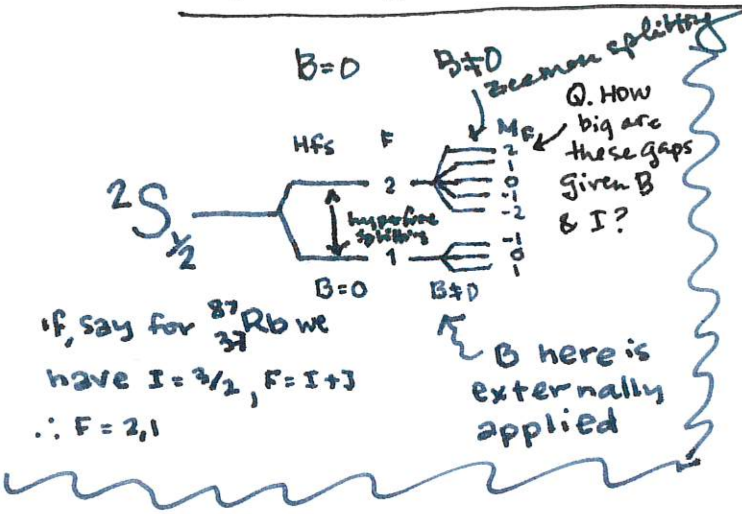
Notes on Zeeman Splitting of a Hyperfine split state



There are 3 experiments in PHYS 480W that involve nuclear quantum #'s; energy gaps arise on account of different orientations of the nuclear magnetic moment $\vec{\mu}_I$ that may be exploited, whether to make possible nuclear magnetic resonance on the one hand or optical pumping on the other. Lasers, atomic clocks, not to mention ~~an enormous wealth of chemical information~~ made possible by 'NMR-Spectroscopy' of all sorts & elaborations (MRI, fMRI, etc) are all 'killer apps' of this area of physics. ~~But~~ However awesome the applications, we as physicists are ~~[illegible]~~ also interested in the inner workings of nature itself (the real reason why we have all these other things). To anthropomorphicise it, we think nature-herself is a marvel and a mystery, fascinating down to depths not yet fathomed. When we turn to the physics of molecules, atoms, and nuclei, as we do here, we ~~[illegible]~~ use quantum mechanics as our guide. Inward bound on yet another fascinating journey, in this note we ask about the physics of Zeeman splittings of hyperfine split states.

Hyperfine states are so called since they are smaller, exquisitely smaller than fine structure splittings arising from the so called 'spin-orbit' effect, $\hat{V}_{so} = A\hat{S}\cdot\hat{L}$, !

where $\hat{S}$ & $\hat{L}$ are angular momentum operators and where $\hat{V}_{so}$ is a perturbing Hamiltonian. Hyperfine splittings come from $\hat{V} = A\hat{I}\cdot\hat{J}$, where the nuclear spin $\vec{I}$ and its operator $\hat{I}$ play the same role as the electron spin in the spin-orbit effect. While the quantum numbers, I, for nuclear spin are just as big or bigger for nuclei ($I = 3/2$ for ${}^{87}_{37}Rb$, for example, $I = 1/2$ for the proton) but their magnetic moments are 'way smaller', $\mu_B \gg \mu_N = e\hbar/M_N$ ....

We want to understand the size of the splittings made by the Zeeman splittings given that the hyperfine split states ($F = 2, 1$ for ${}^{87}_{37}Rb$ nucleus) are degenerate.

The way Quantum Theory handles ~~degenerate~~ perturbation to degenerate states is to identify the perturbing Hamiltonian and to ask how the degenerate states combine to become eigenstates of this new Hamiltonian, and to find the eigenvalues, wh are the splittings we are trying to find. The perturbing Hamiltonians are

$$\hat{V}_z = \underbrace{-\vec{\mu}_J \cdot \vec{B}}_{\text{electron magnet energy}} \underbrace{- \vec{\mu}_I \cdot \vec{B}}_{\text{nuclear magnet energy}}, \qquad (1)$$

but since $\mu_I << \mu_J$ we can ignore the 2nd term and try to solve for the e.f.'s of the 1st term

$$\hat{V}_Z \approx -\vec{\mu}_J \cdot \vec{B} = g_J \mu_B \frac{\vec{J}\cdot\vec{B}}{\hbar} \qquad (2)$$

we invoke the appropriate g-factor, $\mu_j = -g_J \mu_B \frac{\vec{J}}{\hbar}$ after $\mu_e = -g_e \mu_B \frac{\vec{S}}{\hbar}$ $(g_e = 2)$ $\mu_L = -g_L \mu_B \frac{\vec{L}}{\hbar}$ $(g_\ell = 1)$

If only $\vec{J}\cdot\vec{B}$ had a definite value!

