Gauss, recurrence relations, and the agM

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Between 1796 and 1814, Gauss kept a diary in which he recorded his mathematical discoveries. The entry for May 30, 1799, reads, "*Terminum medium arithmetico-geometricum inter* 1 *et* $\sqrt{2}$ *esse* $= \frac{\pi}{\varpi}$ *usque ad figuram undecimam comprobavimus, quare demonstrata prorsus novus campus in analysi certo aperietur*"; that is, "We have verified that the arithmetico-geometric mean between 1 and $\sqrt{2}$ is $= \frac{\pi}{\varpi}$ up to 11 figures, which being proved, a completely new field in analysis will surely be opened up."

Let a and b be two positive numbers. We may assume that $b \le a$. Their *arithmetic mean* is $a' = (a+b)/2$. Their *geometric mean* is $b' = \sqrt{ab}$. Note that $b \le a' \le a$ and $b \le b' \le a$. Also, an elementary argument shows that $b' \le a'$.

Having now produced two numbers a' and b', we could take *their* arithmetic and geometric means. This suggests the following process. Start with two positive numbers a and b, and let $a_0 = \max(a, b)$ and $b_0 = \min(a, b)$, so that $b_0 \le a_0$. Then, for $n \ge 0$, define $a_{n+1} = (a_n + b_n)/2$ and $b_{n+1} = \sqrt{a_n b_n}$. Our previous remarks show that $b_n \le b_{n+1} \le a_{n+1} \le a_n$. Furthermore,

$$a_{n+1} - b_{n+1} \le a_{n+1} - b_n = 2^{-1}(a_n - b_n).$$

Consequently, the sequences $\{a_n\}$ and $\{b_n\}$ approach a common limit, called the *arithmetic-geometric mean* (or "agM") of a and b, denoted by

$$M(a, b) = \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n.$$

It can be shown (*e.g.*, Borwein & Borwein, *Pi and the AGM*, p. 2) that the convergence is very fast. Thus, as Gauss himself computed, if $a = \sqrt{2}$ and $b = 1$ the arithmetic and geometric means a_4 and b_4 agree to 19 decimal places.

We remark the following simple properties of the agM. If $\lambda > 0$, then $M(\lambda a, \lambda b) = \lambda M(a, b)$. Also,

$$M(a, b) = M\left(\frac{a+b}{2}, \sqrt{ab}\right).$$

The other side of Gauss's discovery involves the integral

$$\varpi = 2\int_0^1 \frac{dx}{\sqrt{1-x^4}},$$

which arises when you try to compute the arc-length of the lemniscate $(x^2+y^2)^2 = x^2 - y^2$ (see Siegel, *Topics in Complex Function Theory*, vol. 1, pp. 1–3). (Note that if the exponent 4 were replaced by 2, ϖ would be replaced by π.) The lemniscatic integral had been computed numerically by Stirling, and many of its mathematical properties had been studied by Euler. We know today that it cannot be expressed in terms of elementary functions; it belongs to a class of integrals known as *elliptic integrals* (so called because another member of the class arises from the computation of the arc-length of an ellipse).

Of course, if two numbers agree to 11 decimal places, they must be equal! But Gauss now undertook to find a mathematical proof of that equality. The entry in his diary for December 23, 1799, reads, "*Medium Arithmetico-Geometricum ipsum est quantitas integralis —Dem[onstratum]*": "The arithmetic-geometric mean is itself an integral quantity —Proved!" It's not entirely clear just what it was that Gauss had proved. However, sometime, evidently in 1800, Gauss wrote up a set of notes containing a proof of the relationship between the arithmetic-geometric mean and the elliptic integral. It is possible that this is the proof that his diary entry refers to. GAUSS never published these notes, but they can now be found in volume III of his collected works (pp. 361–374).

To understand GAUSS's proof, start again with $a \geq b > 0$. Can we operate the agM iteration *backwards*? In other words, can we find $\alpha \geq \beta > 0$ with $a = (\alpha + \beta)/2$ and $b = \sqrt{\alpha\beta}$? Equivalently, $2a = \alpha + \beta$ and $b^2 = \alpha\beta$, so that α and β are the solutions of the quadratic equation

$$x^2 - 2ax + b^2 = 0.$$

Thus, we have $\alpha = a + \sqrt{a^2 - b^2}$ and $\beta = a - \sqrt{a^2 - b^2}$.

