Fractional Roman Domination

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Abstract

A function \( f : V(G) \to \{0, 1, 2\} \) is a Roman dominating function if for every vertex with \( f(v) = 0 \), there exists a vertex \( w \in N(v) \) with \( f(w) = 2 \). We introduce two fractional Roman domination parameters, \( \gamma_{Rf} \) and \( \gamma_{Rf} \), from relaxations of two equivalent integer programming formulations of Roman domination (the former using open neighborhoods and the later using closed neighborhoods in the Roman domination integer program). We show \( \gamma_f(G) \leq \gamma_{Rf}(G) \leq 2\gamma_f(G) \), for all graphs \( G \). We define a graph to be fractionally Roman if \( \gamma_{Rf}(G) = 2\gamma_f(G) \). We find some classes of fractionally Roman graphs, and some classes which are not fractionally Roman.

1 Introduction

Our notation follows that of Haynes, Hedetniemi, and Slater \cite{4, 5}. In this paper, graphs \( G = (V, E) \) are finite with \( |V(G)| = n \) vertices and have no multiple edges or loops. The open neighborhood of a vertex \( v \in V(G) \) is defined as \( N_G(v) = \{ u \in V(G) \mid uv \in E \} \), the set of all vertices adjacent to \( v \). The closed neighborhood of a vertex \( v \in V(G) \) is defined as \( N_G[v] = \{ v \} \cup N_G(v) \). When the graph \( G \) is clear from context, \( N_G(v) \) and \( N_G[v] \) will be denoted by \( N(v) \) and \( N[v] \), respectively. The degree of a vertex \( v \), \( |N_G(v)| \), will be denoted by \( d(v) \). For a set \( S \subseteq V(G) \), let \( N(S) = \bigcup_{u \in S} N(u) \) and let \( N[S] = \bigcup_{u \in S} N[u] \). The distance between any two vertices \( u, v \in V(G) \), denoted by \( \text{dist}(u, v) \), is the length of a shortest path from \( u \) to \( v \).

A set of vertices \( S \subseteq V(G) \) is called a dominating set if every vertex \( v \in V(G) \) is either an element of \( S \) or is adjacent to some element of \( S \). That

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is, a set $S$ is a dominating set if $|N[v] \cap S| \geq 1$ for all vertices $v \in V(G)$. The domination number $\gamma(G)$ is the size of a smallest dominating set.

A Roman dominating function on a graph $G$ is a function $f : V \rightarrow \{0, 1, 2\}$ which satisfies the property that whenever $f(v) = 0$, there exists a $u \in N(v)$ for which $f(u) = 2$. The total weight of a minimum Roman dominating function is denoted by $\gamma_R$. Roman domination was introduced in [7] and popularized in [13]. Roman domination received much further attention after the publication of [10], [3], and [1].

The Roman domination number satisfies $\gamma \leq \gamma_R \leq 2\gamma$. The only graph on $n$ vertices with Roman domination number equal to its domination number is $K_n$ (see [1]). Graphs with $\gamma_R = 2\gamma$ are called Roman graphs.

A set $S \subseteq V(G)$ is a 2-packing if the minimum distance between any distinct vertices $u, v$ of $S$ is at least three. A set $S$ is a closed neighborhood packing if for each $u, v \in S$, $u \neq v$ we have $N[u] \cap N[v] = \emptyset$. A set $S$ is a closed neighborhood packing if and only if $S$ is a 2-packing, since vertices $u \neq v$ are in a closed neighborhood packing $S$ if and only if $\text{dist}(u, v) > 2$. The packing number $\rho(G)$ is the size of a largest closed neighborhood packing.

