CHAPTER 13 SECTION 4
HYPERBOLAS

A hyperbola is the set of all points, the difference of whose distances traced from two distinct points (the foci), is constant.

When graphed, two disconnected curves are formed. These are known as the branches of the hyperbola.

A line passing through both foci and intersecting the hyperbola at two points is known as the transverse axis. A line perpendicular to the transverse axis is called the conjugate axis. The midpoint of the transverse axis is known as the center.

A pair of lines called asymptotes will need to be solved for and graphed in order to more accurately graph the hyperbola. These lines cross at the center and will pass diagonally through a rectangle to be constructed as an aid in the graphing process.

The following information applies to hyperbolas with their centers at the origin.

Formulas for solving hyperbolas (with the center at the origin) having a:

A) Horizontal transverse axis.

1.) Standard equation: $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

2.) $c^2 = a^2 + b^2$

3.) Foci: $(-c,0),(c,0)$

4.) Vertices: $(-a,0),(a,0)$

5.) Endpoints: $(0,-b),(0,b)$

6.) Asymptotes: $y = \pm\frac{b}{a}x$

7) Transverse axis: $y = 0$

8.) Conjugate axis: $x = 0$

Formulas for solving hyperbolas (with the center at the origin) having a:

B) Vertical transverse axis

1.) Standard equation: $\frac{y^2}{b^2}-\frac{x^2}{a^2}=1$

2.) $c^2=a^2+b^2$

3.) Foci: $(0,-c),(0,c)$

4.) Vertices: $(0,-b),(0,b)$

5.) Endpoints: $(-a,0),(a,0)$

6.) Asymptotes: $y=\pm\frac{b}{a}x$

7) Transverse axis: ***x = 0***

8.) Conjugate axis: ***y = 0***

Example 1. **Sketch the graph of the hyperbola with the equation** $4x^2-y^2=16$**. Include and label the foci, vertices, endpoints, asymptotes, transverse and conjugate axis, and rectangle constructions.**

Step 1. **Rewrite the equation in standard form.**

$$4x^2-y^2=16$$
$$\frac{4x^2}{16}-\frac{y^2}{16}=\frac{16}{16}$$
$$\frac{x^2}{4}-\frac{y^2}{16}=1$$
$$\frac{x^2}{(2)^2}-\frac{y^2}{(4)^2}=1$$

Step 2. **Analysis.**

a.) **The transverse axis is horizontal because** x^2 **is positive.**

b.) $c^2=a^2+b^2$

c.) $a=2$ **,** $b=4$

Example 1 (Continued):