Now GAUSS considers the symmetric expression $M(1 + x, 1 - x)$, for $0 \leq x < 1$. By homogeneity, we have

$$M(1 + x, 1 - x) = (1 + x)M\left(1, \frac{1 - x}{1 + x}\right).$$

Note that

$$1 - \left(\frac{1 - x}{1 + x}\right)^2 = \frac{4x}{(1 + x)^2},$$

so that

$$\begin{aligned} M(1 + x, 1 - x) &= (1 + x)M\left(1, \frac{1 - x}{1 + x}\right) \\ &= (1 + x)M\left(1 + \frac{2\sqrt{x}}{1 + x}, 1 - \frac{2\sqrt{x}}{1 + x}\right). \end{aligned} \tag{A}$$

GAUSS next assumed that the reciprocal of $M(1 + x, 1 - x)$ could be expanded as a power series

$$\frac{1}{M(1 + x, 1 - x)} = 1 + d_1x^2 + d_2x^4 + d_3x^6 + \cdots.$$

Clearly $M(1+x, 1-x)$ is even; hence the power-series expansion has only terms with even powers of x. Why expand the *reciprocal* of $M(1 + x, 1 - x)$? Presumably to bring the lemniscatic integral into the numerator. Incidentally, using techniques of complex-variable theory, it is not difficult to show that $M(1 + x, 1 - x)^{-1}$ is an *analytic* function of x, thus justifying GAUSS's assumption. (See BORWEIN/BORWEIN, p. 5, Ex. 4; the agM can be expressed as an *infinite product*.)

One might also ask what led GAUSS to consider the symmetric form $M(1+x, 1-x)$. Actually, his notes show that he originally tried to work with $M(1, x)$, but couldn't find any obvious pattern in the coefficients of the corresponding power-series expansion.

The identity (A) gives a corresponding identity of power series:

$$1 + d_1x^2 + d_2x^4 + \cdots = \frac{1}{1 + x} + d_1\frac{(2\sqrt{x})^2}{(1 + x)^3} + d_2\frac{(2\sqrt{x})^4}{(1 + x)^5} + \cdots.$$

The idea is to use this identity to solve for the coefficients d_n.

Note that each radical is raised to an even power. In order to simplify the resulting formulas, let us, after multiplying out the radicals, replace x by $-x$. This gives

$$1 + d_1x^2 + d_2x^4 + \cdots = \frac{1}{1 - x} - \frac{2^2d_1x}{(1 - x)^3} + \frac{2^4d_2x^2}{(1 - x)^5} - \cdots + (-1)^n\frac{2^{2n}d_nx^n}{(1 - x)^{2n+1}} + \cdots. \tag{B}$$

Recall that by NEWTON's binomial series,

$$(1 - x)^{-m} = \sum_{\ell=0}^{\infty}\binom{-m}{\ell}(-1)^\ell x^\ell = \sum_{\ell=0}^{\infty}\frac{m^{\overline{\ell}}}{\ell!}x^\ell = \sum_{\ell=0}^{\infty}\binom{m + \ell - 1}{\ell}x^\ell.$$

(Here $m^{\overline{\ell}}$ is the "rising factorial power": $m^{\overline{\ell}} = m(m+1)\cdots(m+\ell-1)$.)

Now since (B) is an identity in power series, we can equate the coefficient of x^n on both sides. (Let's take $d_0 = 1$.) On the LHS, the coefficient is

$$\begin{cases} d_{n/2} & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

On the RHS, we have

$$\begin{aligned} 1 - 2^2 d_1 \binom{2+n-1}{n-1} &+ 2^4 d_2 \binom{4+n-2}{n-2} + \cdots + (-1)^n 2^{2n} d_n \binom{2n+n-n}{n-n} \\ &= 1 - 2^2 d_1 \binom{n+1}{2} + 2^4 d_2 \binom{n+2}{4} + \cdots + (-1)^n 2^{2n} d_n \binom{2n}{2n}. \end{aligned} \tag{C}$$

