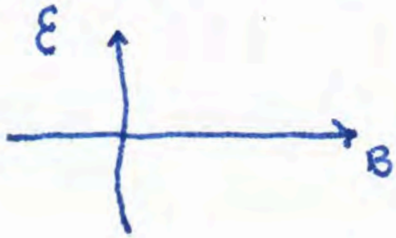
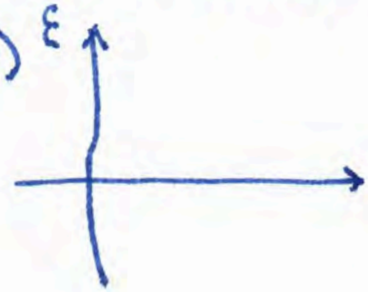
QM - 2 state System - 480 worksheet #4


#1 Draw in the space below partial Grotrian diagram relevant for (a) a proton and (b) a H atom at rest in an external, homogeneous magnetic field B. Label the states. Discuss the relevance to any of the experiments you've done this semester. Are their 2-state systems in both problems?

a)




b)




#2 The problem we want to solve now is this: what happens to a quantum state if the magnetic field has a strong spatially homogeneous part, and a weaker, time dependent part, one that varies sinusoidally with frequency $\omega$, that is $\underline{B} = \underline{B}_0 \hat{z} + [B_1 \cos(\omega t)]\hat{x} + [B_1 \sin(\omega t)]\hat{y}$. "rotating field"

For our two state system, which we take to be the eigen states of the $-\vec{\mu} \cdot \underline{B}_0 \hat{z}$ operator, we want to think of weak, time dependent field as a time dependent perturbation, with a complete Hamiltonian given by

$$\hat{H} = \hat{H}_0 + \hat{H}(t), \qquad (1)$$

where $|\hat{H}(t)| / |\hat{H}_0| \ll 1$. Sounds like a job for time dependent theory right?

The dream is to solve the time dependent Schrodinger Eq. usually just called the Schrodinger Equation (SE)

$$i\hbar\frac{\partial}{\partial t}|\psi\rangle = \hat{H}|\psi\rangle, \qquad (2)$$

lhs ↗ ↖ rhs

and, as everyone knows, we can expand $|\psi\rangle$ in any complete set of basis states (the eigenstates of $\hat{H}_0$? Why NOT!?). Given that

here $\vec{\sigma} = \underline{\underline{\sigma}}_z \cdot \hat{z} + \underline{\underline{\sigma}}_x \hat{x} + \underline{\underline{\sigma}}_y \hat{y}$ a "matrix" vector where the $\sigma$'s are the Pauli Spin matrices - handy for spin ½ particles... $\sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ $\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $\sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$

(3) $$H_0 = -\vec{\mu}\cdot B_0\hat{z}, \quad \vec{\mu} = \gamma\vec{I} = \gamma\frac{\hbar}{2}\vec{\sigma}$$

with eigen kets

(4a) $$|u_1\rangle = |+\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}; \; E_1 = -\gamma\frac{\hbar}{2}B_0,$$

(4b) $$|u_2\rangle = |-\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \; E_2 = +\gamma\frac{\hbar}{2}B_0,$$

which span the space of the qu-bits, an arbitrary solution to SE (Eq. 2) would, could be

(5) $$|\psi(t)\rangle = c_1(t)|u_1\rangle e^{-iE_1t/\hbar} + c_2(t)|u_2\rangle e^{-iE_2t/\hbar}.$$

The object is to find the $c_{1,2}$'s by solving SE, because

$$P_1(t) = |c_1(t)|^2, \quad P_2(t) = 1 - P_1 = 1 - |c_1|^2$$

are probabilities. Well, given initial conditions, we'll be able to say, after such and such a time, what the probability is of finding $|\psi\rangle$ in $|u_1\rangle$, say.

For the sake of brevity (i.e. leave it for the reader to connect the dots) but also to give us an intermediate landing spot to think about something important, we write SE as a matrix equation

leave this too as an exercise for later... Off diagonal elements only!!!

$$(6)\quad \underbrace{i\hbar \frac{\partial}{\partial t}\begin{bmatrix} c_1 e^{-iE_1 t/\hbar} \\ c_2 e^{-iE_2 t/\hbar}\end{bmatrix}}_{lhs} = \underbrace{\left\{ -\frac{\gamma \hbar B}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} 0 & H_{12} \\ H_{21} & 0 \end{bmatrix}\right\}\begin{bmatrix} c_1 e^{-iE_1 t/\hbar} \\ c_2 e^{-iE_2 t/\hbar}\end{bmatrix}}_{rhs}$$