But then the manual (2B-7) writes

$$\hat{V}_z = g_F \mu_B B \frac{\hat{F}_z}{\hbar}, \qquad (3)$$

rather precipitously; curious because g_F depends on the nuclear quantum #'s (F, I) as does M_F, and didn't we pitch that term?

Hokay.... Let's see how this (3) happens...

from (2)

$$\hat{V}_z = g_J \mu_B \frac{J_z B}{\hbar} \quad \ldots\ldots \quad \text{but } \hat{J}_z$$

why, we ask

has no definite value along $\hat{z}$ (?) M_J is NOT A GOOD QUANTUM # (!) what does this mean? It means that a quantum # is 'good' if it has a definite value. Why doesnt it?

give me an example of a physical observable not possessing a definite value

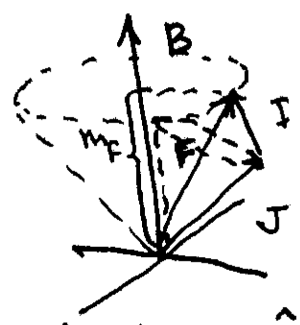
Because for very weak magnetic fields $\vec{I}$ couples* with $\vec{J}$ to give a total angular momentum $\vec{F}$,

$$\vec{F} = \vec{I} + \vec{J}. \tag{4}$$

why say stuff like th? Is this nothing more than e.g. magnet interacting more strongly with the magnet like μ_I more than each responds to the external B? Hold on though

It follows that the $|F M_F\rangle$ states, which are the degenerate states of the hyperfine effect, do have a definite projection

NB: $\hat{J}^2$ & $\hat{I}^2$ ARE GOOD Quantum operators & J&I are good Q#'s IN CONSEQUENCE!! $\therefore |F M_F\rangle = |J I F M_F\rangle$

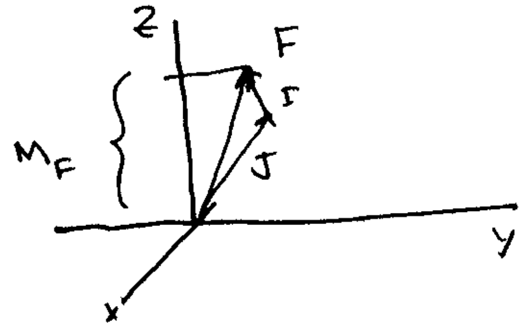


In this pictoral 'vector model' J & I take on whatever orientations needed to create an allowed $\vec{F}$. The result is that the projections along z (of I & J) take on a range of values, not a definite one (or single one)

along $\vec{B}$ (i.e. $\hat{F}_z |F M_F\rangle = M_F \hbar |F M_F\rangle$), but $\vec{J}$ does not, nor does $\vec{I}$, but their valid combinations do! So let's calculate them.

I'll just be too lazy to write this out

THE SHORT ANSWER: INSTEAD OF CALCULATING $\vec{J}\cdot\vec{B}$, CALCULATE



$$\frac{(\vec{J}\cdot\vec{F})(\vec{F}\cdot\vec{B})}{|F^2|}$$ instead. Same thing right?

We attempt to find the projection of $\vec{J}$ along $\vec{F}$ because we may expect this to have a definite value, and we want to see how the nuclear quantum # (F, I) come into the theory of the Zeeman spacings.

The projection of $\vec{J}$ along $\vec{F}$ can be seen by thinking of $\vec{J}$ in terms of ∥ & ⊥ components relative to $\vec{F}$,

$$\vec{J} = J_{\parallel}\frac{\vec{F}}{|\vec{F}|} + \vec{J}_{\perp}\underbrace{\left(\frac{\vec{F}\times\vec{J}}{|\vec{F}||\vec{J}|}\right)}_{\text{expect } \langle\vec{J}_{\perp}\rangle = 0!} \qquad (5)$$

What is $J_{\parallel}$?

