# PHYS 480W WORKSHEET #7 (feedback)

actually "worksheet 7a" another worksheet #'d 7 (henceforth "7b") was given last Friday the 13th of Feb

IAW (ion acoustic waves)

Task #1: Are coherent, propagating electrostatic perturbations possible in a weakly ionized ($n_e \ll n_0$), low temperature ($T_e < \mathcal{U}_i$ (ionization potential of neutral gas)) plasma? Complete the perturbation analysis using Newton's Law, Poisson's Eq., and Continuity as follows

$$\text{NII:}\quad M n_i \left[\frac{\partial v_i}{\partial t} + \underline{v}_i \cdot \nabla v_i\right] = e n_i E \quad \text{(ions)}$$

$$\text{Poisson:}\quad \nabla \cdot \underline{E} = \rho/\epsilon_0 = \frac{e}{\epsilon_0}(n_i - n_e)$$

$$\text{Continuity:}\quad \frac{\partial n_i}{\partial t} + \nabla \cdot (n_i \underline{v}_i) = 0$$

$\tilde{E}_1$ or $\tilde{\Phi}_1 \leftrightarrow \tilde{v}_1$

$\tilde{E}_1$ or $\tilde{\Phi}_1 \leftrightarrow \tilde{n}_i, \tilde{n}_e$

$\tilde{n}_i \leftrightarrow \tilde{v}_i$

3 eq. 3 unk.

where we suppose all 1st order perturbations, e.g. $n_{i_1} = \tilde{n}_{i_1} e^{i(kx-\omega t)}$, for the density, and we'll drop the i subscript as we know we are concerned about the ions; and will take the amplitude of $n_1(x,t) = \tilde{n}_1 e^{i(kx-\omega t)}$ to just be $n_1$

**DEMONSTRATE:** $n_1 = n_0 \frac{k v_1}{\omega}$, to 1st order, describe assumptions! Note, of all the relations above, the one most directly related to the desired outcome is the equation of continuity. Doing Perturbation theory with this equation is simplified by adopting an <u>infinite</u> (no local plasma boundaries), <u>homogeneous</u> (no local gradients (in equilibrium!)), and <u>nondrifting model</u>, in which our perturbed quantities vary only in 1-D; then Continuity becomes.....

$$\frac{\partial n}{\partial t} + \frac{d}{dx}(n_i v) = 0$$

but $\frac{d}{dx}\tilde{v}_1 e^{\overbrace{i(kx-\omega t)}^{\alpha}} = \tilde{v}_1 (ik_x) e^{i\alpha}$

expanding to 1st order "linear in perturbed quantities"

& $\frac{\partial n_1}{\partial t} = -i\omega \tilde{n}_1 e^{i\alpha}$

$$\frac{\partial}{\partial t}(\cancel{n_0}^{0} + n_1) + \underbrace{(\cancel{v_0}^{0} + v_1)\frac{d}{dx}n}_{\text{quadratic in perturbed quantities}} + \underbrace{(n_0 + n_1)\frac{d}{dx}(\cancel{v_0}^{0} + v_1)}_{\text{quadratic in perturbed quantities}} = 0$$

but $\tilde{v}_1 e^{i\alpha} = v_1$ and $n_1 = \tilde{n}_1 e^{i\alpha}$,

$\therefore \; -i\omega n_1 + n_0 i k v_1 = 0$

$\frac{\partial n_1}{\partial t}$ yields

$$\frac{\partial n_1}{\partial t} + n_0 \frac{d}{dx} v_1 = 0 \quad \cdots\cdots \rightarrow \quad \boxed{n_1 = n_0 \frac{k}{\omega} v_1}$$

Hey! dimensionally correct!? ???!!?

Task #2 Let's prefer to include $\phi_1$ in favor of $E_1$? OK let's....

