

CHAPTER 13 SECTION 3

ELLIPSES

Unlike circles or parabola, ellipses have two foci. An ellipse is defined as the set of all points (x,y) the sum of whose distances from the two foci is constant. A line that passes through both foci and intersects the ellipse at two points (the vertices) is known as the major axis. The minor axis is a chord that is perpendicular to the major axis. Their point of intersection is the center.

Note: The major axis, containing the foci, is always longer than the minor axis.

The following formulas and information concern ellipses whose center is at the origin.

Ellipse with the major axis on the X axis and the center at $(0, 0)$.

Standard equation: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where $a > b$.

Vertices of the major axis: $(-a, 0)$ and $(a, 0)$.

Endpoints of the minor axis: $(0, b)$ and $(0, -b)$

Foci: $(-c, 0)$ and $(c, 0)$ where $c^2 = a^2 - b^2$

Ellipse with the major axis on the Y axis and the center at $(0, 0)$.

Standard equation: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where $b > a$

Vertices of the major axis: $(0, b)$ and $(0, -b)$

Endpoints of the minor axis: $(-a, 0)$ and $(a, 0)$.

Foci: $(0, c)$ and $(0, -c)$ where $c^2 = b^2 - a^2$.

The following two examples will show how to deal with these types of problems.

Example 1. Given the ellipse, $4x^2 + y^2 = 36$, find the foci, vertices, endpoints and graph.

Step 1. Rewrite the equation in standard form.

$$4x^2 + y^2 = 36$$

$$\frac{4x^2}{36} + \frac{y^2}{36} = \frac{36}{36}$$

$$\frac{x^2}{9} + \frac{y^2}{36} = 1$$

$$\frac{x^2}{(3)^2} + \frac{y^2}{(6)^2} = 1$$

Step 2. Analysis.

- a.) Since $b > a$, the vertical axis is the major axis.
- b.) Since the major axis is vertical, the formula for the foci, $[(0, -c), (0, c)]$ is $c^2 = b^2 - a^2$.
- c.) $a = 3$, $b = 6$

Step 3. Plot the required points.

- a.) The vertices.
Since $b = 6$, the vertices, $(0, b)$, $(0, -b)$ are:
 $(0, 6)$ and $(0, -6)$.
- b.) The endpoints.
Since $a = 3$, the endpoints, $(-a, 0)$, $(a, 0)$, are:
 $(3, 0)$ and $(-3, 0)$.
- c.) The foci.

