The word ‘sheath’ in connection with plasma, the fourth state of matter, was coined by Irving Langmuir\textsuperscript{1} to describe that thin region of strong electric fields formed by space charge that joins the body of the plasma to its material boundary. The plasma boundary one has in mind here might be the metal walls of a vacuum chamber confining a laboratory plasma, or the wafer of silicon sitting on an electrode immersed in a plasma etching reactor, maybe the glass tube surrounding the plasma discharge of a neon sign, or even the condenser plates in J.J. Thomson’s (insufficiently exhausted) gas filled cathode ray tube. The electrostatic potential of the boundary joins smoothly to the interior plasma potential, and nearly all of the changes occur in the thin region called the sheath, an effect called ‘Debye shielding’.\textsuperscript{2}

The plasma, a dynamic, seething hot (2-10 times hotter than the surface of the sun for the plasmas in the examples just given) collection of electrons, ions, and neutral atoms and molecules, is characterized by weak electric fields and charge neutrality. It reacts to externally applied potentials in a complicated way. The shielding of course cannot be perfect, simply because the temperature is finite, although the reason for this is obscure in the details. Sheath formation requires the acceleration of the plasma ions to up to Mach speeds at the sheath edge, that is, if the ion-neutral collisions are sufficiently rare. This is the Bohm Criterion.\textsuperscript{3} It is possible to examine this phenomena for plasmas with a single ion species using simple fluid theory. This is the purpose of this note.

The sheath region begins where charge neutrality breaks down. The electric potential gradient then becomes quite steep quite abruptly on the boundary side of the sheath edge, where charge of one sign is increasingly repelled and the other accelerated. For simplicity, we imagine planar symmetry where plasma parameters vary in a direction normal to a planar boundary. This is shown schematically in a cartoon in figure 1. The spatial potential profile forms self consistently, at once influenced by and creating charged flows; its formation depends on the kinetic distribution functions of the charged species, and on ion-neutral collisions even if the Debye length (the sheath thickness has the order of magnitude of a few Debye lengths) is the smallest scale length in the plasma. The question we ask is this: how is it that the breakdown of charge neutrality at the sheath edge implies that the ion drift speed becomes greater than the ion sound speed,

\[ C_s = \sqrt{\frac{kT_e}{M}} \] 

at the sheath edge? Algebraically we ask, how is it that if the ion density gradient is less than the electron density gradient,

\[ \frac{dn_i}{dx} < \frac{dn_e}{dx}, \]  

that the ion drift speed becomes greater than the ion sound speed,

\[ |v_i| > C_s? \]  

It is hard to ‘see it at a glance’ to borrow Polya’s phrase.\textsuperscript{5} Typical demonstrations of this result involve setting up complicated differential equation,\textsuperscript{2} making it manageable by expanding the potential in a Taylor series, and taking the limit that the potential is small compared with the electron temperature, \( e\phi/kT_e \ll 1 \) and so on, or by numerically integrating a system of equations,\textsuperscript{4} all of which is warranted if one aims at understanding the potential profile over the region. If however one’s aim is simply to understand the Bohm
Criterion, and how it arises at the sheath edge, then one can find insight into this problem much more simply.

We can take the sheath criterion, equation 1, as the condition that charge neutrality breaks down and space charge appears at the spatial location where the first order changes in the charge concentration no longer cancel out. Within the body of the plasma, these changes do cancel. Next, the equation of ion continuity requires that the changes in ion concentration and flows are related to the sources and sinks of ion charge,

$$\frac{\partial n_i}{\partial t} - \frac{d(n_v)}{dx} = S_+ - S_-, \tag{3}$$

where the sources and sinks can be set to zero if we concern ourselves with a region of no more than a couple of Debye lengths from the sheath edge, where (1) begins to be satisfied. In the case of stationary flow, it is not hard to rearrange the continuity equation, (3), into the form

$$\frac{1}{n_i} \frac{dn_i}{dx} = - \frac{1}{v_i} \frac{dv_i}{dx}, \tag{4}$$

which is convenient for comparing the scale lengths over which things change. Each side has the units of a reciprocal length. This length, often referred to as the gradient scale length, characterizes the length over which the quantity changes by some fractional amount, and (4) says that these two scale lengths are the same. The fractional changes in ion density and speed occur over distances just as short, although in opposite senses—the density is rising toward the sheath edge while the drift speed is falling as we follow the ions up the potential gradient, past the sheath edge into the bulk plasma. It is more fun to take the ride with the ions in the other direction. The ions pick up speed as the electric field does work on them. But why should this speed at the sheath edge have to be sufficiently high in order for the spatial gradients in the ion and electron population begin to differ? The answer is still not obvious, but the sheath criterion does produce a second inequality in connection with (4). Given (1), the continuity equation says that if the ion density gradient has to be less than the electron density gradient then it follows that

$$-n_i \frac{dv_i}{dx} < \frac{dn_e}{dx}, \tag{5}$$

at the sheath edge. One instinctively wants to reverse the inequality because of the minus sign, but since the ion speed is negative here we can leave it as it is.