How are $E_1$ & $\phi_1$ related?

$$\vec{E} = -\nabla\phi$$

or

$$E_x = -\frac{d\phi}{dx}$$

no equil. gradients!

$$\therefore \; \cancel{E_0} + E_1 = -\frac{d}{dx}(\cancel{\phi_0} + \phi_1)$$

$$E_1 = -\frac{d}{dx}\phi_1 = -ik\tilde{\phi}_1 e^{i\alpha} = -ik\phi_1$$ ← - - - later I forget this minus sign!

Task #3 Demonstrate that to 1st Order

$$k^2\phi_1 = \frac{e n_1}{\epsilon_0} - \frac{n e^2}{\epsilon_0 kT_e}\phi_1 ,$$

} Sorta looks like the Poisson Equation, "linearized"

assuming Boltzmann thermal equilibrium for the electrons

Back to Boltzmann

$n_e = n_{e0} e^{x}, \quad n_{e0}\, g(x)$ ← expansion parameter; small departure from "equilibrium"... or "0"

$\approx n_{e0}\,(g(0) + x\, g'(0) + \frac{x^2}{2!} g''(0) \ldots$

↑ Taylor Expansion*

$\approx n_{e0}\,(1 + x)$

$x = \frac{e\phi_1}{kT_e}$

And Poisson's Eq. becomes, to 1st Order

here I forgot the minus sign!

$$\frac{d}{dx} \underbrace{E_1}_{ik\phi_1} = \frac{e}{\epsilon_0}\left(n_{i0} + n_{i1} - n_{e0}\left(1 + \frac{e\phi_1}{kT_e}\right)\right)$$

cancels: quasineutrality

$$\downarrow$$

$$-k^2 \phi_1 = \frac{e}{\epsilon_0}\left(n_{i1} - \frac{e}{kT_e}\phi_1 \cdot {}^{*}n_{e0}\right),$$

$$\boxed{\text{or, } \left(\frac{e^2 n_{e0}}{\epsilon_0 kT_e} - k^2\right)\phi_1 = \frac{e}{\epsilon_0} n_{i1}}$$

[actually it's: $(\text{blah} + k^2)\phi_1 = \text{blah}\cdot\text{blah}\ldots$]

cf. It's 'clear' that the leading term in the box must have an inverse-length-squared dimension (as does the non-Boltzmann $k^2$). But it's just constants, fundamental ones – so it's important

$$\frac{1}{\lambda_D^2} = \frac{e^2 n_{0e}}{\epsilon_0 kT_e}$$

## Task #4 Assuming NII "Boils Down" to

$$v_1 = \frac{e k}{M_i \omega}\phi_1,$$

cf. this & continuity also combine (eliminating $v_1$) a relation between $n_1$ & $\phi_1$ – 2 eq. 2 unk's! we can solve it!

Demonstrate

$$\left[k^2 - \frac{\Omega_p^2}{\omega^2} k^2 + \frac{1}{\lambda_D^2}\right]\phi_1 = 0$$

- given what definitions of $\Omega_p$ & $\lambda_D$?
- implying what 'dispersion relation'? Plot it!
- under what conditions do these electrostatic perturbations propagate?
- are density & potential fluctuations "in phase"?

* fun fact: Brook Taylor took Law degrees from Cambridge (1709, 1714). In 1715 he publish 'Methodus Incrementum Directa et Inversa', containing "Taylor's Theorem" as they are now known ... later LaGrange termed the work, "the main foundation of differential calculus". Taylor was elected fellow of RS in 1712, and in the same year ~~[illegible]~~ sat on the RS committee adjudicating the Leibniz, Newton priority dispute

it? they?