A fractional dominating function is a function $g : V(G) \rightarrow [0, 1]$ such that $g(N[v]) = \sum_{u \in N[v]} g(u) \geq 1$ for all vertices $v \in V(G)$. A minimum fractional dominating function is a fractional dominating function $g$ such that the value $|g| = \sum_{v \in V(G)} g(v)$ is as small as possible. This minimum value is the fractional domination number of $G$, denoted by $\gamma_f(G)$. A function $h : V(G) \rightarrow [0, 1]$ is a fractional packing function provided that
h(N[v]) = \sum_{u \in N[v]} h(u) \leq 1 \text{ for all } v \in V(G). \text{ A maximum fractional packing function} is a fractional packing function } h \text{ such that the value attained by } |h| = \sum_{v \in V(G)} h(v) \text{ is as large as possible. This maximum is called the fractional (closed neighborhood) packing number of } G \text{ and is denoted by } \rho_f(G).

A set of vertices } S \subseteq V(G) \text{ to be a total dominating set if every vertex } v \in V(G) \text{ is adjacent to some element of } S. \text{ That is, a set } S \text{ is a total dominating set if } |N(v) \cap S| \geq 1 \text{ for all vertices } v \in V(G). \text{ The total domination number } \gamma(G) \text{ is the size of a smallest total dominating set. A fractional total dominating function is a function } g : V \to [0, 1], \text{ such that } g(N(v)) = \sum_{u \in N(v)} g(u) \geq 1 \text{ for all vertices } v \in V. \text{ A minimum total fractional dominating function is a fractional total dominating function } g \text{ such that the value } |g| = \sum_{v \in V} g(v) \text{ is as small as possible. This minimum value is called the fractional total domination number of } G, \text{ denoted by } \gamma^\circ_f(G).

For all graphs } G, \text{ we have } \rho(G) \leq \rho_f(G) = \gamma_f(G) \leq \gamma(G). \text{ For all graphs } G \text{ without isolates, we have } \rho^2(G) \leq \rho^\circ_f(G) = \gamma^\circ_f(G) \leq \gamma_t(G). \text{ Also for all graphs } G \text{ without isolates, } \gamma_f(G) \leq \gamma^\circ_f(G), \text{ since every minimum fractional total dominating function is a fractional dominating function.}

2 Roman domination as an integer program

In [10], ReVelle and Rosing formulate Roman domination as an integer program. For a graph } G = (V, E) \text{ with } V = \{v_1, \ldots, v_n\}, \text{ for each } v_i \text{ in } V, \text{ define two } \{0, 1\} \text{ variables } X_i \text{ and } Y_i \text{ to be the first and second legions respectively located at } v_i. \text{ In earlier literature, this integer program is called the Set Covering Deployment Problem, or SCDP.}

Minimize \quad \sum_{i=1}^{n} (X_i + Y_i)
Subject to:
\begin{align*}
X_i &\geq Y_i \text{ for all } i \\
X_i + \sum_{v_i, v_j \in E} Y_j &\geq 1 \text{ for all } i \\
X_i, Y_i &\in \{0, 1\} \text{ for all } i
\end{align*}

The first constraint guarantees that the first legion is stationed at a vertex before the second. The first and second constraints guarantee that every vertex either has a legion stationed on it or has a neighbor with two legions stationed on it. The third constraint allows for only entire legions to be stationed. From an optimal solution to the SCDP problem, \{X_1, Y_1, \ldots, X_n, Y_n\}, we can obtain a minimum Roman dominating function (MRDF) } r \text{ by letting } r(v_i) = X_i + Y_i \text{ for all } i. \text{ Thus, the value of } \sum_{i=1}^{n} (X_i + Y_i), \text{ for any optimal solution is equal to } \gamma_R.
Let us translate the above IP into matrix terms. Let $A$ be the $n \times n$ adjacency matrix of a graph $G$. If we let $v = [X \ Y]^{T}$ be the $2n \times 1$ matrix $[X_1, \ldots, X_n, Y_1, \ldots, Y_n]^{T}$, then (1) is equivalent to (2) below.

$$\text{Minimize} \quad [1^{T} \ 1^{T}] [X \ Y]$$

$$\text{Subject to:} \quad \begin{bmatrix} I_n & A \\ I_n & -I_n \end{bmatrix} [X \ Y] \geq \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$v \in \{0,1\}$ vector \hspace{1cm} (2)