$$c^2 = b^2 - a^2$$

$$c^2 = (6)^2 - (3)^2$$

$$c^2 = 36 - 9$$

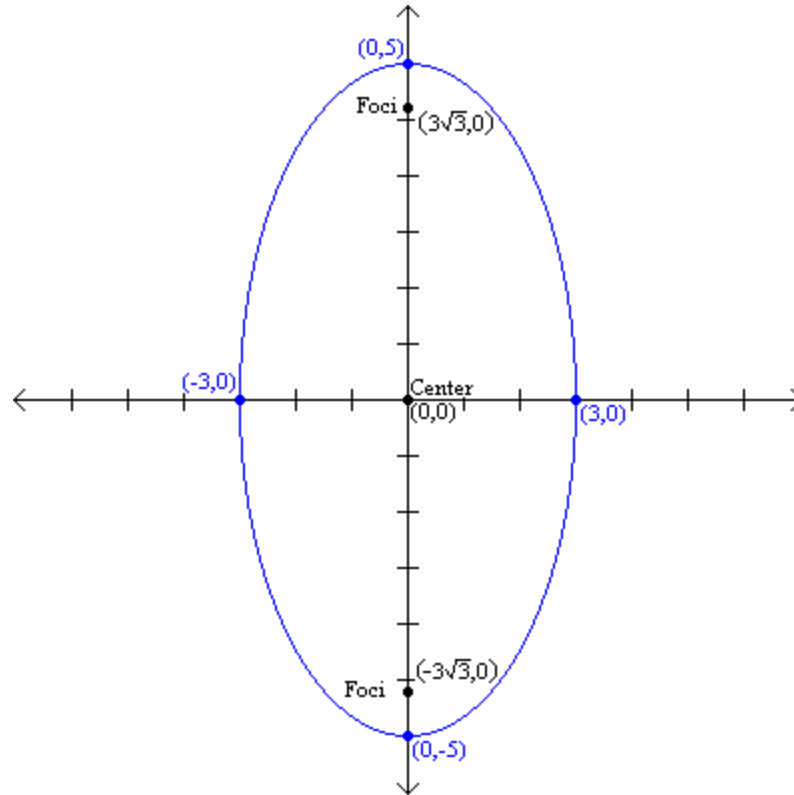
$$c^2 = 27$$

$$c = \pm 3\sqrt{3}$$

The foci, $(0, c)$, $(0, -c)$ are: $(0, 3\sqrt{3})$, $(0, -3\sqrt{3})$.

Example 1 (Continued):

Step 4. Graph.



Example 2. Given the ellipse, $x^2 + 4y^2 = 16$, find the foci, vertices, endpoints and graph.

Step 1. Rewrite the equation in standard form.

$$x^2 + 4y^2 = 16$$

$$\frac{x^2}{16} + \frac{4y^2}{16} = \frac{16}{16}$$

$$\frac{x^2}{16} + \frac{y^2}{4} = 1$$

$$\frac{x^2}{(4)^2} + \frac{y^2}{(2)^2} = 1$$

Example 2 (Continued):

Step 2. Analysis.

- a.) **Since $a > b$, the horizontal axis is the major axis.**
- b.) **Since the major axis is horizontal, the formula for the foci, $[(-c,0) , (c,0)]$ is $c^2 = a^2 - b^2$.**
- c.) $a = 4$, $b = 2$

Step 3. Plot the required points.

- a.) **The vertices.**

**Since $a = 4$, the vertices, $(-a,0)$ and $(a,0)$, are:
 $(4,0)$ and $(-4,0)$.**

- b.) **The endpoints.**

**Since $b = 2$, the endpoints, $(0,b)$ and $(0,-b)$, are:
 $(0,2)$ and $(0,-2)$.**

- c.) **The foci.**

$$c^2 = a^2 - b^2$$

$$c^2 = (4)^2 - (2)^2$$

$$c^2 = 16 - 4$$

$$c^2 = 12$$

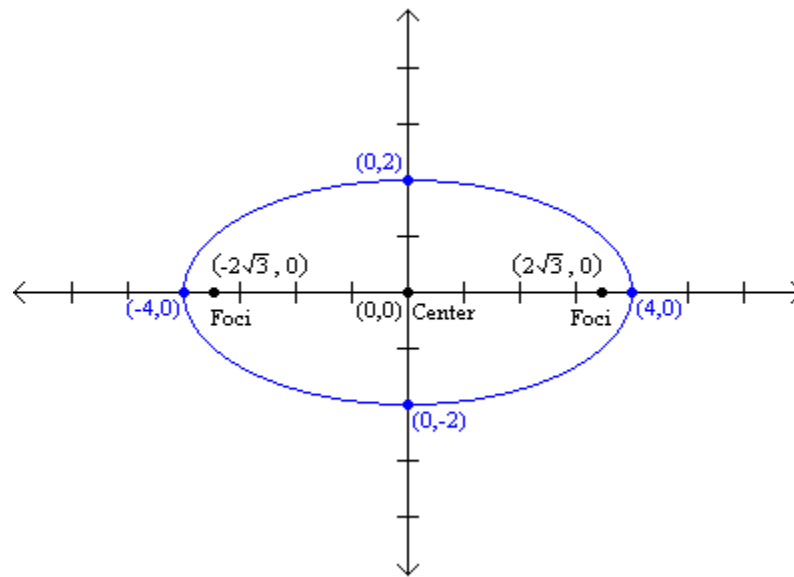
$$c = \pm\sqrt{12} = \pm 2\sqrt{3}$$

The foci, $(c,0)$ and $(-c,0)$, are:

$(2\sqrt{3},0)$ and $(-2\sqrt{3},0)$.

