ADDITIVE ∞-CATEGORIES AND CANONICAL MONOIDAL STRUCTURES I

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PLAN

1. $\mathcal{E}_\infty$-MONOIDS AND $\mathcal{E}_\infty$-GROUPS IN $\infty$-CATegORIES

2. PREADDITIVE AND ADDITIVE $\infty$-CATegORIES

3. MONOIDS IN PRESENTABLE $\infty$-CATegORIES

4. $\infty$-CATegorical VARIanT OF ALGEBRAIC K-theory
Notation and terminology as in *Higher Topos Theory* and *Higher Algebra*.

$\infty$-category = simplicial set with inner-horn-filling-property

Alternative choices:
- quasi-category
- weak Kan complex
- inner Kan complex
- quategory
- . . .
Segal’s picture on $\mathbb{E}_\infty$-monoids:

- $\mathcal{F}\text{in}_*$: category of finite pointed sets, pointed maps
- $\langle n \rangle \in \mathcal{F}\text{in}_*$: finite pointed set $\langle \{0,1,\ldots,n\},0 \rangle$
- $\rho_i: \langle n \rangle \to \langle 1 \rangle$: pointed map with $\rho_i^{-1}(1) = \{i\}$
- $\mathcal{C}$: $\infty$-category with finite products

**Definition**

An $\mathbb{E}_\infty$-**monoid** in $\mathcal{C}$ is a functor $M: N(\mathcal{F}\text{in}_*) \to \mathcal{C}$ such that the canonical maps

$$\rho_*: M_n \to M_1 \times \ldots \times M_1, \quad n \geq 0,$$

are equivalences.
$\mathcal{E}_\infty$-GROUPS IN $\infty$-CATEGORIES

An $\mathcal{E}_\infty$-monoid $M \colon N(\mathcal{F}\text{in}_*) \to \mathcal{C}$ gives rise to

- multiplication maps $\mu \colon M_1 \times M_1 \to M_1$,
- shear maps $\sigma = (\mu, \pi_2) \colon M_1 \times M_1 \to M_1 \times M_1$.

**DEFINITION**

An $\mathcal{E}_\infty$-monoid in $\mathcal{C}$ is an $\mathcal{E}_\infty$-group if equivalently:

1. There is an **inversion** map $M_1 \to M_1$.
2. The shear maps are equivalences.
3. The underlying commutative monoid in $\text{Ho}(\mathcal{C})$ is a group object.
Let $C$ be a pointed $\infty$-category with finite coproducts and finite products. Then $C$ is **preadditive** if the canonical maps $X \sqcup Y \to X \times Y$ are equivalences.

For such pointed $\infty$-category $C$ with finite coproducts and finite products the following are equivalent:

1. The $\infty$-category $C$ is preadditive.
2. The homotopy category $\text{Ho}(C)$ is preadditive.
3. The forgetful functor $\text{Mon}_{E_\infty}(C) \to C$ is an equivalence.
If $C$ has finite products, then $\text{Mon}_{\mathbb{E}_\infty}(C)$ is preadditive.

**Corollary**

If $C$ has finite products, then the forgetful functor $\text{Mon}_{\mathbb{E}_\infty}(\text{Mon}_{\mathbb{E}_\infty}(C)) \rightarrow \text{Mon}_{\mathbb{E}_\infty}(C)$ is an equivalence.

In preadditive $C$, canonical $\mathbb{E}_\infty$-monoid structures arise from fold maps $\nabla : X \oplus X \rightarrow X$. Associated to these we have shear maps $\sigma = (\nabla, \pi_2) : X \oplus X \rightarrow X \oplus X$. 
**Additive ∞-categories**

**Definition**

A preadditive ∞-category is **additive** if the shear maps \( \sigma : X \oplus X \to X \oplus X \) are equivalences.

**Proposition**

1. A preadditive ∞-category \( C \) is additive iff \( \text{Ho}(C) \) is additive iff \( \text{Grp}_{E_\infty}(C) \to C \) is an equivalence.

2. If \( C \) has finite products, then \( \text{Grp}_{E_\infty}(C) \) is additive and the functor \( \text{Grp}_{E_\infty}(\text{Grp}_{E_\infty}(C)) \to \text{Grp}_{E_\infty}(C) \) is an equivalence.
Let $\mathcal{P}_L$ be the $\infty$-category of presentable $\infty$-categories with morphisms the left adjoint functors.

