# Convolution Representation of Linear Time-Invariant Continuous-Time Systems

## Impulse Response

Recall the definition of an impulse function:

$$\delta(t) = 0 \quad , \quad t \neq 0$$

$$\int \delta(\lambda)\, d\lambda = 1 \quad .$$

The impulse response is,

$$h(t) = \delta(t) \quad .$$

For a causal system, $h(t) = 0$ for all $t < 0$ .

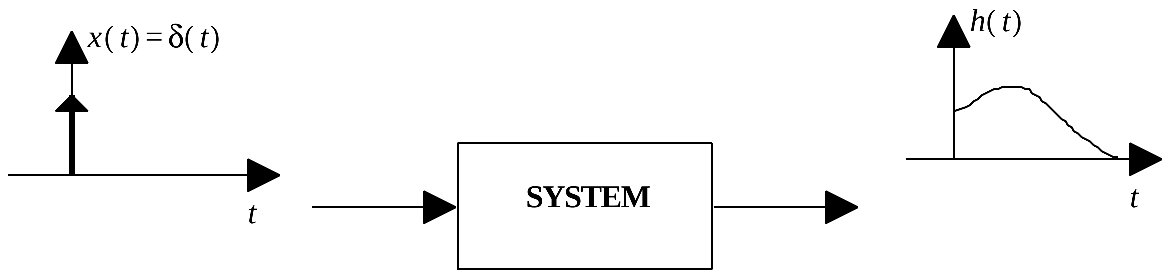



Figure 1: System impulse response.

If $x(t)$ is a causal function ( $x(t) = 0$ for all $t < 0$ ), by the sifting property,

$$x(t) = \int_{0^-}^{\infty} x(\lambda)\,\delta(t-\lambda)\, d\lambda \quad .$$

The output is,

$$\begin{aligned} y(t) &= x(t) * h(t) \quad , t \geq 0 \\ &= \int_{0}^{\infty} x(\lambda)\, h(t-\lambda)\, d\lambda \end{aligned}$$

which is the convolution representation.

Therefore,

$$x(t)*v(t)=\begin{cases}0 & ,t<0\\ \int_0^t x(\lambda)v(t-\lambda)\,d\lambda & ,t>0\end{cases}$$

The integral exists if $x(t)$ and $v(t)$ are integrable. That is,

$$\int_0^t |x(\lambda)|\,d\lambda<\infty$$

$$\int_0^t |v(\lambda)|\,d\lambda<\infty \quad ,t>0.$$

The following steps are recommended when performing convolutions of two functions $x(t)$ and $v(t)$.

Step 1. Graph $x(l)$ and $v(-l)$ (or vise-versa) as functions of $l$. That is, fold $v(l)$ about the vertical axis.

Step 2. Upon folding $v(l)$, denote the point where $l = 0$ as $t$. All other points of $v(-l)$ are relative to the reference point $t$. Remember, we're now in the $l$-axis.

Step 3. Define an interval for which the product $x(l)v(t-l)$ has the same analytical form (e.g. from $t = 0$ to $t = 1$).

Step 4. Integrate the product $x(l)v(t-l)$ as a function of $l$ over the integral defined in Step 3.

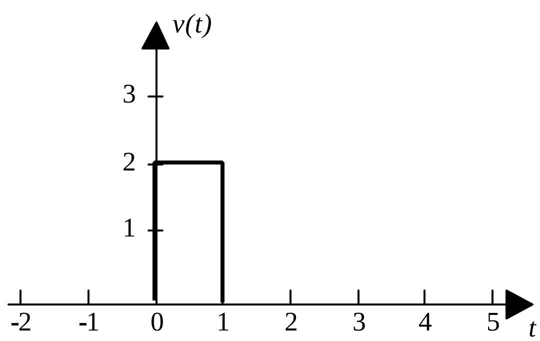
Step 5. Repeat Steps 3 and 4 as many times as necessary until $x(l)*v(t)$ is computed for all $t > 0$.