We can relax the condition that $v$ be a $\{0,1\}$ vector, and instead require that the entries be non-negative. Then the integer program (2) becomes a linear program. The value of $1^{T}v$ for any optimal solution of (3) is equal to the fractional (open neighborhood) Roman domination number, $\gamma_{R}^{f}(G)$.

$$\text{Minimize} \quad [1^{T} \ 1^{T}] [X \ Y]$$

$$\text{Subject to:} \quad \begin{bmatrix} I_n & A \\ I_n & -I_n \end{bmatrix} [X \ Y] \geq \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$v \geq 0$ \hspace{1cm} (3)

### 2.1 Open neighborhood Roman packing

The dual linear program of fractional (open neighborhood) Roman domination (3) is given below, where $[1 \ 0]^{T}$ is the $2n \times 1$ matrix with the 1 as the first $n$ entries and 0 as the next $n$ entries, and $u^{T}$ as the $2n \times 1$ matrix $[W \ Z]^{T}$.

$$\text{Maximize} \quad [1^{T} \ 0^{T}] [W \ Z]$$

$$\text{Subject to:} \quad \begin{bmatrix} I_n & I_n \\ A & -I_n \end{bmatrix} [W \ Z] \leq \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$u \geq 0$ \hspace{1cm} (4)

By linear programming duality, the value of the above linear program is the fractional (open neighborhood) Roman domination number. The integer program is then:
Maximize \[ \begin{bmatrix} 1^T & 0^T \end{bmatrix} \begin{bmatrix} W \\ Z \end{bmatrix} \]

Subject to:
\[ \begin{bmatrix} I_n & I_n \\ A & -I_n \end{bmatrix} \begin{bmatrix} W \\ Z \end{bmatrix} \leq \begin{bmatrix} 1 \\ 1 \end{bmatrix} \]
\[ u \in \{0, 1\} \text{ vector} \quad (5) \]

The value of this integer program will be called the \textit{(open neighborhood) Roman packing number}, denoted by \( \rho^o_R \). Thus, we have \( \rho^o_R(G) \leq \gamma^o_R(G) \leq \gamma_R(G) \), for all graphs \( G \).

For a concrete interpretation of (open neighborhood) Roman packing functions, suppose that a graph \( G \) represents a network of sites where energy generators can be placed; the goal is to maximize the production of energy. A vertex \( v \) with a generator produces one unit of energy; however, the operation of this generator also emits one unit of harmful radiation to each neighbor of \( v \) (though not at \( v \) itself). A given site can safely be exposed to a single unit of radiation; to counter higher levels of exposures, an undeveloped vertex (i.e. one without a generator) can host a single “reducer” to shield the site from one unit of radiation. The (open neighborhood) Roman packing number, \( \rho^o_R(G) \), is the maximum amount of energy produced, such that no vertex is exposed to more than one unit of harmful radiation.

Fractional (open neighborhood) Roman packing functions can be interpreted in a similar manner. Let \( W(v) \) and \( Z(v) \) denote the weight of a generator and reducer, respectively, at a vertex \( v \). In the integer case \( W(v) \) and \( Z(v) \) are \( \{0, 1\} \) valued variables. In the fractional case \( W(v) \) and \( Z(v) \) may take on values anywhere in \( [0, 1] \), provided that \( W(v) + Z(v) \leq 1 \) for all \( v \in V(G) \). At each vertex \( v \), \( W(v) \) units of energy are produced there. Also, \( W(v) \) units of harmful radiation are emitted to all neighbors of \( v \) (none to \( v \) itself). As before, any single vertex can safely handle at most one unit of harmful radiation. For each vertex \( v \), the harmful radiation present at \( v \) is reduced by \( Z(v) \) units. The fractional (open neighborhood) Roman packing number is the maximum amount of energy produced, such that no vertex is exposed to more than one unit of harmful radiation. That is, \( \gamma^o_R(G) = \max \{ \sum_{u \in V(G)} W(u) \} \), where the maximum is taken over all fractional (open neighborhood) Roman packing functions.