Step 2. **Analysis.**

d.) $c^2 = a^2 + b^2$

$c^2 = (2)^2 + (4)^2$

$c^2 = 4 + 16 = 20$

$c = \pm\sqrt{20} = \pm 2\sqrt{5}$

e.) **Foci:** $\left(-2\sqrt{5}, 0\right), \left(2\sqrt{5}, 0\right)$

f.) **Vertices:** $(-2, 0), (2, 0)$

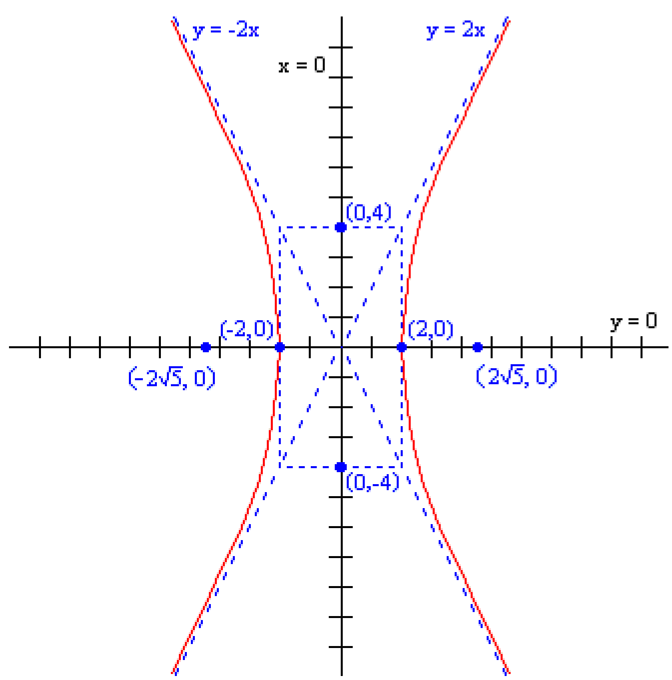
g.) **Endpoints:** $(0, -4), (0, 4)$

h.) **Asymptotes:** $y = \pm\frac{4}{2}x = \pm 2x$

i.) **Transverse axis:** **y = 0**

j.) **Conjugate axis:** **x = 0**

Step 3. **Graph**



Example 2. **Sketch the graph of the hyperbola with the equation** $9y^2 - x^2 = 9$**. Include and label the foci, vertices, endpoints, asymptotes, transverse and conjugate axis, and rectangle constructions.**

Step 1. **Rewrite the equation in standard form.**

$$9y^2 - x^2 = 9$$
$$\frac{9y^2}{9} - \frac{x^2}{9} = \frac{9}{9}$$
$$\frac{y^2}{1} - \frac{x^2}{9} = 1$$
$$\frac{y^2}{(1)^2} - \frac{x^2}{(3)^2} = 1$$

Step 2. **Analysis.**

a.) **The transverse axis is vertical because** y^2 **is positive.**

b.) $c^2 = a^2 + b^2$

c.) $a = 3$ **,** $b = 1$

d.) $c^2 = a^2 + b^2$

$$c^2 = (3)^2 + (1)^2$$
$$c^2 = 9 + 1 = 10$$
$$c = \pm\sqrt{10}$$

e.) **Foci:** $\left(0, -\sqrt{10}\right), \left(0, \sqrt{10}\right)$

f.) **Vertices:** $(0, -1), (0, 1)$

g.) **Endpoints:** $(-3, 0), (3, 0)$

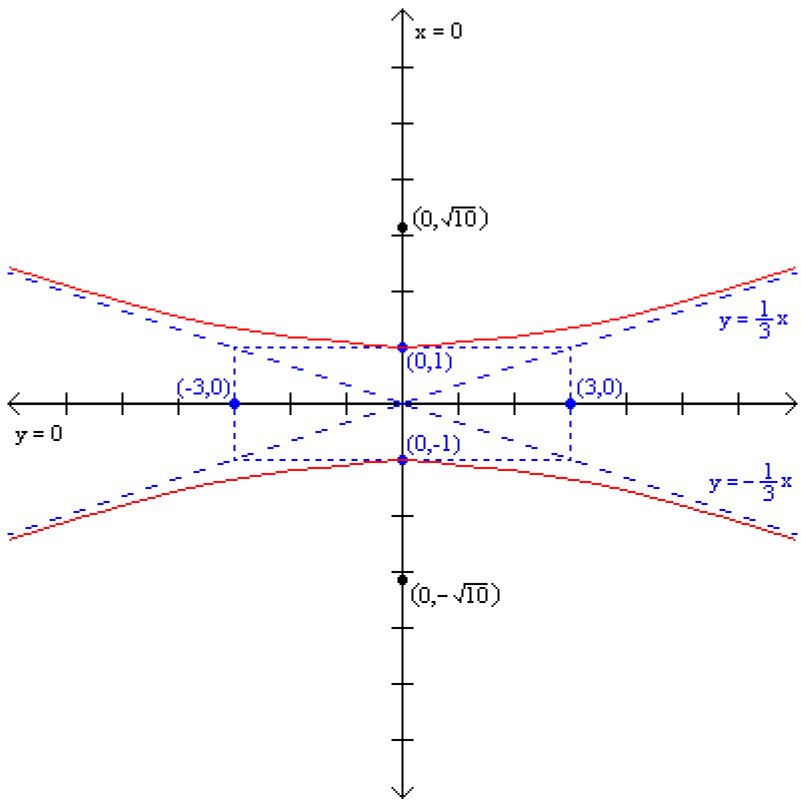
h.) **Asymptotes:** $y = \pm\frac{1}{3}x$

i.) **Transverse axis:** **x = 0**

j.) **Conjugate axis:** **y = 0**

Example 2 (Continued):

Step 3. **Graph**



Formulas for solving hyperbolas (with the center <u>not</u> at the origin) having a:

A) Horizontal transverse axis.