Example 2 (Continued):

Step 4. Graph.



As with circles, the center of an ellipse does not have to be located at the origin. The following formulas cover this possibility. They should be used in lieu of the previous formulas given.

The standard equation of an ellipse whose center is not the origin.

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

Vertex: (h, k)

Vertices:

Horizontal: $(h-a, k)$ and $(h+a, k)$

Vertical: $(h, k+b)$ and $(h, k-b)$

Endpoints:

Horizontal: $(h, k+b)$ and $(h, k-b)$

Vertical: $(h-a, k)$ and $(h+a, k)$

Foci:

Horizontal: $(h-c, k)$ and $(h+c, k)$ where $c^2 = a^2 - b^2$

Vertical: $(h, k+c)$ and $(h, k-c)$ where $c^2 = b^2 - a^2$

Example 3. Find the center, foci, vertices, endpoints and graph the ellipse:

$$x^2 + 4y^2 + 6x - 8y + 9 = 0$$

Step 1. Rewrite the equation in standard form.

$$x^2 + 4y^2 + 6x - 8y + 9 = 0$$

$$x^2 + 6x + 4y^2 - 8y = -9$$

$$(x^2 + 6x) + (4y^2 - 8y) = -9$$

$$(x^2 + 6x) + (4)(y^2 - 2y) = -9$$

$$(x^2 + 6x + 9) + (4)(y^2 - 2y + 1) = -9 + 9 + 4$$

$$(x + 3)^2 + 4(y - 1)^2 = 4$$

$$\frac{(x - (-3))^2}{4} + \frac{4(y - 1)^2}{4} = \frac{4}{4}$$

$$\frac{(x - (-3))^2}{(2)^2} + \frac{(y - 1)^2}{1} = 1$$

Step 2. Analysis.

- a.) **Since $a > b$, the horizontal axis is the major axis.**
- b.) **Since the major axis is horizontal, the formula for the foci is $[(h - c, k), (h + c, k)]$, where $c^2 = a^2 - b^2$.**
- c.) **$a = 2$, $b = 1$, $h = -3$, $k = 1$**

Step 3. Plot the required points.

a.) **The vertex**

Since the formula for the vertex is (h, k) and $h = -3$ and $k = 1$, the vertex is $(-3, 1)$

b.) **The vertices.**

Since $h = -3$, $k = 1$ and $a = 2$, the vertices, $(h - a, k)$ and $(h + a, k)$, are: $(-3 - 2, 1) = (-5, 1)$ and $(-3 + 2, 1) = (-1, 1)$.

Example 3 (Continued):

Step 3. Plot the required points.

c.) The endpoints.

Since $h = -3$, $k = 1$ and $b = 1$, the endpoints, $(h, k + b)$ and $(h, k - b)$ are: $(-3, 1 + 1) = (-3, 2)$ and $(-3, 1 - 1) = (-3, 0)$.

d.) The foci.

Since $a = 2$ and $b = 1$

$$c^2 = a^2 - b^2$$

$$c^2 = (2)^2 - (1)^2$$

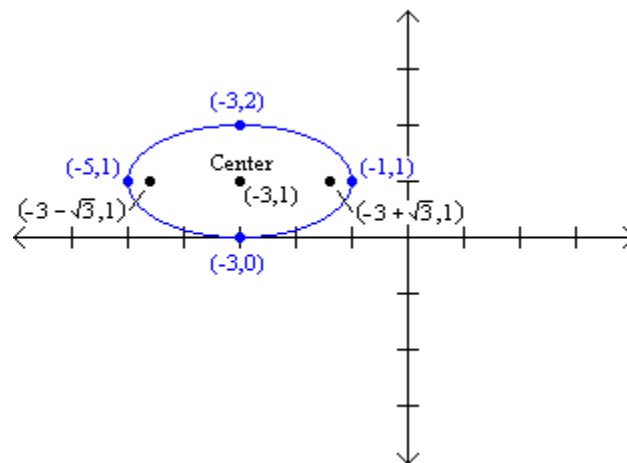
$$c^2 = 4 - 1$$

$$c^2 = 3$$

$$c = \pm\sqrt{3}$$

Since $h = -3$, $k = 1$ and $c = \pm\sqrt{3}$ the foci, $(h - c, k)$ and $(h + c, k)$, are: $(-3 - \sqrt{3}, 1)$ and $(-3 + \sqrt{3}, 1)$.