As an example, a fractional (open neighborhood) Roman dominating function \( f \) of the graph \( G \) is given in Figure 2(a) and a fractional (open neighborhood) Roman packing function \( p \) of \( G \) is given in Figure 2(b). Note that in Figure 2(a), the function values are ordered pairs which give stationary and traveling army sizes in the first and second coordinates, respectively, and in Figure 2(b), the function values are ordered pairs which
give generator and reducer sizes in the first and second coordinates, respectively.

![Figure 2: (a) Fractional (open neighborhood) Roman dominating and (b) fractional (open neighborhood) Roman packing functions of a Roman graph.](image)

In the graph $G$ above, the weight of $f$ and $p$ are both $\frac{30}{7}$. Thus, $f$ is a minimum fractional (open neighborhood) Roman dominating function of $G$ and $p$ is a maximum fractional (open neighborhood) Roman packing function of $G$. Furthermore, $\gamma_{R}^{f}(G) = \frac{30}{7} \approx 4.286$. Note that $G$ is a Roman graph with $\gamma_{R}(G) = 2\gamma(G) = 6$.

### 3 Closed neighborhood fractional Roman domination

In the previous section, traveling armies (or portions of) could not defend the location which they were stationed at, a situation which is ambiguous in the integer Roman domination problem. In the integer case, if there is a traveling army stationed at a location, then, there is necessarily a full stationary army stationed there as well. In the event of an attack on this location, the traveling army need not defend the home. We give the integer programming formulation of closed neighborhood Roman domination below. By the remark above, the value of (6) is equal to the value of (2); that is, if we replace $A$ with the closed neighborhood matrix $N = A + I$ in (2), the value of the IP is unchanged.
Minimize \[ \begin{bmatrix} 1^T & 1^T \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} \]

Subject to:
\[ \begin{bmatrix} I_n & N \\ I_n & -I_n \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} \geq \begin{bmatrix} 1 \\ 0 \end{bmatrix} \]
\[ v \in \{0, 1\} \text{ vector} \]

We can relax the condition that \( v \) be a \( \{0, 1\} \) vector, and instead require that the entries be non-negative. Then the integer program (6) becomes a linear program. The value of \( 1^Tv \) for any optimal solution of (7) is equal to the fractional (closed neighborhood) Roman domination number, \( \gamma_{RF}(G) \).

Minimize \[ \begin{bmatrix} 1^T & 1^T \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} \]

Subject to:
\[ \begin{bmatrix} I_n & N \\ I_n & -I_n \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} \geq \begin{bmatrix} 1 \\ 0 \end{bmatrix} \]
\[ v \geq 0 \] (7)

3.1 Closed neighborhood Roman packing

The circumstances for the dual LP also change when closed neighborhoods are used. So now, any vertex with a generator will emit radiation to all vertices in its closed neighborhood. The dual linear program of fractional (closed neighborhood) Roman domination (7) is given below, where \( \begin{bmatrix} 1 & 0 \end{bmatrix}^T \) is the \( 2n \times 1 \) matrix with the 1 as the first \( n \) entries and 0 as the next \( n \) entries.

Maximize \[ \begin{bmatrix} 1^T & 0^T \end{bmatrix} \begin{bmatrix} W \\ Z \end{bmatrix} \]

Subject to:
\[ \begin{bmatrix} I_n & I_n \\ N & -I_n \end{bmatrix} \begin{bmatrix} W \\ Z \end{bmatrix} \leq \begin{bmatrix} 1 \\ 1 \end{bmatrix} \]
\[ u \geq 0 \] (8)

The value of the above linear program is the fractional (closed neighborhood) Roman domination number. The integer program is then:
Maximize  \[ \begin{bmatrix} 1^T & 0^T \end{bmatrix} \begin{bmatrix} W \\ Z \end{bmatrix} \]

Subject to:
\[ \begin{bmatrix} I_n & I_n \\ N & -I_n \end{bmatrix} \begin{bmatrix} W \\ Z \end{bmatrix} \leq \begin{bmatrix} 1 \\ 1 \end{bmatrix} \]
\[ u \in \{0, 1\} \text{ vector} \quad (9) \]

The value of this integer program will be called the (closed neighborhood) Roman packing number, denoted by \( \rho_R \). Thus, we have \( \rho_R(G) \leq \gamma_{RF}(G) \leq \gamma_R(G) \), for all graphs \( G \).

