Appendix

I really want to apologize for how long these notes have become. But to follow Lincoln, I really dont have the time to make them shorter. They are a blurting out of many important things. If I made them shorter (the ♫'s) I think the reader wouldn't ~~[illegible]~~ easily follow me through the transitions!

This appendix exists to answer the question, 'hey doctor S., how do I know, why should I believe that

$$\vec{J}\cdot\vec{B} = \frac{(\vec{J}\cdot\vec{F})(\vec{F}\cdot\vec{B})}{|F^2|} \; ? \qquad (1)$$

You <u>are</u> right to question this – it is not obviously true. At some point we'll run out of plausible 'picture reasoning' to help us. We'll need pure quantum reasoning. I feel the need to supply the answer, that it is my duty to do so. The reasoning is purely quantum thinking. It's starts with a commutator identity whose proof I've clamped eyes on but havent proved for myself [see ~~SCHIFF~~ sorry SLATER vol II, pp 369–371]. For any operator $\vec{A}$, and generic angular momentum operator $\vec{J}$, it can be proved that

(2) $$[J^2, [J^2, \vec{A}]] = 2\hbar^2 (J^2 \vec{A} + \vec{A} J^2) - 4\hbar^2 (\vec{A}\cdot\vec{J})\vec{J}$$

Mandl* uses this in connection with the spin-orbit effect. Maddeningly, he leaves stuff out and I cannot follow unless I attempt a "brute-force" calculation or two before

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* I like Mandl, "Quantum Mechanics", the Quantum text of the Manchesters Physics Series – he used to argue with Bell when they were both at CERN. Yes, THAT BELL

can't see the light' so to speak! So, I'll follow his treatment on a problem that even slow people such as I can see is a perfectly analogous problem: what are the Zeeman Splittings of the spin-orbit split states (for single electron atoms, such as the $^2P_{1/2}$ & $^2P_{3/2}$ states)??

Well, that's easy

$$\vec{\mu}_e = -g_e \mu_B \frac{\vec{S}}{\hbar} \quad 3a\,i)$$

$$\vec{\mu}_l = -g_l \mu_B \frac{\vec{L}}{\hbar} \quad 3a\,ii)$$

BUT $g_e = 2$, $g_l = 1$ 3a iii)

(3a) $$\hat{V}_Z = -\vec{\mu}_e \cdot B - \vec{\mu}_L \cdot B$$

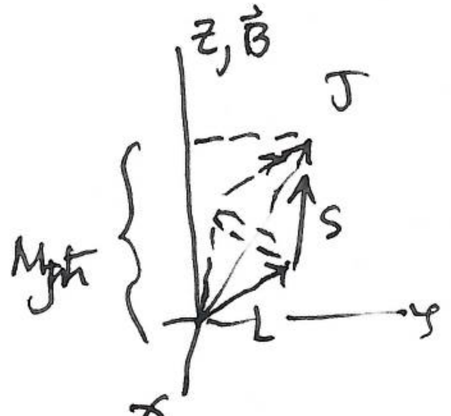
(3b) $$\approx +2\frac{\mu_B}{\hbar}\vec{S}\cdot\vec{B} + 1\frac{\mu_B}{\hbar}\vec{L}\cdot\vec{B}$$

(3c) $$= \frac{\mu_B}{\hbar}(\underbrace{\vec{L}+\vec{S}}_{\vec{J}} + \vec{S})\cdot\vec{B}$$

(3d) $$= \frac{\mu_B}{\hbar}(\vec{J}+\vec{S})\cdot\vec{B}$$

which is happy & sad at once. The $|JM_J\rangle$ eigenstates (the degenerate states of $^{2S+1}\mathcal{L}_J$ terms such as $^2P_{3/2, 1/2}$ ...

seem 'almost' perfect for the Zeeman perturbation. The problem is that $\vec{S}\cdot\vec{B} = S_z B$ and S_z or the M_s in the $|SM_s\rangle$ kets ARE NO LONGER GOOD QUANTUM #S! Why?



$\vec{J} = \vec{L} + \vec{S}$

That's because whereas J_z has a definite value, the projections of $\vec{S}$ & $\vec{L}$ along $\hat{z}$ no longer do. This 'vector model' is simply a classical crutch to lean on, what is really the case is that the $|JM\rangle$ kets ~~are~~ comprise or <u>are</u> a linear superposition of the $|LM_L\rangle$ & $|SM_s\rangle$ kets whose L&S sum to J and M_L & M_s sum to M_J after the fashion

(4) $$|JM_J\rangle = \sum_{M_L}\sum_{M_S} C^{L,S,J}_{M_L,M_S,M_J} |LM_L\rangle |SM_S\rangle$$

Clebsch-Gordon Coefficients!

and for any $|JM\rangle$ ket, that superposition has more than one bilinear term – it has a bunch. The point is that

(6) $$\hat{S}_z |JM\rangle \neq \text{something} * |JM\rangle,$$

~~but~~ not even a $|JM\rangle$ ket at all! So $\hat{S}_z$ is not possessed even of a physically meaningful quantum number at all!