This gives an infinite sequence of equations which can be solved, successively, for d_1, d_2, ... In fact, let's write out the first few equations:

$$\begin{aligned}
n = 0: &\qquad 1 = 1 \\
n = 1: &\qquad 0 = 1 - 2^2 d_1 \\
n = 2: &\qquad d_1 = 1 - 2^2 \cdot 3 d_1 + 2^4 d_2 \\
n = 3: &\qquad 0 = 1 - 2^2 \binom{4}{2} d_1 + 2^4 \binom{5}{4} d_2 - 2^6 d_3 \\
n = 4: &\qquad d_2 = 1 - 2^2 \binom{5}{2} d_1 + 2^4 \binom{6}{4} d_2 - 2^6 \binom{7}{6} d_3 + 2^8 d_4 \\
&\qquad \text{etc.}
\end{aligned}$$

(I've written out these equations using modern notation, with subscripts, binomial coefficients, and so forth. Gauss himself calls the unknown coefficients A, B, C, ... rather than d_1, d_2, d_3, ..., and when he writes out this system of equations he just gives the numerical value of each coefficient. He clearly sees the general pattern, however; see, for example, his *Works*, vol. III, p. 368. It's not easy, in his notation, to follow just what he is doing, so in the following exposition I have expanded many details.)

In equation n, the coefficient of highest index is d_n, so that that equation can be solved for d_n in terms of previous coefficients. If we do this, we find, for example,

$$d_1 = \frac{1}{4} = \left(\frac{1}{2}\right)^2, \quad d_2 = \frac{9}{64} = \left(\frac{3}{8}\right)^2, \quad d_3 = \frac{25}{256} = \left(\frac{5}{16}\right)^2, \quad d_4 = \frac{1225}{16384} = \left(\frac{35}{128}\right)^2,$$
$$d_5 = \frac{3969}{65536} = \left(\frac{63}{256}\right)^2, \quad d_6 = \frac{53361}{1048576} = \left(\frac{231}{1024}\right)^2, \quad d_7 = \frac{184041}{4194304} = \left(\frac{429}{2048}\right)^2, \ldots.$$

We note that these are all perfect squares. Gauss probably noticed that if we take the *ratios* of successive terms,

$$\frac{d_2}{d_1} = \left(\frac{3}{4}\right)^2, \qquad \frac{d_3}{d_2} = \left(\frac{5}{6}\right)^2, \qquad \frac{d_4}{d_3} = \left(\frac{7}{8}\right)^2, \ldots,$$

and the pattern is revealed! Clearly,

$$\frac{d_n}{d_{n-1}} = \left(\frac{2n-1}{2n}\right)^2,$$

so that

$$d_n = \left(\frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots 2n}\right)^2.$$

So far, of course, we have only found the solution "empirically". GAUSS still had to *prove* that this solution was in fact correct. This is not so simple. The idea will be to *combine* the equations (C) in such a way as to simplify them (as in "Gaussian elimination"!). Specifically, GAUSS formed

$$n^2[\text{Eq. } n] - (n-1)^2[\text{Eq. } (n-2)]$$

for $n \geq 2$.

How did GAUSS think of this? Well, he doesn't say. But one virtue of this combination is that —if our conjecture about the values of the coefficients d_n is correct— it will eliminate the terms on the LHS.

From eq. 1, $d_1 = \frac{1}{4}$. Then $4 \cdot$ Eq. 2 − Eq. 0 gives

$$\begin{aligned} 0 &= 3 - 4\cdot 4\cdot 3\cdot d_1 + 4\cdot 16\cdot d_2 \\ &= 3 - 3\cdot 4\cdot d_1 - 3\cdot 4\cdot 3\cdot d_1 + 4\cdot 16\cdot d_2 \\ &= 3(1-4d_1) + 4(-9d_1 + 16d_2). \end{aligned}$$