As an example, a fractional (closed neighborhood) Roman dominating function \( f \) of the graph \( G \) is given in Figure 2(a) and a fractional (closed neighborhood) Roman packing function \( p \) of \( G \) is given in Figure 2(b). Note that in Figure 3(a), the function values are ordered pairs which give stationary and traveling army sizes in the first and second coordinates, respectively, and in Figure 3(b), the function values are ordered pairs which give generator and reducer sizes in the first and second coordinates, respectively.

Figure 3: (a) Fractional (closed neighborhood) Roman dominating and (b) fractional (closed neighborhood) Roman packing functions of a Roman graph.

In the graph \( G \) above, the weight of \( f \) and \( p \) are both \( \frac{24}{7} \). Thus, \( f \) is a minimum fractional (closed neighborhood) Roman dominating function of \( G \) and \( p \) is a maximum fractional (closed neighborhood) Roman packing function of \( G \). Furthermore, \( \gamma_{RF}(G) = \frac{24}{7} \approx 3.429 \).
4 Results

Our results are mainly for the open neighborhood version of fractional Roman domination, as the open version is a more natural analogue of the integer valued version of Roman domination. The following two propositions follow from linear programming duality and therefore, their proofs are omitted.

**Proposition 4.1.** For any graph $G$, $\rho_R(G) \leq \gamma_{RF}(G) \leq \gamma_{R\bar{f}}(G) \leq \gamma_R(G)$.

**Proposition 4.2.** For any graph $G$, $\bar{\rho}_R(G) \leq \gamma_{R\bar{f}}(G) \leq \gamma_R(G)$.

**Proposition 4.3.** For any graph $G$, $\gamma^o_{Rf}(G) \leq \gamma_{R\bar{f}}(G) \leq 2\gamma_{Rf}(G) \leq \gamma_R(G)$.

Further, these bounds are sharp.

**Proof.** To show the lower bound on $\gamma_{R\bar{f}}(G)$, let $f(v) = (s(v), t(v))$ be a minimum fractional (open neighborhood) Roman dominating function. Then for all $v \in V(G)$, we have $s(v) + \sum_{x \in N(v)} t(x) \geq 1$. Then define $g : V \rightarrow [0, 1]$ by $g(v) = \min\{1, t(v) + \sum_{x \in N(v)} \frac{s(x)}{d(x)}\}$. Then for any vertex $v \in V(G),

$$
\sum_{x \in N(v)} g(x) \geq \sum_{x \in N(v)} t(x) + \frac{s(v)}{d(v)}
= s(v) + \sum_{x \in N(v)} t(x)
\geq 1
$$

Hence, $g$ is a fractional total dominating function, and clearly $|g| \leq |f|$. Thus, we have $\gamma^o_{Rf}(G) \leq |g| \leq \gamma_{R\bar{f}}(G)$. To show the upper bound on $\gamma_{R\bar{f}}(G)$, let $g(v)$ be a minimum fractional dominating function. Then, $f(v) = (g(v), g(v))$ is a fractional (open neighborhood) Roman dominating function with weight $2\gamma_{Rf}(G)$. Therefore, $\gamma_{R\bar{f}}(G) \leq 2\gamma_{Rf}(G)$. The lower bound for $\gamma_R$ follows, since $\gamma_{R\bar{f}}(G)$ is a lower bound for $\gamma_R$. To see that these bounds are sharp, $\gamma^o_{Rf}(K_{1,1}, t) = \gamma_R(K_{1,1}) = 2$, for all integer $t \geq 1$. □

**Corollary 4.4.** For all graphs $G$, $\gamma_{Rf}(G) \leq \gamma_{R\bar{f}}(G) \leq 2\gamma_{Rf}(G)$.

In [2], it was shown that for any graph $G$ on $n$ vertices and maximum degree $\Delta$, $\gamma_R(G) \geq \left\lceil \frac{2n}{1+\Delta} \right\rceil$. This result was also stated as Proposition 5 in [1] (without the ceiling function, since $\gamma_R$ is integer valued).