1.) Standard equation: $\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$

2.) Center: (h,k)

3.) $c^2=a^2+b^2$

4.) Foci: $(h-c,k),(h+c,k)$

5.) Vertices: $(h-a,k),(h+a,k)$

6.) Endpoints: $(h,k-b),(h,k+b)$

7.) Asymptotes: $y=\pm\frac{b}{a}x$

8) Transverse axis: ***y = k***

9.) Conjugate axis: ***x = h***

Formulas for solving hyperbolas (with the center <u>not</u> at the origin) having a:

B) ***Vertical transverse axis***

1.) ***Standard equation:*** $\frac{(y-k)^2}{b^2} - \frac{(x-h)^2}{a^2} = 1$

2.) $c^2 = a^2 + b^2$

3.) ***Center:*** (h,k)

4.) ***Foci:*** $(h,k-c),(h,k+c)$

5.) ***Vertices:*** $(h,k-b),(h,k+b)$

6.) ***Endpoints:*** $(h-a,k),(h+a,k)$

7.) ***Asymptotes:*** $y = \pm\frac{b}{a}x$

8) ***Transverse axis:*** ***x = h***

9.) ***Conjugate axis:*** ***y = k***

Example 3. **Sketch the graph of the hyperbola with the equation** $y^2 - 4x^2 + 4y + 24x - 41 = 0$**. Include and label the foci, vertices, endpoints, asymptotes, transverse and conjugate axis and rectangle constructions.**

Step 1. **Rewrite the equation in standard form.**

$$y^2 - 4x^2 + 4y + 24x - 41 = 0$$
$$y^2 + 4y - 4x^2 + 24x = 41$$
$$(y^2 + 4y) - 4(x^2 - 6x) = 41$$
$$(y^2 + 4y + 4) - 4(x^2 - 6x + 9) = 41 + 4 - 36$$
$$(y+2)^2 - 4(x-3)^2 = 9$$
$$\frac{(y-(-2))^2}{9} - \frac{4(x-3)^2}{9} = \frac{9}{9}$$
$$\frac{(y-(-2))^2}{(3)^2} - \frac{(x-3)^2}{\left(\frac{3}{2}\right)^2} = 1$$

Example 3 (Continued):

Step 2. **Analysis.**

a.) **The transverse axis is vertical.**

b.) $c^2 = a^2 + b^2$

c.) $a = \frac{3}{2}$, $b = 3$, $h = 3$, $k = -2$

d.) **Center =** $(3,-2)$

e.) $c^2 = a^2 + b^2$

$$c^2 = \left(\frac{3}{2}\right)^2 + (3)^2$$

$$c^2 = \frac{9}{4} + \frac{36}{4} = \frac{45}{4}$$

$$c = \pm\frac{\sqrt{45}}{2} = \pm\frac{3\sqrt{5}}{2}$$

f.) **Foci :** $\left(3,-2+\frac{3\sqrt{5}}{2}\right) = \left(3,\frac{-4+3\sqrt{5}}{2}\right)$

$$\left(3,-2-\frac{3\sqrt{5}}{2}\right) = \left(3,\frac{-4-3\sqrt{5}}{2}\right)$$

g.) **Vertices:** $(3,-2+3) = (3,1)$

$$(3,-2-3) = (3,-5)$$

h.) **Endpoints :** $\left(3+\frac{3}{2},-2\right) = \left(\frac{9}{2},-2\right)$