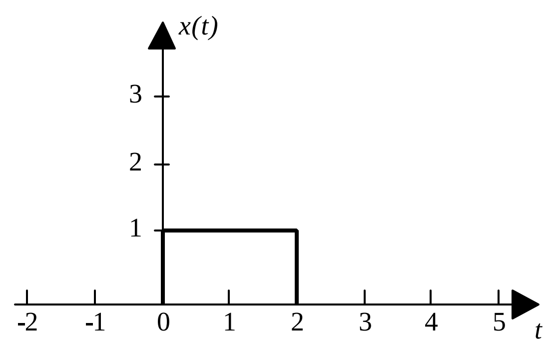
<u>EXAMPLE</u>

Convolve the following functions:

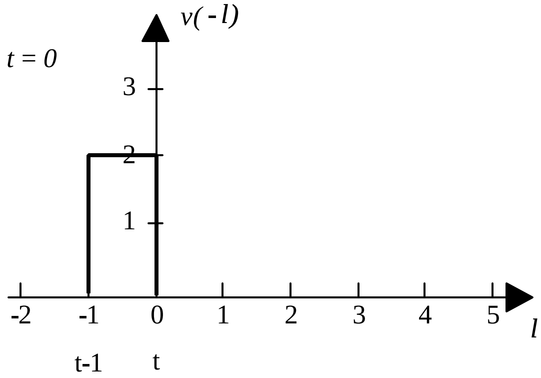
$$x(t) = \Pi\left(\frac{t-1}{2}\right)$$

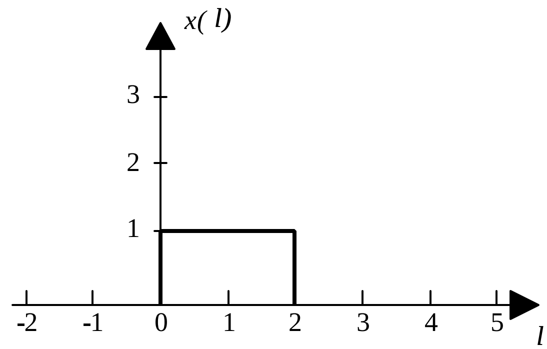
$$v(t) = 2\Pi\left(t - \frac{1}{2}\right)$$



Choose to flip $v(t)$ about the y-axis. This will then be the convolution at $t$ = 0.




The first region of integration is $t$<0 for which no overlapping area exists. We construct a table as follows:

| Time shift , $t$<br>Region of Overlap | $l$ lower limit<br>of integration | $l$ upper limit<br>of integration | Area of Overlap<br>(Integral) |
|---|---|---|---|
| $t$ < 0 | --- | --- | 0 |
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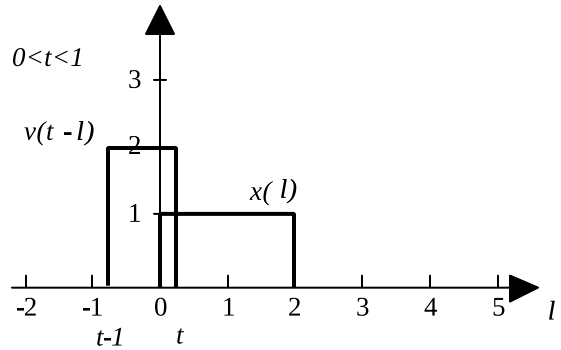
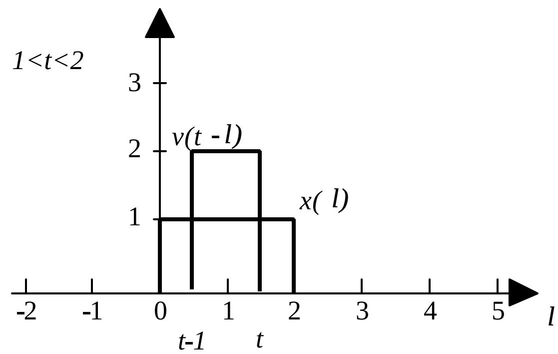
For ease of viewing the "overlap" intervals, the two functions are drawn on the same axes:




The function $v(t - l)$ slides over the function $x(l)$. The second region region of overlap and function to be integrated is $0 \le t < 1$ with the lower limit of the overlapping area determined by $x(l)$ and the upper limit $t$ of the area is determined by $v(t - l)$ .

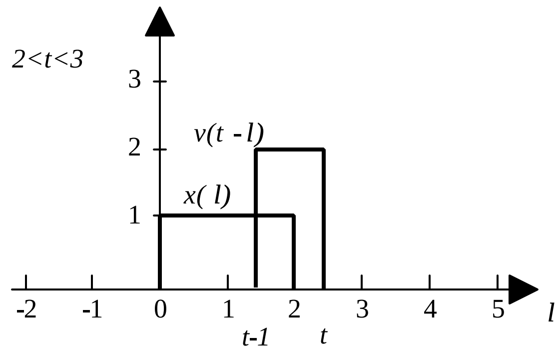
| Time shift , $t$<br>Region of Overlap | $l$ lower limit<br>of integration | $l$ upper limit<br>of integration | Area of Overlap<br>(Integral) |
|---|---|---|---|
| $t < 0$ | --- | --- | 0 |
| $0 \le t < 1$ | 0 | $t$ | $\int_{\lambda\, lower}^{\lambda\, upper} (2)(1)\, d\lambda$ |
| | | | |
| | | | |
| | | | |

The third region of overlap is $1 \le t < 2$ as shown below. There is complete overlap of the two areas. Therefore, the integration limit of the overlap area are from $t - 1$ to t defined by $v(t - l)$ in the region $1 \le t < 2$.