**Proposition 4.5.** Let $G$ be a regular graph of degree $k$ on $n$ vertices, then $\gamma_{R\bar{f}}(G) = \frac{2n}{1+k}$. 

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Proof. Suppose $G$ is regular of degree $k$ on $n$ vertices. Let $f(v) = (s(v), t(v))$ with $s(v) = t(v) = \frac{1}{1+k}$ for all $v \in V(G)$. Then $s(v) + k \cdot t(v) = 1$ and $s(v) \geq t(v)$ for all $v \in V(G)$. Thus, $f(v) = (s(v), t(v))$ is a fractional (open neighborhood) Roman dominating function with weight $\frac{2n}{1+k}$. Let $p(v) = (g(v), r(v))$ with $g(v) = \frac{2}{1+k}$ and $r(v) = 1 - \frac{2}{1+k} = \frac{k}{1+k}$ for all $v \in V(G)$. Then for each $v \in V(G)$, the amount of harmful radiation is equal to $k \cdot \frac{2}{1+k} - \left(\frac{k}{1+k}\right) = 1$. Thus, $p$ is a fractional (open neighborhood) Roman packing function with weight $\frac{2n}{1+k}$. Therefore, $f$ is a minimum fractional (open neighborhood) Roman dominating function, and $\gamma^R_f(G) = \frac{2n}{1+k}$.

Proposition 4.6. Let $G$ be a regular graph of degree $k$ on $n$ vertices, then 
\[ \gamma^R_f(G) = \frac{2n}{2+k}. \]

Proof. Suppose $G$ is regular of degree $k$ on $n$ vertices. Let $f(v) = \left(\frac{1}{k+2}, \frac{1}{k+2}\right)$ and $p(v) = \left(\frac{2}{k+2}, \frac{k}{k+2}\right)$ for all $v \in V(G)$. Then $f$ and $p$ are fractional (closed neighborhood) Roman dominating and packing functions, respectively, with $|f| = |p| = \frac{2n}{2+k}$. Thus, $\gamma^R_f(G) = \frac{2n}{2+k}$. \qed

Note that for regular graphs $G$ of degree 2 (i.e. cycles), $\gamma^f(G) = \gamma^R_f(G) = \frac{n}{2}$. However, for regular graphs of degree at least 3, $\gamma^f(G) < \gamma^R_f(G)$.

4.1 Fractional Roman graphs

Recall that a graph for which $\gamma^R(G)$ attains the upper bound of $2\gamma(G)$ is called a Roman graph. We define call a graph for which $\gamma^R_f(G)$ attains the upper bound of $2\gamma_f(G)$ to be a fractional Roman graph (or fractionally Roman).

Proposition 4.7. Every $k$-regular graph $G$ is fractionally Roman.

Proof. Suppose $G$ is regular of degree $k$ on $n$ vertices. Then, $\gamma^R^\circ_f(G) = \frac{2n}{1+k} = 2\gamma_f(G)$.

A class of trees called spiders play a critical role in Roman domination. [1] and [4] define a healthy spider to be a star $K_{1,t}$ ($t \geq 2$) with all of its edges subdivided. For $t \geq 1$, a wounded spider is a star $K_{1,t}$ with at most $t - 1$ edges subdivided. In a wounded or healthy spider, a vertex of degree $t$ will be called the head vertex, and vertices distance two from the head will be called foot vertices. In a healthy spider the head and foot vertices are well defined. In a wounded spider, the head and foot vertices are well defined except when the wounded spider is $P_2$ or $P_4$. In the case of $P_4,$
fix one vertex $v$ of degree two to be the head vertex and then the vertex
distance two from $v$ will be the induced foot vertex. For $P_2$, we will consider
both vertices as head vertices.