And we have arrived at the principal result of this note. The velocity gradient has a certain positive value (since speed is becoming less negative, its derivative is positive definite), and the density has a certain positive value. The only way for the lhs of (5) to be less than the electron density gradient is for the magnitude $|v_i|$ to be greater than something, that is

$$-v_i > \frac{n_i}{A} \frac{dv_i}{dx} = m_i \frac{dv_i}{dx} \frac{dn_e}{dx}. \tag{6}$$

Of course now we have to show that the rhs of this inequality (6) is in fact equal to the ion sound speed. If we substitute the electron density gradient into the expression (6), we see that the right hand side is the ratio of two derivatives. Energy considerations lead to these. Let’s now consider the electrons. They are repelled or confined by the shape of the potential and thus are in electrostatic equilibrium and so describable in terms of a Boltzmann equation,

$$n_e = n_o e^{e\phi/kT_e}, \tag{7}$$

where $n_o$ is the electron density in the bulk plasma far from the sheath edge. This may look a bit odd without a minus sign in the exponent. But it does get the energy right for the
negative charge: the electron density is greatest where the electron potential energy is least, and vice versa, as required in thermal equilibrium. The electron density gradient is then
\[ \frac{dn_e}{dx} = \frac{e}{kT_e} \frac{d\phi}{dx} e^{e\phi/kT_e} = \frac{n_e e}{kT_e} \frac{d\phi}{dx}, \] (8)
which gives us one of the derivatives in (6); the conservation of energy for the ion fluid gives us the other:
\[ \frac{d(U + K)}{dx} = 0, \] (9)
where \( U = e\phi \) and \( K = \frac{1}{2} M v_i^2 \). So we write
\[ \frac{d\phi}{dx} + M v_i \frac{dv_i}{dx} = 0, \] (10)
and we see that both spatial derivatives in (6) are proportional to the potential gradient (or electric field strength), which cancels out, leaving only constants. At last we may write down
\[ -v_i > -n_e \frac{d\phi}{dx} \frac{kT_e}{e n_e M v_i} \frac{d\phi}{dx} = -\frac{kT_e}{M v_i}, \] (11)
or,
\[ \sqrt{v_i^2} = \sqrt{\frac{kT_e}{M}}, \] (12)
which is what we wished to demonstrate. The electron and ion densities cancel out at the sheath edge.

One further question. Is there some simple picture that we can take from experience to predict this result? The cardinal fact of this heuristic sketch is that the continuity equation leads to the conclusion that the ion density gradient can only be less than some value if the ion velocity is sufficiently high. We can see it algebraically, but can we see it physically? I will argue that the answer is no, it's not easy to see, but I will give a physical picture anyway.

All our physical intuition and experience in the flow of fluids basically comes from playing with and in water. The problem is that it is incompressible. So, when one turns on a faucet, the column of the water narrows farther down from the orifice. Why? Certainly, the number of water molecules is conserved, and the divergence of the flow is zero. But the density is constant, so that as the water falls and picks up speed, the only way to keep the number of particles entering and leaving a portion of the water column is for the cross sectional area of the downstream end to get smaller, since the speed of the water rises. In the plasma however, the density can change, the flow is not incompressible. It is like a fluid column falling under the force of gravity (analogous to the electric field in the plasma) which remains perfectly constant in its cross sectional area (as shown in figure 2 below), something foreign to common experience.

If one thinks about it (one can satisfy oneself mathematically that this is true), the only way that the cross sectional area can remain constant is if (4) is true. But why, if the drop in the density for a given vertical displacement is less than some critical value, must magnitude of the ion velocity then be greater than something? It is for this reason: the amount by which the speed changes depends only on the arbitrary choice of vertical displacement, the \( \Delta y \). The change arises on account of the work done by gravity over this \( \Delta y \). Now this change is not linear, it isn’t simple, nevertheless, the change in velocity grows with \( \Delta y \).
But, and here’s the point, the fractional change in $v$ is reduced as we make $v$ bigger and bigger. And since the fractional change in $v$ is identical to the fractional change in density, the way to make the ion density change less than something given some ion density there, is to increase the incident speed of the ions there above some threshold, which turns out to be the ion sound speed.

What have we achieved here? Perhaps some insight into the Bohm Criterion. But there is more than this. I claim that it is not possible to summon up a common experience of the motion of fluids in order to ‘see’ the result at a glance, and that all the insight is actually derived from the simple equations themselves. The equations themselves must tutor our intuition rather than the other way around. This is not uncommon in the learning of physics, nor should we expect it to be always otherwise.

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5 G. Polya, How to Solve it, Princeton University Press, 2nd Ed. p.xvii (1973)
Figure Captions

fig. 1: The plasma is shown here schematically as bounded by a negatively biased boundary wall. Ions flow to the wall down the potential hill ($\phi(x)$), while electrons are repelled. Net space charge appears at the sheath edge, where the gradients in the ion density and electron density diverge.

fig. 2: For incompressible flow (a), falling water, say, from a faucet, narrows as it falls a distance $\Delta y$; since $v_{in} > v_{out}$, the only way the amount of water within the geometric volume bounded by the input and output surfaces stays constant is if the lower output surface contracts (flux in = flux out). But if the flow is not incompressible (b), and if the density diminishes as the fluid falls, and if the fractional change in density is identical to the fractional change in the fluid flow velocity, then the input and output surface are stays the same, the column diameter remains constant.
FIG. 1: Severnfig1
FIG. 2: Severnfig2