$$\left(3-\frac{3}{2},-2\right) = \left(\frac{3}{2},-2\right)$$

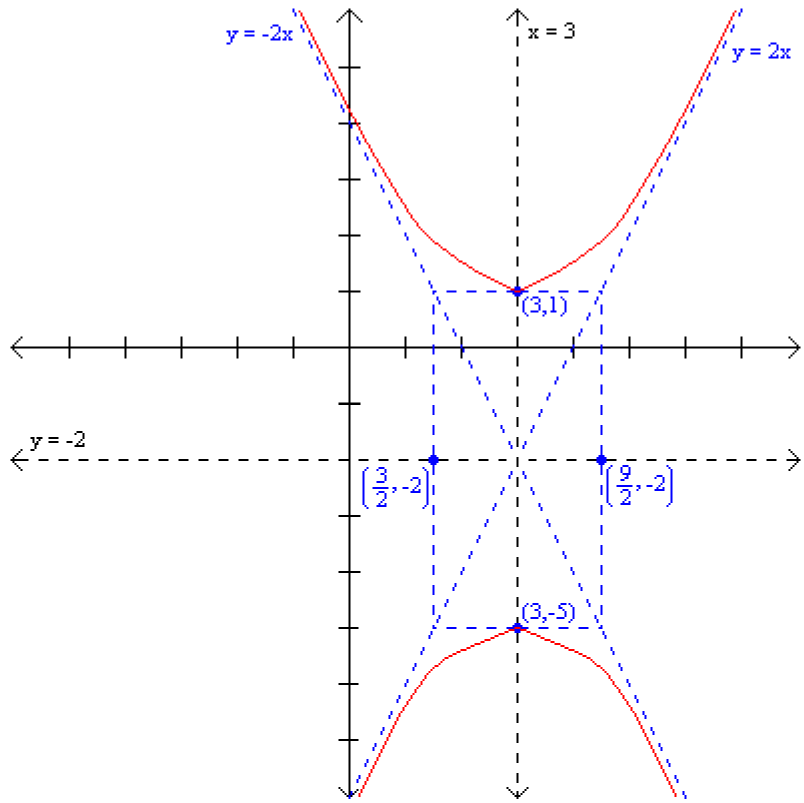
i.) **Asymptotes:** $y = \pm\frac{3}{\frac{3}{2}}x = \pm 2x$

j.) **Transverse axis =** **x = 3**

k.) **Conjugate axis =** **y = -2**

Example 3 (Continued):

Step 3. **Graph**



Example 4. **Sketch the graph of the hyperbola with the equation** $x^2 - 4y^2 - 2x - 16y - 19 = 0$**. Include and label the foci, vertices, endpoints, asymptotes, transverse and conjugate axis and rectangle constructions.**

Step 1. **Rewrite the equation in standard form.**

$$x^2 - 4y^2 - 2x - 16y - 19 = 0$$

$$x^2 - 2x - 4y^2 - 16y = 19$$

$$\left(x^2 - 2x\right) - 4\left(y^2 + 4y\right) = 19$$

$$\left(x^2 - 2x + 1\right) - 4\left(y^2 + 4y + 4\right) = 19 + 1 - 16$$

$$\left(x-1\right)^2 - 4\left(y+2\right)^2 = 4$$

$$\frac{\left(x-1\right)^2}{4} - \frac{4\left(y-(-2)\right)^2}{4} = \frac{4}{4}$$

$$\frac{\left(x-1\right)^2}{\left(2\right)^2} - \frac{\left(y-(-2)\right)^2}{\left(1\right)^2} = 1$$

Example 4 (Continued):

Step 2. **Analysis.**

a.) **The transverse axis is horizontal.**

b.) $c^2 = a^2 + b^2$

c.) $a = 2$ **,** $b = 1$**,** $h = 1$**,** $k = -2$

d.) **Center =** $(1, -2)$

e.) $c^2 = a^2 + b^2$

$c^2 = (2)^2 + (1)^2$

$c^2 = 4 + 1 = 5$

$c = \pm\sqrt{5}$

f.) **Foci :** $(1+\sqrt{5}, -2), (1-\sqrt{5}, -2)$

g.) **Vertices:** $(1+2, -2) = (3, -2)$

$(1-2, -2) = (-1, -2)$

h.) **Endpoints :** $(1, -2+1) = (1, -1)$

$(1, -2-1) = (1, -3)$

i.) **Asymptotes:** $y = \pm\frac{1}{2}x$

j.) **Transverse axis = y = -2**

k.) **Conjugate axis = x = 1**

Example 4 (Continued):

Step 3. **Graph**