But to assign a value of ~~#~~

(7) $$\hat{V}_z = \frac{\mu_B}{\hbar}(\vec{J}+\vec{S})\cdot\vec{B}$$

it seems we need to know what

(8) $$\langle JM|\, S_z \,|JM\rangle$$

is. Indeed we do. So back to 1 and let $\vec{A}=\vec{S}$. Then

(9) $$[J^2,[J^2,\vec{A}]] = 2\hbar^2(J^2\vec{A}-\vec{A}J^2) - 4\hbar^2(\vec{S}\cdot\vec{J})\vec{J},$$

and we aim to PROVE that

(10) $$\langle JM|S_z|JM\rangle = \hbar M_J \frac{\langle JM|\vec{S}\cdot\vec{J}|JM\rangle}{\hbar^2(J(J+1)}$$

so you can see the "plausibility" of saying

(11) $$\vec{S}\cdot\vec{B} = \frac{(\vec{S}\cdot\vec{J})\overbrace{(\vec{J}\cdot\vec{B})}^{\hbar M_J B}}{\underbrace{|J^2|}_{\hbar^2(J(J+1))}},$$

if one has THE PROOF in one's back pocket!

Hokay, here goes. Since $\vec{J} = \vec{L} + \vec{S}$

(12) $$\vec{S}\cdot\vec{J} = \vec{S}\cdot\vec{S} + \vec{L}\cdot\vec{S}$$

(13) but $$\frac{2\vec{L}\cdot\vec{S}}{2} = \frac{\vec{J}\cdot\vec{J} - \vec{L}\cdot\vec{L} - \vec{S}\cdot\vec{S}}{2}$$

(14) $\therefore$ $$\vec{S}\cdot\vec{J} = \frac{J^2 - L^2 - S^2}{2} + S^2 = \frac{J^2 + S^2 - L^2}{2}$$

So now

(15) $$\hat{V}_z = \frac{\mu_B B}{\hbar}\left(\hat{J}_z + \frac{(\hat{S}\cdot\hat{J})\hat{J}_z}{\hbar^2(J(J+1))}\right) = \mu_B B \overbrace{\left(1 + \frac{J^2 + S^2 - L^2}{2(J(J+1))}\right)}^{g_J!}\frac{\hat{J}_z}{\hbar}$$

(16) $$\equiv g_J \mu_B B \frac{\hat{J}_z}{\hbar},$$

and $\hat{V}_z$ is diagonal in $|JM\rangle$ basis after all despite the scare!

That is if we "believe" the PROOF.

Do you believe? I do not.

So, down we go into the belly of the whale, whether you're Ismael, Jonah, or just a physics student like me trying to understand stuff for himself.....

Let us try it out in the $|JM\rangle$ basis, and let J be J :) What is

(17) $\langle JM | S_z | JM \rangle$?

we have

(18) $[J^2, [J^2, \vec{A}]]$

which is code for

Look on!

(19) $$\langle JM| \Big\{ \left[J^2 (J^2 \vec{A} + \vec{A} J^2) - (J^2 \vec{A} + \vec{A} J^2) J^2 \right] - \left[(J^2 \vec{A} + \vec{A} J^2) J^2 - J^2 (J^2 \vec{A} + \vec{A} J^2) \right] \Big\} |JM\rangle$$

and which becomes

(20) $$2\langle JM|\left\{ \underset{①}{J^2(J^2\vec{A}+\vec{A}J^2)} - \underset{②}{(J^2\vec{A}+\vec{A}J^2)J^2} \right\}|JM\rangle \,!$$

Now it's fair-dinkum to use the left bra to operate on the J^2 operator on the left, and the same on the right, hence

(21) $$2\langle JM|\;① = 2\hbar^2(J(J+1)\langle JM|\,(J^2\vec{A}+\vec{A}J^2) - \ldots$$

(22) $$\&\quad ②\,|JM\rangle = 2\hbar^2(J(J+1)\left\{(J^2\vec{A}+\vec{A}J^2)|JM\rangle\right. !$$

So (20) becomes

$$4\hbar^2(J(J+1)\langle JM|\,(J^2\vec{A}+\vec{A}J^2) - (J^2\vec{A}+\vec{A}J^2)|JM$$

$$= 0 \,!!$$

For any $\vec{A}$! WOW! So (2), the theorem becomes, for $|JM\rangle$ basis

$$(J^2\vec{A}+\vec{A}J^2) = 2(\vec{A}\cdot\vec{J})\vec{J}$$

or

$$(J^2\vec{S}+\vec{S}J^2) = 2(\vec{S}\cdot\vec{J})\vec{J}$$

our theorem (2) becomes

$$\langle JM|(\vec{S}\cdot\vec{J})\vec{J}\,|JM\rangle = \frac{1}{2}\langle JM|\left(J^2\vec{S} + \vec{S}J^2\right)|JM\rangle$$

or, nibling on the edges

$$\cancel{\langle JM|\vec{S}\cdot\vec{J}|} \quad = \frac{1}{2}\left(J(J+1)\hbar^2\langle JM|\,2\vec{S}\,|JM\rangle\right)$$

It will not make any sense to proceed unless we take the inner product of the vector momenta ~~of~~ with a unit vector to assign say the $\hat{z}$ direction ($\hat{z} = \frac{\vec{B}}{B}$ maybe?)

then

$$\langle JM|(\vec{S}\cdot\vec{J})\underbrace{\left(\vec{J}\cdot\frac{\vec{B}}{B}\right)}_{\hat{J}_z}|JM\rangle = J(J+1)\hbar^2\langle JM|S_z|JM\rangle$$

and then

$$\boxed{\frac{m\hbar}{J(J+1)\hbar^2}\langle JM|\vec{S}\cdot\vec{J}|JM\rangle = \langle JM|S_z|JM\rangle}$$

I quit - I need to put in another load of laundry and eat some dinner ......