So in fact this looks promising. Let's write down $n^2[\text{Eq. } n] - (n-1)^2[\text{Eq. } (n-2)]$ in general:

$$\text{LHS}: \quad \begin{cases} n^2 d_{n/2} - (n-1)^2 d_{(n-2)/2} & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

$$\begin{aligned} \text{RHS}: \quad & n^2\left[1 - 2^2 d_1\binom{n+1}{2} + 2^4 d_2\binom{n+2}{4} + \cdots + (-1)^n 2^{2n} d_n\binom{2n}{2n}\right] \\ & - (n-1)^2\left[1 - 2^2 d_1\binom{n-1}{2} + 2^4 d_2\binom{n}{4} + \cdots + (-1)^{n-2}2^{2n-4}d_{n-2}\binom{2n-4}{2n-4}\right] \\ = {}& [n^2-(n-1)^2] - 2^2 d_1\left[n^2\binom{n+1}{2} - (n-1)^2\binom{n-1}{2}\right] + 2^4 d_2\left[n^2\binom{n+2}{4} - (n-1)^2\binom{n}{4}\right] + \cdots \\ & + (-1)^{n-2}2^{2n-4}d_{n-2}\left[n^2\binom{2n-2}{2n-4} - (n-1)^2\binom{2n-4}{2n-4}\right] \\ & + (-1)^{n-1}n^2 2^{2n-2}d_{n-1}\binom{2n-1}{2n-2} + (-1)^n n^2 2^{2n}\binom{2n}{2n}. \end{aligned}$$

Here the coefficient of $d_0 = 1$ is $n^2 - (n-1)^2 = 2n-1$. For $1 \leq k \leq n-2$, the coefficient of d_k on the RHS will be

$$\begin{aligned} & (-1)^k 2^{2k}\left[n^2\binom{n+k}{2k} - (n-1)^2\binom{n+k-2}{2k}\right] \\ &= (-1)^k\frac{2^{2k}}{(2k)!}\left[n^2\overbrace{(n+k)(n+k-1)\cdots(n-k+1)}^{2k\text{ factors}} - (n-1)^2\overbrace{(n+k-2)\cdots(n-k-1)}^{2k\text{ factors}}\right] \\ &= (-1)^k\frac{2^{2k}}{(2k)!}\underbrace{(n+k-2)\cdots(n-k+1)}_{2k-2\text{ factors}}\left[n^2(n+k)(n+k-1) - (n-1)^2(n-k)(n-k-1)\right] \\ &= (-1)^k\frac{2^{2k}}{(2k)!}(n+k-2)\cdots(n-k+1)\left[n^2(n^2+(2k-1)n+k^2-k) - (n-1)^2(n^2-(2k+1)n+k^2+k)\right] \\ &= (-1)^k\frac{2^{2k}}{(2k)!}(n+k-2)\cdots(n-k+1)\left[(n^2-(n-1)^2)(n^2-(2k+1)n+k^2+k) + n^2(4kn-2k)\right] \\ &= (-1)^k\frac{2^{2k}}{(2k)!}\underbrace{(n+k-2)\cdots(n-k+1)}_{2k-2\text{ factors}}(2n-1)\left[n^2-(2k+1)n+k^2+k+2kn^2\right] \\ &= (-1)^k\frac{2^{2k}}{(2k)!}(n+k-2)\cdots(n-k+1)(2n-1)\left[(2k+1)n^2-(2k+1)n+k^2+k\right]. \end{aligned}$$

Recall that this holds for $1 \le k \le n-2$. In fact, it *also* holds for $k = n-1, n$. For if $k = n-1$, the formula becomes

$$\begin{aligned}
&(-1)^{n-1}\frac{2^{2n-2}}{(2n-2)!}\underbrace{(2n-3)\cdots 2}_{2n-4\text{ factors}}(2n-1)\left[(2n-1)n^2-(2n-1)n+(n-1)^2+(n-1)\right]\\
&\quad=(-1)^{n-1}\frac{2^{2n-2}}{(2n-2)!}(2n-3)\cdots 2(2n-1)\left[n^2(2n-2)\right]\\
&\quad=(-1)^{n-1}n^2(2n-1)2^{2n-2}.
\end{aligned}$$

And if $k = n$, the formula gives

$$\begin{aligned}
&(-1)^n\frac{2^{2n}}{(2n)!}(2n-2)!(2n-1)\left[(2n+1)n^2-(2n+1)n+n^2+n\right]\\
&\quad=(-1)^n\frac{2^{2n}}{(2n)!}(2n-2)!(2n-1)[n^2\cdot 2n]\\
&\quad=(-1)^n n^2 2^{2n}.
\end{aligned}$$

(Actually, it's obvious that the general formula must continue to hold even for $k = n-1$ and $k = n$, since, for those values of k, the binomial coefficient $\binom{n+k-2}{2k} = 0$.)