In [1] it is shown that if $T$ is a tree on two or more vertices, then
$\gamma_R(T) = \gamma(T) + 1$ if and only if $T$ is a wounded spider. Also in [1] it is
shown that if $T$ is a healthy spider, $\gamma_R(T) = \gamma(T) + 2$.

![Figure 4: Wounded spiders (a)-(c) and a healthy spider (d)](image)

**Proposition 4.8.** If $T$ is a spider, then $T$ is not fractionally Roman, unless
$T$ is the healthy spider $P_5$, or the wounded spider $T = K_{1,t}$. If $T$ is the
healthy spider $P_5$, or the wounded spider $T = K_{1,t}$, then $T$ is fractionally
Roman. Further, for all spiders $T$, $\rho_R^\flat(T) = \gamma_R(T)$.

**Proof.** First, we prove the last claim. Let $T$ be a healthy spider. Let all
t foot vertices receive a generator, then pick two vertices adjacent to the
foot vertices and place a generator there. All other vertices, including the
head vertex, will receive a reducer. Then $\rho_R(G) = t + 2 = \gamma_R(G)$.

Now, let $T$ be the wounded spider $K_{1,t}$ with $0 \leq k < t$ edges subdivided.
If $T$ is a star, then let the head and one other vertex have a generator and
place reducers on all remaining vertices. Otherwise, let the $k$ foot vertices
receive generators. Then pick any two vertices adjacent to the head which
will both receive generators. All other vertices, including the head vertex,
will receive a reducer. Then $\rho_R(G) = k + 2 = \gamma_R(G)$.

Case (1a). Let $T$ be a healthy spider with $t \geq 3$. Then, $\gamma_f(T) = \gamma(T) = t$ and
$\gamma_R(T) = \gamma(T) + 2 = t + 2$. Thus, $2\gamma_f(T) = 2t > \gamma_R(T) = t + 2$.
However, $\gamma^\flat(R)(G) \leq \gamma_R(T)$, therefore, $T$ is not fractionally Roman. Case
(1b). Let $T$ be a wounded spider, which is not a star. Let $k$ be the number
of edges of $K_{1,t}$ which were subdivided to form $T$. Note that $1 \leq k < t$.
Then, $\gamma_f(T) = \gamma(T) = k + 1$ and $\gamma_R(T) = \gamma(T) + 1 = k + 2$. Thus,
$2\gamma_f(T) = 2k + 2 > \gamma_R(T) = k + 2$, therefore, $T$ is not fractionally Roman.

Case (2a). Let $T$ be the healthy spider $P_5$. Then $\gamma_R(P_5) = \rho_R^\flat(P_5) = 4$
and $\gamma_f(P_5) = 2$. Thus, $\gamma^\flat(R)(P_5) = 4 = 2\gamma_f(P_5)$ and $P_5$ is fractionally
Roman. Case (2b). Let $T$ be the wounded spider $T = K_{1,t}$. We have
$\gamma_R(T) = \rho_R^\flat(T) = 2$ and $\gamma_f(T) = 1$. Thus, $\gamma^\flat(R)(T) = 2 = 2\gamma_f(T)$ and $K_{1,t}$
is fractionally Roman. □
5 Open questions and conjectures

**Conjecture 5.1.** For any isolate-free graph $G$, $\gamma_t(G) \leq \rho_R^\varnothing(G)$.

**Conjecture 5.2.** For all trees $T$, $\gamma_R(T) = \rho_R^\varnothing(T)$.

Note that the above conjecture would imply that the (open neighborhood) fractional Roman domination number is equal to the Roman domination number for trees. A similar result holds for the domination number and the 2-packing number. If $G$ is a strongly chordal graph, then $\gamma(G) = \rho(G)$ (see [11] for a survey).

**Conjecture 5.3.** For a graph $G$, if every minimum fractional dominating function is a maximum fractional packing function, then $G$ is fractionally Roman.

For a study of graphs where every minimum fractional dominating function is a maximum fractional packing function, see [12]. We have seen several examples of fractionally Roman graphs which are not Roman.

**Open Problem 5.4.** Are there Roman graphs which are not fractionally Roman?

References