$J_{\parallel} = \frac{J\cdot\vec{F}}{|\vec{F}|}$, so the Hamiltonian may be written

$\hat{z}$

$$\hat{V}_z = +g_J\frac{\mu_B}{\hbar}\frac{\vec{J}\cdot\vec{F}}{|\vec{F}|}\frac{\vec{F}\cdot\vec{B}}{|\vec{F}|} = +g_J\frac{\mu_B B}{\hbar}\frac{\vec{J}\cdot\vec{F}}{|F^2|} \qquad (6)$$

which is handy, since the $|F M_F\rangle$ kets are automatically the eigenstates of $\underline{\hat{V}}_z$ with e.v.'s

$$= +g_F\,\mu_B B M_F$$

clearly greg, this needs some elaboration where now we must show, or understand how theory predicts that

$$g_F = g_J \left[\frac{F\cdot(F+1) - I(I+1) + J(J+1)}{2F\cdot(F+1)} \right], \quad (7)$$

something the manual supplies without comment.

So let's go back to (6) and $\hat{V}_z$. What is $\vec{J}\cdot\vec{F}$ in the $|FM_F\rangle$ basis?

$$\left\{ \begin{array}{l} \vec{F} = \vec{J} + \vec{I} \\ \vec{F}\cdot\vec{F} = \vec{J}\cdot\vec{J} + \vec{I}\cdot\vec{I} + 2\vec{J}\cdot\vec{I} \end{array} \right.$$

(8) $\vec{F} = \vec{J} + \vec{I}$

(9) $\therefore \quad \vec{J}\cdot\vec{F} = \vec{J}\cdot\vec{J} + \vec{J}\cdot\vec{I}$

(10) $\therefore \quad \vec{J}\cdot\vec{F} = \vec{J}\cdot\vec{J} + \left[\dfrac{F^2 - J^2 - I^2}{2} \right]$

(11) $= \dfrac{2}{2}J^2 + \dfrac{F^2}{2} - \dfrac{J^2}{2} - \dfrac{I^2}{2}$

(12) $= \dfrac{F^2 + J^2 - I^2}{2}$

$$\left\{ \begin{array}{l} I^2 = \vec{I}\cdot\vec{I} \\ \& \ \vec{F}\cdot\vec{F} = F^2; \\ \hat{F}^2 |FM_F\rangle = \\ \quad F(F+1)\hbar^2 |FM_F\rangle \\ \therefore \ |\vec{F}|^2 = F(F+1)\hbar^2 \\ \quad \therefore \end{array} \right.$$

So, in the $|FM_F\rangle$ basis

(13) $\underline{\underline{\hat{V}}}_z =$

	$\vert FM_F\rangle$	$\vert F M_F-1\rangle$	$\vert F M_F-2\rangle$	$\cdots$	$\vert F -F\rangle$
$\langle FM_F\vert$	$V_{z_{11}}$	$V_{z_{12}} \to 0$			
$\langle F M_F-1\vert$	$V_{z_{21}} \to 0$	$V_{z_{22}}$			
$\langle F M_F-2\vert$			$V_{z_{33}}$		
$\vdots$				$\ddots$	
$\langle F -F\vert$					

matrix elements calculated

Along the diagonal we have

$$V_{z_{ii}} = \langle F M_F | \left\{ +g_J \frac{\mu_B B}{\hbar} \left\{ \frac{\hat{F}^2 + \hat{J}^2 - \hat{I}^2}{2|\vec{F}^2|} \right\} \hat{F}_z | F M_F \rangle \right\} \quad (14)$$

$$= \langle F M_F | \left\{ \underbrace{+g_J \frac{\mu_B B}{\cancel{\hbar}} \frac{(F(F+1) + J(J+1) - I(I+1))\cancel{\hbar^2}}{2F(F+1)\cancel{\hbar^2}}}_{\text{Real \#s}} \cancel{\hbar} M_F \right\} | F M_F \rangle \quad (15)$$

$$\therefore \quad V_{z_{ii}} = \left\{ +g_F \mu_B B M_F \frac{\cancel{\hbar}}{\cancel{\hbar}} \right\} \underbrace{\langle F M_F | F M_F \rangle}_{1} \quad (16)$$

$$\left(\delta_{FF'} \delta_{M_F' M_F} = \langle F' M_F' | F M_F \rangle \right)$$

g_F: AS DEFINED IN EQ. 7 !!!

Further, since the $|F M_F\rangle$ kets are eigenstates of the operator $(\hat{V}_z)$ in the curly brackets in eq. 14, that by the ortho·normality of the kets, $\hat{V}_z$ is diagonal and ~~eg~~ eq. 16 are the eigenvalues (ev's)

Yay! were done (almost).