Next, Gauss breaks up the coefficient of d_k on the RHS into two parts. (Compare the numerical example we worked out for $n = 2$.)

$$\begin{aligned}
&(-1)^k\frac{2^{2k}}{(2k)!}\overbrace{(n+k-2)\cdots(n-k+1)}^{2k-2\text{ factors}}(2n-1)\left[(2k+1)n^2-(2k+1)n+k^2+k\right]\\
&\quad=(-1)^k(2n-1)\frac{2^{2k}}{(2k)!}(n+k-2)\cdots(n-k+1)\left[(2k+1)n^2-(2k+1)n-(2k+1)(k^2-k)+2k^3\right]\\
&\quad=(-1)^k(2n-1)(2k+1)\frac{2^{2k}}{(2k)!}\overbrace{(n+k-2)\cdots(n-k+1)[(n+k-1)(n-k)]}^{2k\text{ factors}}\\
&\qquad+(-1)^k(2n-1)\frac{2^{2k}}{(2k)!}\underbrace{(n+k-2)\cdots(n-k+1)}_{2k-2\text{ factors}}\cdot 2k^3\\
&\quad=A_k+B_k,\ \text{say.}
\end{aligned}$$

We have expressed the RHS of our equation as

$$\text{RHS} = (2n-1) + (A_1+B_1)d_1 + (A_2+B_2)d_2 + \cdots + (A_n+B_n)d_n.$$

Now Gauss rearranges this expression into the form

$$[(2n-1)+B_1d_1] + [A_1d_1 + B_2d_2] + \cdots + [A_{n-1}d_{n-1} + B_nd_n] + A_nd_n.$$

(Of course, the point is to bring out the relation between d_k and d_{k-1}. How did Gauss find the right way to break up into $A_k + B_k$? Presumably by working out the first few terms numerically.)

But $A_n = 0$ (since A_k contains the factor $n-k$). Also, $B_1 = -(2n-1)\frac{2^2}{2!}\cdot 2 = -(2n-1)2^2$. Let us therefore look at $A_{k-1}d_{k-1} + B_kd_k$

$$= (2n-1)\Bigg[(-1)^{k-1}(2k-1)\frac{2^{2k-2}}{(2k-2)!}\overbrace{(n+k-2)\cdots(n-k+1)}^{2k-2\text{ factors}}d_{k-1}$$

$$+ (-1)^k \frac{2^{2k}}{(2k)!} \underbrace{(n+k-2)\cdots(n-k+1)}_{2k-2 \text{ factors}} 2k^3 d_k \Bigg]$$

$$= (-1)^k (2n-1) \underbrace{(n+k-2)\cdots(n-k+1)}_{2k-2 \text{ factors}} \left[-(2k-1)\frac{2^{2k-2}}{(2k-2)!} d_{k-1} + 2k^3 \frac{2^{2k}}{(2k)!} d_k \right]$$

$$= (-1)^k (2n-1)(n+k-2)\cdots(n-k+1) \frac{1}{(2k-1)!} \left[-(2k-1)^2 2^{2k-2} d_{k-1} + k^2 2^{2k} d_k \right]$$

$$= (-1)^k \frac{(2n-1)(n+k-2)\cdots(n-k+1)2^{2k-2}}{(2k-1)!} \left[-(2k-1)^2 d_{k-1} + (2k)^2 d_k \right].$$

Thus $n^2[\text{Eq. } n] - (n-1)^2[\text{Eq. } n-2]$ is

$$\{n^2 d_{n/2} - (n-1)^2 d_{(n-2)/2}\}[n \text{ even}]$$

$$= (2n-1)\Bigg[\{1 - 2^2 d_1\} + \cdots + (-1)^k \frac{\overbrace{(n+k-2)\cdots(n-k+1)}^{2k-2 \text{ factors}} 2^{2k-2}}{(2k-1)!} \{-(2k-1)^2 d_{k-1} + (2k)^2 d_k\}$$

$$+ \cdots + (-1)^n \frac{(2n-2)! 2^{2n-2}}{(2n-1)!} \{-(2n-1)^2 d_{n-1} + (2n)^2 d_n\} \Bigg]$$

Here '$[n \text{ even}]$' is the characteristic function:

$$[n \text{ even}] = \begin{cases} 1 & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

It now follows clearly that

$$-(2n-1)^2 d_{n-1} + (2n)^2 d_n = 0 \qquad \text{for } n \geq 1;$$

thus

$$d_n = \left(\frac{1 \cdot 3 \cdots (2n-1)}{2 \cdot 4 \cdots 2n} \right)^2.$$

In other words, we have shown that

$$\frac{1}{M(1+x, 1-x)} = \sum_{n=0}^{\infty} \left(\frac{1 \cdot 3 \cdots (2n-1)}{2 \cdot 4 \cdots 2n} \right)^2 x^{2n}.$$

(Note that the coefficient of x^0 is 1.)

