z, which is

$$\mathbf{Z} = \frac{\mathbf{V}}{\mathbf{I}} = \frac{\mathbf{V^+}(\mathbf{1}+G)}{\mathbf{I^+}(\mathbf{1}-G)}$$

. Substituting $\dfrac{V^+}{I^+} = Z_o$, so

$$\mathbf{Z} = \frac{\mathbf{V}}{\mathbf{I}} = \mathbf{Z_o}\frac{\mathbf{1}+G}{\mathbf{1}-G}$$

. Defining normalized $z = \dfrac{Z}{Z_o}$

$$\frac{\mathbf{Z}}{\mathbf{Z_o}} = \mathbf{z} = \frac{\mathbf{1}+G}{\mathbf{1}-G}$$

, so if we know z, we know Γ and vice versa.

To get z from Γ, if we define $\Gamma = \Gamma_r + j\Gamma_i = a + jb$

8 Moreno, T., *Microwave Transmission Design Data*, Dover, 1958, pg. 64

9 Rizzi, P., *Microwave Engineering Passive Circuits*, Prentice Hall, 1988, pg. 189-190

$$z = \frac{1+a+jb}{1-a-jb} = \frac{[(1+a)+jb][(1-a)+jb]}{[(1-a)-jb][(1-a)+jb]} = \frac{(1+a)(1-a)-b^2+j[b(1-a)+b(1+a)]}{(1-a)^2+b^2}$$

$$z = \frac{1-a^2-b^2 +j2b}{(1-a)^2+b2}$$, so

$$r = \frac{1-a^2-b^2}{(1-a)^2+b^2} = \frac{1-\Gamma_r^2-\Gamma_i^2}{(1-\Gamma_r)^2+\Gamma_i^2}$$ and

$$x = \frac{2b}{(1-a)^2+b^2} = \frac{2\Gamma_i}{(1-\Gamma_r)^2+\Gamma_i^2}$$

To get Γ if we know z, solving for Γ,

$$z = \frac{1+\Gamma}{1-\Gamma}$$

$$(1-\Gamma)z = 1+\Gamma$$

$$z - z\Gamma = 1+\Gamma$$

$$z-1 = z\Gamma+ \Gamma = (z+1)\Gamma$$, so

$\Gamma = \frac{z-1}{z+1}$. The magnitude $|\Gamma| = \rho$ of the reflection coefficient Γ is determined by

$$\Gamma\Gamma^* = \rho^2 = \frac{(z-1)(z^*-1)}{(z+1)(z^*+1)} = \frac{(r-1+jx)(r-1-jx)}{(r+1+jx)(r+1-jx)} = \frac{(r-1)^2+x^2}{(r+1)^2+x^2}$$, so

$|\Gamma| = \rho = \sqrt{\frac{(r-1)^2+x^2}{(r+1)^2+x^2}}$. We can determine SWR from ρ by

$SWR = \frac{1+\rho}{1-\rho}$. If we know ρ we know SWR and vice versa. Solving for ρ,

$$\rho = \frac{SWR-1}{SWR+1} .$$

If x = 0, $\rho = \frac{r-1}{r+1}$ and for r = 1, SWR = r; for r = 1, SWR = $\frac{1}{r}$.

Consider a transmission line with arbitrary mismatched complex load establishing a given SWR. At the points of minimum and maximum voltage (and hence, minimum and maximum impedance) on the transmission line, the impedance is real and is related to the SWR by the simple relationships

$\mathbf{r_{max} = SWR}$, or $\mathbf{R_{max} = SWR \times Z_o}$.

$\mathbf{r_{min} = \frac{1}{SWR}}$, or $\mathbf{R_{min} = \frac{Z_o}{SWR}}$.

The numerical fraction of power reflected at an arbitrary impedance defined by ρ is

RL = $r^{\mathbf{2}}$ and the numerical fraction of power transmitted is

TL = $t^{\mathbf{2}}$ **= (1 -** $r^{\mathbf{2}}$**)**.

For example: SWR = 2, $\rho^2 = 0.11$ (-12.3 dB) and $\tau^2 = 0.89$ (-0.26 dB).

Defining z = r + jx, we can determine Γ from z by

$$G = \frac{\mathbf{z-1}}{\mathbf{z+1}} = \frac{\mathbf{r-1+jx}}{\mathbf{r+1+jx}} = \frac{\mathbf{(r-1+jx)(r+1-jx)}}{\mathbf{(r+1+jx)(r+1-jx)}} = \frac{\mathbf{(r-1)(r+1)+x^{2`}+jx(r+1-r+1)}}{\mathbf{(r+1)^2+x^2}}$$

$$G = \frac{\mathbf{r^2-1+x^2+jx(2)}}{\mathbf{(r+1)^2+x^2}} \ .$$

The real part of $\Gamma = \Gamma_r + j\Gamma_i$ is

$G_{\mathbf{r}} = \frac{\mathbf{r^2+x^2-1}}{\mathbf{(r+1)^2+x^2}}$ and the imaginary part of Γ is

$G_{\mathbf{i}} = \frac{\mathbf{j2x}}{\mathbf{(r+1)^2+x^2}}$. These relationships can be used to create a Smith chart plot of Γ.