A note on the plasma sheath and the Bohm criterion

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The Bohm criterion is an inequality signifying that the ion flow speed at the plasma boundary must be at least as great as the ion sound speed in order for a sheath to form at the boundary. A physical explanation for this phenomenon is given, and the phenomenon is compared with the flow of falling water.
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The word "sheath" in connection with plasma, the fourth state of matter, was coined by Irving Langmuir[1] to describe the thin region of strong electric fields formed by space charge that joins the body of the plasma to its material boundary. The plasma boundary might be the metal wall of a vacuum chamber confining a laboratory plasma, the wafer of silicon sitting on an electrode immersed in a plasma etching reactor, the glass tube surrounding the plasma discharge of a neon sign, or even the condenser plates in J. J. Thomson's (insufficiently exhausted) gas filled cathode ray tube. The electrostatic potential of the boundary joins smoothly to the interior plasma potential, and nearly all of the voltage difference occurs in a very thin region called the sheath, an effect called Debye shielding or Debye screening.[2] The Debye length[3] is the characteristic scale length of the sheath which is usually much smaller than the length of the plasma itself. Like any good electrical conductor, the plasma shields external static electric fields from its interior. The plasma, a dynamic collection of electrons, ions, and neutral atoms and molecules, is characterized by weak electric fields, charge neutrality, and high temperatures (2–10 times hotter than the surface of the sun for the plasmas in the examples we will discuss). It reacts to externally applied potentials in a complicated way. For reasons that are not simple to explain, shielding cannot be perfect because of thermal effects. The formation of the sheath requires that plasma ions be accelerated up to speeds equal to or greater than Mach 1 at the sheath edge, an inequality referred to as the Bohm criterion[4] for sheath formation, typically expressed as

$$v_i \geqslant C_s = (kT_e/M_i)^{1/2}, \qquad (1)$$

which holds if ion-neutral collisions are sufficiently rare. Here M_i is the ion mass, kT_e is the electron thermal energy, and C_s is the ion sound speed. It is noteworthy that the ion sound speed depends principally on the electron rather than the ion temperature for $T_e \gg T_i$, a situation that is often true in plasmas. It is typically not true that the ion thermal speed is also the ion sound speed. But it is not to this curious aspect of plasma physics that we want to draw the reader's attention. The purpose of this note is to examine with simple fluid theory, why the ions have to go so fast at the sheath edge in order for the sheath to form.

The sheath region begins where charge neutrality begins to breaks down. The electric potential gradient becomes very steep abruptly on the boundary side of the sheath edge, and because of the curvature of the potential, electrons are repelled from and ions are accelerated to the boundary. This state of affairs is pictured in Fig. 1. For simplicity, we imagine that all physical variables (for example, the electrostatic potential, plasma density, and the ion velocity) vary only in a direction normal to the (planar) boundary. The potential profile forms self-consistently, at once influenced by and leading to charged flows. From a kinetic theory point of view, sheath formation depends on the distribution functions of the charged species, and on ion-neutral collisions. But whether we are looking at the plasma microscopically or as a fluid, the questions of interest are how is it that the breakdown of charge neutrality at the sheath edge implies that the ion drift speed must become at least as great as the ion sound speed, C_s, at the sheath edge? How is it that if the ion density gradient is less than the electron density gradient,

$$\frac{dn_i}{dx} < \frac{dn_e}{dx}, \qquad (2)$$

then the ion drift speed has to be greater than the ion sound speed, $v_i > C_s$? Typical demonstrations of this result involve solving complicated transcendental, nonlinear differential equations,[2,5] which arise from Poisson's equation and the Boltzmann distribution which characterizes the electrons. This approach is necessary in order to solve for the electrostatic potential in the vicinity of the sheath edge on the plasma side. Extending these solutions into the sheath itself involves the method of matched asymptotic expansions[6] or numerical methods. If our goal is to understand the Bohm criterion and how it arises at the sheath edge, then we can obtain insight without having to solve the complicated equations. What follows is a heuristic sketch of how to do so.

