480W Notes on perturbation theory

Do Pressure Fluctuations Propagate Through Air?

Fall 2017

Let's discuss a complete set of relations between velocity, density, and pressure fluctuations in a fluid. From the outset let's look steadily at our goal: a self consistent set of 3 equations governing velocity, density, and pressure fluctuations in a neutral gas. This is what we want. So, we'll regard the fluctuations as first order perturbation to the equilibrium quantities, the relations between which we suppose we already know, even know them by name: Newton's second law for fluids, the equation of continuity, and an equation of state (adiabatic):

$$\rho\left\{\frac{\partial \mathbf{v}}{\partial t}+(\mathbf{v}\cdot\nabla)\mathbf{v}\right\} = -\nabla p \qquad Newton's\ Second\ Law\ for\ Fluids \tag{1}$$

$$\frac{\partial \rho}{\partial t}+\nabla\cdot(\rho\mathbf{v}) = 0 \qquad Equation\ of\ Continuity \tag{2}$$

$$pV^{\gamma} = Const. \qquad Equation\ of\ \text{``}State''. \tag{3}$$

Beside describing their equilibria (can we agree that we won't treat the case of equilibrium flow, just to make life easy?), these equations tell the fate of small fluctuations in pressure, density, and velocity about their equilibrium values. One can see this by first making the following approximation for each of ρ, p, and $\mathbf{v}$,

$$p \approx p_o + p_1, \tag{4}$$

where we suppose that p_1/p_o, the ratio of the amplitude of the pressure fluctuations to atmospheric pressure, is $\ll 1$. We further suppose that that each of these first order quantities do in fact propagate as

$$p_1(x,t) = p_1 e^{i(kx-\omega t)}, \tag{5}$$

and so on for the the other variables, where for simplicity we have assumed a plane wave which propagates in the the $\hat{\mathbf{x}}$ direction (i.e., $\mathbf{v_1} \rightarrow v_1$).

It is important to note that we are looking for wave propagation in the same direction that the motion of the medium occurs—that is, we are looking for *longitudinal waves*.

The game we are here playing is to use the system of equations above to arrive at a new system of equations which the perturbed, first order quantities are bound to satisfy, supposing that there is a simple wave solution. If we can solve such a system of equations, than we will have arrived at what is called a "dispersion relation" for the wave, which is simply a function $\omega(k)$, which describes how the frequency depends on the wavelength of the perturbations. Oh and, we will have justified our assumptions, or at least arrived at a definite answer which may be put to the test.

Let's see how it works out for Newton's Second Law. One writes

$$(\rho_o + \rho_1)\left(\frac{\partial v_1}{\partial t} + (v_1 \frac{dv_1}{dx})\right) = -\frac{d}{dx}(p_o + p_1), \tag{6}$$

and throws out all the terms that are quadratic in small quantities. Such extermination leaves out everything but

$$\rho_o \frac{\partial v_1}{\partial t} = -\frac{dp_1}{dx}, \tag{7}$$

where it has been assumed that there is no equilibrium bulk motion of the air ($\mathbf{v_o} = 0$, and that there are no spatial gradients in any equilibrium quantity. This ruthlessness is called "linearization" for obvious reasons. Recalling that perturbed quantities vary in space and time like plane waves, this equation becomes

$$\rho_o i \omega v_1 = ikp_1. \tag{8}$$

In just the same way, the equations of continuity and equation of state yield

$$\rho_1 = \rho_o k v_1/\omega, \quad \texttt{(is this even possible?)} \tag{9}$$

$$\frac{p_1}{p_0} = \gamma \frac{\rho_1}{\rho_0}. \tag{10}$$

These three equations in the three first order quantities can be solved; for example, solving the perturbed density, we arrive at

$$\left(\frac{\omega^2}{k^2} - \gamma \frac{p_0}{\rho_0}\right)\rho_1 = 0, \tag{11}$$

which implies that

$$v_\phi = \frac{\omega}{k} = \sqrt{\gamma \frac{p_0}{\rho_0}}. \tag{12}$$

Our guess was justified, and our reward: dispersionless plane waves.