On higher quasi-categories

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Quasi-categories and complete Segal spaces

Quasi-categories

Definition

A **quasi-category** is a simplicial set X such that for every solid diagram

$$\begin{array}{ccc} \Lambda^k_n & \longrightarrow & X \\ \downarrow & \nearrow & \\ \Delta_n & & \end{array} \qquad n \geq 2, \quad 0 < k < n,$$

there exists a dotted arrow making the triangle commute.

Theorem (Joyal)

There exists a model structure on simplicial sets whose cofibrations are the monomorphisms and whose fibrant objects are the quasi-categories.

Complete Segal spaces

Definition

A simplicial space X (i.e., a bisimplicial set) is a **complete Segal space** if

1. X is Reedy fibrant;
2. $X_n = \mathrm{Map}(\Delta_n, X) \to X_1 \times_{X_0} \cdots \times_{X_0} X_1 = \mathrm{Map}(I_n, X)$ is a simplicial weak equivalence;
3. $X_0 = \mathrm{Map}(\Delta_0, X) \to X_{1,\mathrm{eq}} = \mathrm{Map}(J, X)$ is a simplicial weak equivalence.

Proposition (Rezk)

There exists a model structure on simplicial spaces whose cofibrations are the monomorphisms and whose fibrant objects are the complete Segal spaces.

Comparison of the two notions (Joyal-Tierney)

Theorem

- *If X is a complete Segal space, then $X_{\bullet,0}$ is a quasi-category.*
- *If X is a quasi-category, then*

$$([m],[n]) \mapsto \mathrm{Hom}_{\widehat{\Delta}}(\Delta_m \times \Pi_1(\Delta_n), X)$$

is a complete Segal space.

Theorem

These constructions define two pairs of Quillen equivalences between quasi-categories and complete Segal spaces.

A-localizer theory

Definition and main theorem

Definition

An A-**localizer** is a class $\mathcal{W}$ of arrows of $\widehat{A}$ satisfying the following conditions:

1. $\mathcal{W}$ satisfies the 2-out-of-3;
2. trivial fibrations ($r(\mathrm{Monos})$) are in $\mathcal{W}$;
3. $\mathcal{W} \cap \mathrm{Monos}$ is stable under pushout and transfinite composition.

Theorem (Cisinski)

Let $\mathcal{W}$ be a class of morphisms of $\widehat{A}$. The following assertions are equivalent:

- $(\widehat{A}, \mathcal{W}, \mathrm{Monos}, r(\mathcal{W} \cap \mathrm{Monos}))$ *is a combinatorial model category;*
- $\mathcal{W}$ *is an (accessible) localizer.*

Examples

Theorem (Cisinski)

Let $\mathcal{W}_{\mathrm{KQ}}$ *be the* Δ*-localizer generated by* $\{\Delta_n \to \Delta_0; n \geq 0\}$*.*
The $\mathcal{W}_{\mathrm{KQ}}$*-model structure is the Kan-Quillen model structure.*

Theorem (Joyal)

Let $\mathcal{W}_{\mathrm{J}}$ *be the* Δ*-localizer generated by* $\{I_n \to \Delta_n; n \geq 0\}$*.*
The $\mathcal{W}_{\mathrm{J}}$*-model structure is the model structure for quasi-categories.*

Proposition

Let $\mathcal{W}_{\mathrm{R}}$ *be the* $(\Delta \times \Delta)$*-localizer generated by*

$$\{f; \forall n \geq 0 \;\; f_{n,\bullet} \in \mathcal{W}_{\mathrm{KQ}}\} \cup \{I_n \to \Delta_n; n \geq 0\} \cup \{J \to \Delta_0\}.$$

The $\mathcal{W}_{\mathrm{R}}$*-model structure is the model structure for complete Segal spaces.*

Simplicial completion

Definition

Let $\mathcal{W}$ be an A-localizer. The **simplicial completion** $\mathcal{W}_\Delta$ of $\mathcal{W}$ is the $(A \times \Delta)$-localizer generated by

$$\{f\,;\forall n \geq 0 \;\; f_{\bullet,n} \in \mathcal{W}\} \cup \{X \times \Delta_1 \to X; X \in \mathrm{Ob}(\widehat{A})\}.$$