We can take the sheath criterion, Eq. (2), as the condition that charge neutrality breaks down and space charge appears at the location where the first order changes in the charge concentration no longer cancel. Within the body of the plasma, these changes do cancel. The equation of ion continuity requires that the changes in the ion concentration and

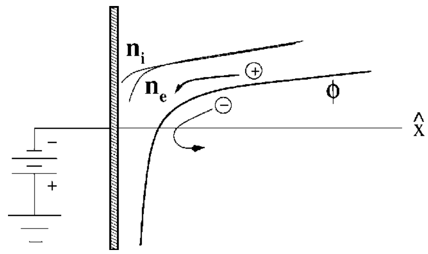

Fig. 1. Schematic of the plasma bounded by a negatively biased boundary wall. Ions flow to the wall down the potential hill $\phi(x)$, while electrons are repelled. Net space charge appears at the sheath edge, where the gradients in the ion density and electron density diverge.

the ion flows are related to the sources and sinks of ion charge,

$$\frac{\partial n}{\partial t}+\frac{d(nv)}{dx}=S_+-S_-, \tag{3}$$

where the sources and sinks can be set to zero if we concern ourselves with a region of no more than a few Debye lengths from the sheath edge where Eq. (2) begins to be satisfied. For stationary flow, it is not difficult to rearrange the continuity equation (3) into the form

$$\frac{1}{n_i}\frac{dn_i}{dx}=-\frac{1}{v_i}\frac{dv_i}{dx}, \tag{4}$$

which is convenient for comparing the scale lengths over which the density and velocity change. Each side has units of a reciprocal length and defines the characteristic length over which each quantity changes by some fractional amount. Equation (4) implies that these two scale lengths are the same. The fractional changes in the ion density and speed occur over distances just as short, although in opposite senses—the density is increasing toward the sheath edge and the drift speed is decreasing as we follow the ions up the potential gradient, past the sheath edge into the bulk plasma. It is more fun to take a ride with the ions in the other direction. The ions pick up speed as the electric field does work on them. But why should this speed at the sheath edge have to be sufficiently high for the spatial gradients in the ion and electron population to begin to differ? The answer is still not obvious. But the sheath criterion yields a second inequality in connection with Eq. (4). Given Eq. (2), the continuity equation says that if the ion density gradient has to be less than the electron density gradient, it follows that

$$-\frac{n_i}{v_i}\frac{dv_i}{dx}<\frac{dn_e}{dx} \tag{5}$$

at the sheath edge. We might want to reverse the inequality because of the minus sign, but because the ion speed is negative here, we can leave it as it is.

We have arrived at our principal result. The velocity gradient has a positive value (because the velocity is becoming less negative, its derivative is positive definite), and the density is positive. The only way for the left-hand side of Eq. (5) to be less than the electron density gradient is for the magnitude $|v_i|$ to satisfy

$$-v_i>n_i\frac{dv_i/dx}{dn_e/dx}. \tag{6}$$

Now we have to show that the right-hand side of the inequality (6) is equal to the ion sound speed. First let's consider the spatial density gradient of the electrons. The electrons are energetically confined by the potential and thus are in electrostatic equilibrium, and so they are describable in terms of the Boltzmann distribution,

$$n_e=n_0e^{e\phi/kT_e}, \tag{7}$$

where n_0 is the electron density in the bulk plasma far from the sheath edge. The exponential factor looks a bit odd without the minus sign in the exponent. But it gets the probability right for the negative charge: the electron density is greatest where the electron potential energy is least, and vice versa, as required in thermal equilibrium. The important point is that the electron density gradient depends on the potential gradient through the Boltzmann relation,

$$\frac{dn_e}{dx}=\frac{e}{kT_e}\frac{d\phi}{dx}e^{e\phi/kT_e}=\frac{n_ee}{kT_e}\frac{d\phi}{dx}. \tag{8}$$

The spatial gradient of the velocity at the sheath edge also depends on the potential gradient because of conservation of energy, which for the ions may be written as

$$\frac{d(U+K)}{dx}=0, \tag{9}$$

where $U=e\phi$ and $K=\frac{1}{2}Mv_i^2$. Hence we write

$$e\frac{d\phi}{dx}+Mv_i\frac{dv_i}{dx}=0. \tag{10}$$

When we substitute our expressions for the ion velocity and electron density gradients back into the inequality for the sheath criterion (Eq. (5)), we see that the potential gradient (or electric field strength) cancels, leaving only constants. If we remember that at the sheath edge, the electron and ion densities are still almost equal, the inequality can be written as

$$-v_i>-n_i\frac{e(d\phi/dx)kT_e}{en_eMv_i(d\phi/dx)}=-\frac{kT_e}{Mv_i}, \tag{11}$$

or

$$\sqrt{v_i^2}=\sqrt{\frac{kT_e}{M}}, \tag{12}$$

which is what we wished to demonstrate.