#3 Perform the time derivative on the lhs and the matrix multiplication on the rhs and find

$$i\hbar\begin{bmatrix} \dot{c}_1 e^{-iE_1 t/\hbar} \\ \dot{c}_2 e^{-iE_2 t/\hbar}\end{bmatrix} = \begin{bmatrix} H_{12} c_2 e^{-iE_2 t/\hbar} \\ H_{21} c_1 e^{-iE_1 t/\hbar}\end{bmatrix}$$

simply identify & equating matrix elements we get 2 coupled 1st order ODE's that solve our problem

(7a) $$i\hbar \dot{C}_1 = H_{12} C_2 e^{i\omega_{12} t}$$
(7b) $$i\hbar \dot{C}_2 = H_{21} C_1 e^{i\omega_{21} t}$$

$$\omega_{12} \equiv \frac{E_1 - E_2}{\hbar}$$

#4 Before we could ask whether a "stationary" perturbation could make the state change. Answer, no. The Zeeman effect, e.g. simply removes degeneracy.
∴ But what about time dependent perturbations? What role is played by the off diagonal matrix elements of a time dependent perturbation? Discuss briefly

BTW
$$H(t) = -\frac{\gamma\hbar B_1}{2}\begin{bmatrix} 0 & e^{-i\omega t} \\ e^{i\omega t} & 0 \end{bmatrix}$$
this from $H(t) = -\mu_x B_x(t) - \mu_y B_y(t)$ and those Paulis spin matrices!!!

It's all precipitous now. The coupled 1st order ODE's in the C's (Eq. 7a) & 7b)) can be turned into 2nd order ODE's, uncoupled yielding (and we only need to do this for 1 of them!)

(8) $$\ddot{C}_1 + i\Omega\dot{C}_1 + \left(\frac{\gamma}{2}B_1\right)^2 C_1 = 0.$$

$\Omega \equiv \omega + \gamma B_0$

Taking $C_1 = \text{something} \cdot e^{ipt}$ etc.

gives

(9) $$p = \frac{\Omega \pm \underbrace{\sqrt{\Omega^2 + (\gamma B_1)^2}}_{\Delta}}{2},$$

$\Delta = \sqrt{\Omega^2 + (\gamma B_1)^2}$

and the solution is starting to look as though it has the Rabi frequency in it. But the initial conditions matter. Let's take $C_1(t) = 0$, $C_2(t) = 1$ @ $t=0$. To satisfy the first,

(10) $$C_1(t) = A e^{i\left(\frac{\Omega}{2} + \frac{\Delta}{2}\right)t} + B e^{-i\left(\frac{\Omega}{2} \mp \frac{\Delta}{2}\right)t}$$

oop "−"

(need Both Solutions

$\therefore B = -A$

you can see it coming now

$$\Rightarrow C_1(t) = a\, e^{\frac{i\Omega t}{2}} \sin\left(\frac{\Delta t}{2}\right);$$

But now to get "a"? Well, from $C_2$ & 7a,

now we can do this.

$$C_2 = \frac{i\hbar\dot{C}_1}{H_{12}e^{i\omega_{12}t}} = \frac{i\hbar\dot{C}_1}{-\frac{\gamma\hbar B_1}{2}e^{i(\omega_{12}-\omega)t}}$$

$$\Rightarrow C_2 = -\frac{a}{\frac{\gamma B_1}{2}} \cdot \left[i\frac{\Omega}{2}\sin\left(\frac{\Delta t}{2}\right) + \frac{\Delta}{2}\cos\left(\frac{\Delta t}{2}\right)\right] e^{\frac{i\Omega}{2}t}$$

From which we can get, at $t=0$, $C_2 \to 1$

(11) $$a = -\frac{\gamma B_1}{i\Delta} = i\frac{\gamma B_1}{\Delta}$$

Thus

(12a) $$C_1 = i\frac{\gamma B_1}{\Delta} e^{i\frac{\Omega t}{2}} \sin\left(\frac{\Delta t}{2}\right),$$

(12b) $$C_2 = \frac{i\gamma B_1}{\Delta} e^{i\frac{\Omega t}{2}} \left( i\frac{\Omega}{2}\sin\left(\frac{\Delta t}{2}\right) + \frac{\Delta}{2}\cos\left(\frac{\Delta t}{2}\right)\right),$$

and one can show that

$$P_1 = \left(\frac{\gamma B_1}{\Delta}\right)^2 \sin^2\left(\frac{\Delta t}{2}\right) \quad P_2 = 1 - P_1$$

**#5** Plot $P_1$ & $P_2$ for a few cycles for the case that $\omega + \gamma B_0$ is NOT $\phi$ (a little of resonance) $\left[\left(\frac{\gamma B_1}{\Delta}\right)^2 < 1\right]$ and draw in the space below the motion of the sample magnetic moment $M(t)$ and its "Motion" relative to "$B_{eff}$" With words, "Draw parallels"