What a mathematical *tour de force*! It is amazing that Gauss had the intestinal fortitude to push this computation all the way through.

The rest of the proof is comparatively easy (and standard). In modern terminology, the *complete elliptic integral* (of the "1st kind") is

$$K(x) = \int_0^{\pi/2} \frac{d\varphi}{\sqrt{1 - x^2 \sin^2 \varphi}} = \int_0^{\pi/2} (1 - x^2 \sin^2 \varphi)^{-1/2} \, d\varphi$$

$$= \sum_{n=0}^{\infty} (-1)^n \binom{-1/2}{n} x^{2n} \int_0^{\pi/2} \sin^{2n} \varphi \, d\varphi$$

$$= \sum_{n=0}^{\infty} \frac{1 \cdot 3 \cdots (2n-1)}{2^n n!} x^{2n} \int_0^{\pi/2} \sin^{2n} \varphi \, d\varphi.$$

Using the standard reduction formula

$$\int \sin^n x\,dx = -\frac{\sin^{n-1} x \cos x}{n} + \frac{n-1}{n}\int \sin^{n-2} x\,dx$$

(which is derived by integration by parts), we find, in particular, that

$$\int_0^{\pi/2} \sin^n \varphi\,d\varphi = \frac{n-1}{n}\int_0^{\pi/2} \sin^{n-2}\varphi\,d\varphi,$$

so that

$$\begin{aligned}\int_0^{\pi/2} \sin^{2n}\varphi\,d\varphi &= \frac{2n-1}{2n}\int_0^{\pi/2}\sin^{2n-2}\varphi\,d\varphi \\ &= \frac{(2n-1)(2n-3)}{2n(2n-2)}\int_0^{\pi/2}\sin^{2n-4}\varphi\,d\varphi \\ &= \frac{(2n-1)(2n-3)\cdots 1}{(2n)(2n-2)\cdots 2}\int_0^{\pi/2} d\varphi \\ &= \frac{\pi}{2}\cdot\frac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdots(2n)}.\end{aligned}$$

Hence

$$K(x) = \frac{\pi}{2}\sum_{n=0}^{\infty}\left(\frac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdots(2n)}\right)^2 x^{2n},$$

and

$$\frac{\pi}{2}\frac{1}{M(1+x,1-x)} = K(x).$$

To relate this theorem to Gauss's original observation, note that

$$M(1+x,1-x) = M(1,\sqrt{1-x^2})$$

(since the arithmetic mean of $1+x$ and $1-x$ is 1, while their geometric mean is $\sqrt{1-x^2}$). Replacing $\sqrt{1-x^2}$ by x, we therefore have

$$\frac{\pi}{2}\frac{1}{M(1,x)} = \int_0^{\pi/2}\frac{d\varphi}{\sqrt{1-(1-x^2)\sin^2\varphi}}.$$

Now take $x=\sqrt{2}$. (There is a difficulty here, since we initially supposed that $|x|<1$, but it can be overcome by using the concept of "analytic continuation".) Making the change of variable $x=\sin\varphi$ in the resulting integral, we get

$$\frac{\pi}{2}\frac{1}{M(1,\sqrt{2})} = \int_0^1 \frac{dx}{\sqrt{1-x^4}},$$

as desired.

Gauss later found other proofs, which are simpler and perhaps better, in that they lead to an understanding of *why* there is a connection between the agM and the elliptic integral. Still, I like this first proof a lot. As for Gauss's "new field in analysis", you can find out about that in sources such as Borwein/Borwein, *Pi and the AGM*, and David Cox, "The Arithmetic-Geometric Mean of Gauss" (*L'Enseignement Mathématique*, **30**, 1984, pp. 275–330).