Is there a simple picture that we can take from experience to predict this result? The result of this heuristic sketch is that the continuity equation leads to the conclusion that the ion density gradient can only be less than some value (the electron density gradient) if the ion velocity is sufficiently high (higher than Mach 1). We can see it algebraically, but can we see it based on our experience of nature? I will argue that the answer is no, but I will give a physical picture anyway.

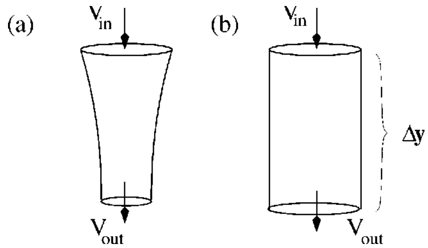

Fig. 2. (a) Incompressible flow, say water falling from a faucet, narrows as it falls a distance Δy. Because $v_{in} > v_{out}$, the only way the amount of water within the geometric volume bounded by the input and output surfaces can stay constant is if the lower output surface contracts (flux in = flux out). (b) If the flow is compressible, and if the density diminishes as the fluid falls, and if the fractional change in density is identical to the fractional change in the fluid flow velocity, then the input and output surface area stays the same, and the cross sectional area of the column remains constant.

All our intuition and experience in the flow of fluids comes from playing with water. The problem is that it is incompressible. When we open a faucet, the column of water narrows as it falls. Why? Because the number of water molecules is conserved, the divergence of the flow is zero. But the density is also constant, so as the water falls down a gravitational potential gradient and picks up speed, the only way to keep constant the number of particles entering and leaving a portion of the water column is for the cross sectional area of the downstream end to be smaller than the upstream end. In a plasma, the density is not necessarily constant, and the flow is not incompressible. The cross sectional area of a column of falling water that behaves as the plasma ion fluid does while it is falling down an electrostatic potential gradient into the sheath would stay constant as it fell, as shown in Fig. 2 or even spread out. Water does not behave that way when falling. Common experience with fluids does not help us with the Bohm criterion.

So why, if the drop in the density for a given vertical displacement is less than some critical value, must the magnitude of the ion velocity be greater than a critical value? The reason is that the ion density and velocity gradient scale lengths are the same. The Bohm criterion inequality may be understood as follows. The spatial amount by which the speed changes depends only on the arbitrary choice of the vertical displacement, Δy. The change arises on account of the work done by gravity, say, or rather an electric field, over a given Δy. As the velocity rises, The fractional change in v is reduced. To reduce the fractional change in the velocity below some critical value (because the fractional ion density gradient must be less than a critical value), the ion flow speed at that location must exceed a critical value, which turns out to be the ion sound speed. The potential difference between the potential in the bulk of the plasma and at the sheath rises to a value high enough for the ions to attain this speed, a potential structure called the presheath.[7]

What have we achieved? Certainly many thorny issues have been overlooked and surprises await when we apply the Bohm criterion to plasmas with more than one ion species. It has been discovered recently that the ions in this case do not reach the sheath edge with at their single ion Bohm speeds.[8] Perhaps we have achieved some insight into the Bohm criterion, if only for the case of plasmas with a single ion species. But I think that there is more. I claim that it is not possible to use our experience with the motion of fluids to "see" the result at a glance, and that all our insight in this case is derived from or suggested by the equations themselves. The equations must tutor our intuition rather than the other way around. Nor should we expect it to be always otherwise.

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[3]The notation for the Debye length is usually given as $\lambda_D = (\epsilon_o kT_e / ne^2)^{1/2}$, where n is the plasma density and kT_e is the electron temperature in energy units. For relatively cool plasmas such as at the Sun's surface, $n \sim 10^{14}$ cm^{-3}, $kT_e \sim 1$ eV, and $\lambda_D \sim 10^{-4}$ cm, which is much smaller than the thickness of this layer, the photosphere, which is about 0.5×10^8 cm thick. A good website for gaining some perspective about the plasma state of matter is ⟨http://www.plasmas.org⟩.

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