Following the marginal note in the directions for task #4, from "N II" & "Continuity"

$$v_1 = \frac{e k}{M_i \omega} \phi_1 , \qquad \text{N II},$$

$$n_1 = n_{e0} \frac{k}{\omega} v_1 , \qquad \text{Continuity},$$

"subbing"

$v_1$ from Continuity into N II we have,

$$\frac{\omega}{k}\frac{n_1}{n_{e0}} = \frac{e k}{M_i \omega}\phi_1$$

$$\boxed{n_1 = \frac{e n_{e0}}{M_i}\frac{k^2}{\omega^2}\phi_1}$$

— comment... no, a question so $\frac{e\phi_1}{M_i} \cdot \frac{k^2}{\omega^2}$ is really dimensionless?

then

$n_1$ in Poisson (prev. page...)

$$\left(\frac{1}{\lambda_D^2} - k^2\right)\phi_1 = \frac{e}{\epsilon_0} \cdot \frac{e n_{0e}}{M_i}\frac{k^2}{\omega^2}\phi_1$$

$$= \frac{e^2 n_{0e}}{\epsilon_0 M_i} \cdot \frac{k^2}{\omega^2}\phi_1$$

$$= \frac{\Omega^2}{\omega^2} \cdot k^2 \phi$$

--- wait is $\frac{e^2 n_{0e}}{\epsilon_0 M_i}\frac{1}{\omega^2}$ really dimensionless

Yes! but that means $\Omega_p^2 \equiv \frac{e^2 n_{0e}}{\epsilon_0 M_i}$ is a frequency depending on a plasma parameter ($n_{0e}$) & constants, so it's important... Yes, the "ion plasma $\Omega_p$ frequency"

$$\therefore \left\{\frac{1}{\lambda_D^2} - k^2 - \frac{\Omega_p^2}{\omega^2}k^2\right\}\phi_1 = 0$$

— Comment ... I think I made a 'sign' error! FOUND IT !!! SORRY!! This is "+"

quo erat demonstratum

$$\boxed{\underbrace{\left\{k^2 - \frac{\Omega_p^2}{\omega^2}k^2 + \frac{1}{\lambda_D^2}\right\}}_{\text{MUST VANISH!}}\phi_1 = 0}$$

$\exists\, \phi_1 \neq 0$

2 words...

we endup with something like

$$\underbrace{D(\omega, k)}\cdot \phi_1 = 0$$

equation of constraint between $\omega$ & $k$ } "dispersion relation for ion "acoustic" waves

$$D(\omega, k) = 0$$

$$k^2 - \frac{\Omega_p^2}{\omega^2} k^2 + \frac{1}{\lambda_D^2} = 0$$

$$k^2 + \frac{1}{\lambda_D^2} = \frac{\Omega_p^2 k^2}{\omega^2}$$

$$1 + \frac{1}{k^2\lambda_D^2} = \frac{\Omega_p^2}{\omega^2}$$

or

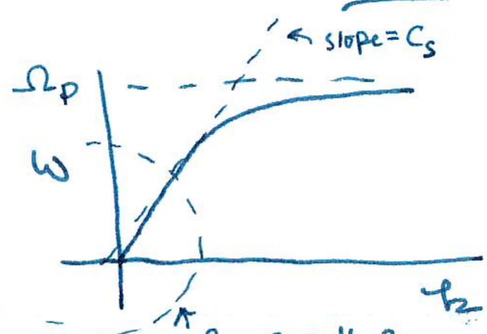
$$\frac{\omega^2}{\Omega_p^2} = \cancel{\frac{1}{1+k^2\lambda_D^2}} = \frac{1}{1+\frac{1}{k^2\lambda_D^2}}$$

$$= \frac{k^2\lambda_D^2}{1+k^2\lambda_D^2}$$

amazingly $\lambda_D^2\Omega_p^2 \equiv C_s^2$

$$\therefore \quad \omega^2 = \frac{k^2\lambda_D^2\cdot\Omega_p^2}{1+k^2\lambda_D^2}$$

$$\boxed{\omega^2 = \frac{C_s^2 k^2}{1+k^2\lambda_D^2}}$$




for small $k$, $\frac{\omega}{k}$ & $\frac{d\omega}{dk}$ are the same! And independent of $k$

interpret!