Proposition (Cisinski)

We have $\mathcal{W} = p^{*-1}(\mathcal{W}_\Delta)$ *where* $p : A \times \Delta \to A$ *is the projection.*

Theorem (Cisinski)

Let $\mathcal{W}$ *be an (accessible)* A*-localizer. Then there are "two" Quillen equivalences between the* $\mathcal{W}$*-model structure and the* $\mathcal{W}_\Delta$*-model structure.*

Quasi-categories and complete Segal spaces

Theorem (Follows from Joyal-Tierney)

The localizer of complete Segal spaces is the simplicial completion of the localizer of quasi-categories.

Consequences

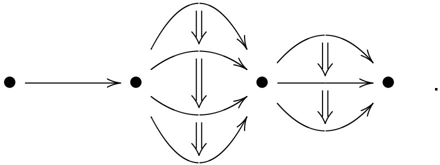
- We recover the two Quillen equivalences of Joyal and Tierney.
- The model structures of quasi-categories and complete Segal spaces can be canonically deduced from one another.

n-quasi-categories

The category Θ_n of Joyal

Objects

Globular pasting schemes of dimension $d \leq n$:



Morphisms

Strict n-functors between the free strict n-categories on these globular pasting schemes.

Proposition (Makkai-Zawadowski, Berger, Weber)

The inclusion $\Theta_n \subset n\text{-}\mathcal{C}\mathrm{at}$ *induces a fully faithful nerve* $n\text{-}\mathcal{C}\mathrm{at} \to \widehat{\Theta_n}$.

A conjecture of Cisinski and Joyal

Cellular spines

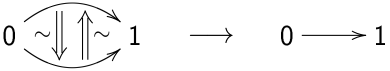
To each representable S of $\widehat{\Theta_n}$, one can associate a **spine** $I_S \subset S$.

Conjecture

Let $\mathcal{W}$ be the Θ_n-localizer generated by the cellular spines. Then the $\mathcal{W}$-model structure is a model for (∞, n)-categories.

The conjecture is slightly wrong

For instance, the equivalence of 2-categories



does not belong to $\mathcal{W}$.

n-quasi-categories

Definition

Let D_i be the free-living i-arrow and J_i be the free-living invertible i-arrow. There is a canonical equivalence of $(i+1)$-categories $J_{i+1} \to D_i$.

Definition

Let $\mathcal{W}_{\mathrm{QCat}_n}$ be the Θ_n-localizer generated by

$$\{I_T \to T; T \in \Theta_n\} \cup \{J_{i+1} \to D_i; 0 < i < n\}.$$

The $\mathcal{W}_{\mathrm{QCat}_n}$-model structure is called the **model structure of** n-**quasi-categories**.

A fibrant object of this structure is called an n-**quasi-category**.

n-quasi-categories and Θ_n-spaces

Θ_n-spaces (Rezk)

Definition

Let $\mathcal{W}_{\mathrm{Rezk}_n}$ be the $(\Theta_n \times \Delta)$-localizer generated by

$$\{f\,;\forall S \in \Theta_n \ \ f_{S,\bullet} \in \mathcal{W}_{\mathrm{KQ}}\}$$

and

$$\{I_T \to T;\, T \in \Theta_n\} \cup \{J_{i+1} \to D_i;\, 0 \le i < n\}.$$

The $\mathcal{W}_{\mathrm{Rezk}_n}$-model structure is called the **model structure of Θ_n-spaces**.
A fibrant object of this structure is called a Θ_n-**space**.

Theorem (Rezk)

The model structure of Θ_n-spaces is cartesian.

n-quasi-categories and Θ_n-spaces

Theorem

The localizer of Θ_n-spaces is the simplicial completion of the localizer of n-quasi-categories.

Corollary

- *If X is a Θ_n-space, then $X_{\bullet,0}$ is an n-quasi-category.*
- *If X is an n-quasi-category, then*

$$(S,[n]) \to \mathrm{Hom}_{\widehat{\Theta_n}}(S \times \Pi_1(\Delta_n), X)$$

  is a Θ_n-space.

Moreover, these constructions define two pairs of Quillen equivalences between n-quasi-categories and Θ_n-spaces.

Thanks for your attention!