480W Notes on perturbation theory in the context of the physics of fluids

Do Pressure Fluctuations Propagate Through Air?*

Tutorial Questions

Let's discuss a complete set of relations between velocity, density, and pressure fluctuations in a fluid. At length we want to understand wave propagation in a plasma, a charged fluid with equal amount of positive and negative charge (ions and electrons) globally, hot enough not to simply recombine, exhibiting collective effects. For now, let's focus simpler neutral atomic or molecular fluids, trying to understanding propagating waves arising from a self consistent set of 3 equations governing velocity, density, and pressure fluctuations in a neutral gas. We'll regard the fluctuations as first order perturbation to the equilibrium quantities, the relations between which we suppose we already know, even know them by name: Newton's second law for fluids, the equation of continuity, and an equation of state (adiabatic):

$$\rho\left\{\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v}\right\} = -\nabla p \quad \textit{Newton's Second Law for Fluids} \tag{1}$$

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\mathbf{v}) = 0 \quad \textit{Equation of Continuity} \tag{2}$$

$$pV^{\gamma} = Const. \quad \textit{Equation of ``State''}. \tag{3}$$

Beside describing their equilibria (can we agree that we won't treat the case of equilibrium flow, just to make life easy?), these equations tell the fate of small fluctuations in pressure, density, and velocity about their equilibrium values. One can see this by first making the following approximation for each of ρ, p, and $\mathbf{v}$,

$$p \approx p_o + p_1, \tag{4}$$

where we suppose that p_1/p_o, the ratio of the amplitude of the pressure fluctuations to atmospheric pressure, is $\ll 1$. We further suppose that that each of these first order quantities do in fact propagate as

$$p_1(x,t) = p_1 e^{i(kx-\omega t)}, \tag{5}$$

and so on for the the other variables, where for simplicity we have assumed a plane wave which propagates in the the $\hat{\mathbf{x}}$ direction (i.e., $\mathbf{v_1} \to v_1$).

It is important to note that we are looking for wave propagation in the same direction that the motion of the medium occurs—that is, we are looking for *longitudinal waves*.

The game we are here playing is to use the system of equations above to arrive at a new system of equations which the perturbed, first order quantities are bound to satisfy, supposing that there is a simple wave solution. If we can solve such a system of equations, than we will have arrived at what is called a "dispersion relation" for the wave, which is simply a function $\omega(k)$, which

*the oddside margin has been made very small leaving a space on the right hand side for pursuing algebra calculations...

describes how the frequency depends on the wavelength of the perturbations. Oh and, we will have justified our assumptions, or at least arrived at a definite answer which may be put to the test.

Let's see how it works out for Newton's Second Law. One writes

$$(\rho_o + \rho_1)\left(\frac{\partial v_1}{\partial t} + (v_1\frac{dv_1}{dx})\right) = -\frac{d}{dx}(p_o + p_1), \tag{6}$$

and throws out all the terms that are quadratic in small quantities. Such extermination leaves out everything but

$$\rho_o\frac{\partial v_1}{\partial t} = -\frac{dp_1}{dx}, \tag{7}$$

where it has been assumed that there is no equilibrium bulk motion of the air, i.e., $\mathbf{v_o} = 0$, and that there are no spatial gradients in any equilibrium quantity. This ruthlessness is called "linearization" for obvious reasons. Recalling that perturbed quantities vary in space and time like plane waves, this equation becomes

$$\rho_o i\omega v_1 = ikp_1. \tag{8}$$

tutorial question #1 In just the same way, show that the equations of continuity (eq.(2)) and equation of state (eq. (3)) yield

$$\begin{aligned}\rho_1 &= \rho_o k v_1/\omega, \quad \texttt{(is this even possible?)} \quad (9)\\ \frac{p_1}{p_0} &= \gamma\frac{\rho_1}{\rho_0}. \quad (10)\end{aligned}$$

tutorial question #2 Next, because we now had three interconnected in terms of the three first order quantities, we can solve for all three, or any one of them. Solve for the perturbed density, arriving at

$$\left(\frac{\omega^2}{k^2} - \gamma\frac{p_0}{\rho_0}\right)\rho_1 = 0. \tag{11}$$

How can we interpret this result to imply that

$$v_\phi = \frac{\omega}{k} = \sqrt{\gamma\frac{p_0}{\rho_0}}? \tag{12}$$

Was our initial guess or assumption of plane waves justified?