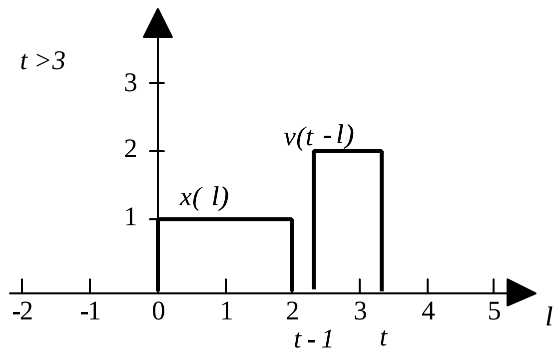

| Time shift , $t$<br>Region of Overlap | $l$ lower limit<br>of integration | $l$ upper limit<br>of integration | Area of Overlap<br>(Integral) |
|---|---|---|---|
| $t < 0$ | --- | --- | 0 |
| $0 \le t < 1$ | 0 | $t$ | $\int_{\lambda lower}^{\lambda upper} (2)(1)\, d\lambda$ |
| $1 \le t < 2$ | $t - 1$ | $t$ | $\int_{\lambda lower}^{\lambda upper} (2)(1)\, d\lambda$ |
| | | | |
| | | | |

The fourth region of overlap is $2 \le t < 3$ over which the upper limit of integration is 2 as defined by $x(l)$ and the lower limit is $t - 1$ as defined by $v(t - l)$ .




| Time shift , $t$<br>Region of Overlap | $l$ lower limit<br>of integration | $l$ upper limit<br>of integration | Area of Overlap<br>(Integral) |
|---|---|---|---|
| $t < 0$ | --- | --- | 0 |
| $0 \le t < 1$ | 0 | $t$ | $\int_{\lambda\, lower}^{\lambda\, upper} (2)(1)\, d\lambda$ |
| $1 \le t < 2$ | $t - 1$ | $t$ | $\int_{\lambda\, lower}^{\lambda\, upper} (2)(1)\, d\lambda$ |
| $2 \le t < 3$ | $t - 1$ | 2 | $\int_{\lambda\, lower}^{\lambda\, upper} (2)(1)\, d\lambda$ |
| | | | |

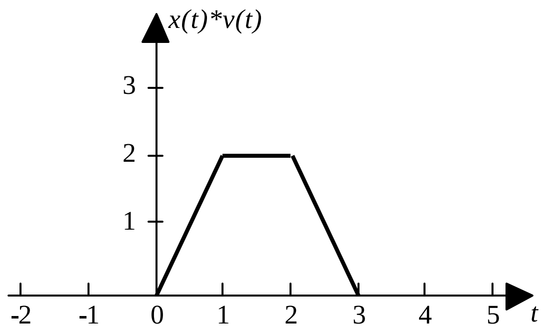
The last region is trivial in that there is no overlap between the two functions. The resultant area of overlap is 0.




| Time shift , $t$<br>Region of Overlap | $l$ lower limit<br>of integration | $l$ upper limit<br>of integration | Area of Overlap<br>(Integral) |
|---|---|---|---|
| $t < 0$ | --- | --- | 0 |
| $0 \le t < 1$ | 0 | $t$ | $\int_{\lambda\, lower}^{\lambda\, upper} (2)(1)\, d\lambda$ |
| $1 \le t < 2$ | $t - 1$ | $t$ | $\int_{\lambda\, lower}^{\lambda\, upper} (2)(1)\, d\lambda$ |
| $2 \le t < 3$ | $t - 1$ | 2 | $\int_{\lambda\, lower}^{\lambda\, upper} (2)(1)\, d\lambda$ |
| $t \ge 3$ | --- | --- | 0 |

The result of the convolution is:

$$x(t)*v(t)=\begin{cases}0 & ,t<0\\ 2t & ,0\le t<1\\ 2 & ,1\le t<2\\ -2t+6 & ,2\le t<3\\ 0 & ,t\ge 3\end{cases}$$




Result of the convolution.