Step 4. Graph.



Example 4. Find the center, foci, vertices, endpoints and graph the ellipse:

$$9x^2 + 4y^2 - 36x + 8y + 31 = 0$$

Step 1. Rewrite the equation in standard form.

$$9x^2 + 4y^2 - 36x + 8y + 31 = 0$$

$$9x^2 - 36x + 4y^2 + 8y + 31 = 0$$

$$(9x^2 - 36x) + (4y^2 + 8y) = -31$$

$$9(x^2 - 4x) + 4(y^2 + 2y) = -31$$

$$9(x^2 - 4x + 4) + 4(y^2 + 2y + 1) = -31 + 36 + 4$$

$$9(x - 2)^2 + 4(y + 1)^2 = 9$$

$$\frac{9(x - 2)^2}{9} + \frac{4(y + 1)^2}{9} = \frac{9}{9}$$

$$\frac{(x - 2)^2}{1} + \frac{4(y + 1)^2}{9} = 1$$

$$\frac{(x - 2)^2}{1} + \frac{(y + 1)^2}{\frac{9}{4}} = 1$$

$$\frac{(x - 2)^2}{1} + \frac{(y - (-1))^2}{\left(\frac{3}{2}\right)^2} = 1$$

Step 2. Analysis.

- a.) Since $b > a$, the vertical axis is the major axis.
- b.) Since the major axis is vertical, the formula for the foci is $[(h, k + c), (h, k - c)]$, where $c^2 = b^2 - a^2$.
- c.) $a = 1$, $b = \frac{3}{2}$, $h = 2$, $k = -1$

Step 3. Plot the required points.

- a.) The vertex

Since the formula for the vertex is (h, k) , where $h = 2$ and $k = -1$, the vertex is $(2, -1)$.

Example 4 (Continued):

Step 3. Plot the required points.

b.) The vertices.

Since $h = 2$, $k = -1$, and $b = \frac{3}{2}$, the vertices, $(h, k + b)$

and $(h, k - b)$, are: $\left(2, -1 + \frac{3}{2}\right) = \left(2, \frac{1}{2}\right)$ and

$$\left(2, -1 - \frac{3}{2}\right) = \left(2, -\frac{5}{2}\right).$$

c.) The endpoints.

Since $h = 2$, $k = -1$ and $a = 1$, the endpoints, $(h - a, k)$

and $(h + a, k)$ are: $(2 - 1, -1) = (1, -1)$ and

$$(2 + 1, -1) = (3, -1).$$

d.) The foci.

Since $a = 1$ and $b = \frac{3}{2}$

$$c^2 = b^2 - a^2$$

$$c^2 = \left(\frac{3}{2}\right)^2 - (1)^2$$

$$c^2 = \frac{9}{4} - \frac{4}{4}$$

$$c^2 = \frac{5}{4}$$

$$c = \pm \sqrt{\frac{5}{4}} = \pm \frac{\sqrt{5}}{2}$$

Since $h = 2$, $k = -1$, and $c = \pm \frac{\sqrt{5}}{2}$ the foci, $(h, k + c)$ and

$(h, k - c)$, are: $\left(2, -1 - \frac{\sqrt{5}}{2}\right) = \left(2, -\frac{2 - \sqrt{5}}{2}\right)$ and

$$\left(2, -1 + \frac{\sqrt{5}}{2}\right) = \left(2, -\frac{2 + \sqrt{5}}{2}\right).$$

Example 4 (Continued):

Step 4. Graph.