**Proposition**

Let $\mathcal{C}$ be a presentable $\infty$-category. Then the $\infty$-categories $\text{Mon}_{\infty}(\mathcal{C})$ and $\text{Grp}_{\infty}(\mathcal{C})$ are presentable.

- $\text{Mon}_{\infty}(\mathcal{C})$ is an accessible localization of $\mathcal{C}^{\mathcal{N}(\mathcal{F}_{\text{in*}})}$.
- $\text{Grp}_{\infty}(\mathcal{C})$ is an accessible localization of $\text{Mon}_{\infty}(\mathcal{C})$.

**Corollary (Group completion)**

Given $\mathcal{C} \in \mathcal{P}_L$ then there are adjunctions:

$$\mathcal{C} \rightleftarrows \text{Mon}_{\infty}(\mathcal{C}) \rightleftarrows \text{Grp}_{\infty}(\mathcal{C})$$
The assignments $\mathcal{C} \mapsto \text{Mon}_{E_\infty}(\mathcal{C})$ and $\mathcal{C} \mapsto \text{Grp}_{E_\infty}(\mathcal{C})$ are obviously functorial in product-preserving functors.

**Proposition**

The assignments $\mathcal{C} \mapsto \text{Mon}_{E_\infty}(\mathcal{C})$ and $\mathcal{C} \mapsto \text{Grp}_{E_\infty}(\mathcal{C})$ define functors $\text{Mon}_{E_\infty}(-): \mathcal{P}r^L \to \mathcal{P}r^L$ and $\text{Grp}_{E_\infty}(-): \mathcal{P}r^L \to \mathcal{P}r^L$.

**Theorem**

1. The functor $\text{Mon}_{E_\infty}(-): \mathcal{P}r^L \to \mathcal{P}r^L$ is a localization with local objects the preadditive, presentable $\infty$-categories.
2. The functor $\text{Grp}_{E_\infty}(-): \mathcal{P}r^L \to \mathcal{P}r^L$ is a localization with local objects the additive, presentable $\infty$-categories.
Some Immediate Consequences

Let $\mathcal{C}, \mathcal{D}$ be presentable $\infty$-categories, and let $\mathcal{S}$ be the $\infty$-category of spaces (‘free homotopy theory on $\Delta^0$’).

**Corollary (‘Preadditivization’)**

If $\mathcal{D}$ is preadditive, then $\mathcal{C} \to \text{Mon}_{\mathcal{E}_\infty}(\mathcal{C})$ induces a canonical equivalence $\text{Fun}^L(\text{Mon}_{\mathcal{E}_\infty}(\mathcal{C}), \mathcal{D}) \to \text{Fun}^L(\mathcal{C}, \mathcal{D})$. In particular, we obtain $\text{Fun}^L(\text{Mon}_{\mathcal{E}_\infty}(\mathcal{S}), \mathcal{D}) \simeq \mathcal{D}$.

**Corollary (‘Additivization’)**

If $\mathcal{D}$ is additive, then $\mathcal{C} \to \text{Grp}_{\mathcal{E}_\infty}(\mathcal{C})$ induces a canonical equivalence $\text{Fun}^L(\text{Grp}_{\mathcal{E}_\infty}(\mathcal{C}), \mathcal{D}) \to \text{Fun}^L(\mathcal{C}, \mathcal{D})$. In particular, we obtain $\text{Fun}^L(\text{Grp}_{\mathcal{E}_\infty}(\mathcal{S}), \mathcal{D}) \simeq \mathcal{D}$. 
A refined picture of the stabilization

\[ \mathcal{P}_{rPt} \] pointed presentable \( \infty \)-categories
\[ \mathcal{P}_{rPre} \] preadditive presentable \( \infty \)-categories
\[ \mathcal{P}_{rAdd} \] additive presentable \( \infty \)-categories
\[ \mathcal{P}_{rSt} \] stable presentable \( \infty \)-categories

**Theorem (Stabilization)**

1. The stabilization of presentable \( \infty \)-categories factors as
\[ \mathcal{P}_{rL} \rightleftarrows \mathcal{P}_{rPt} \rightleftarrows \mathcal{P}_{rPre} \rightleftarrows \mathcal{P}_{rAdd} \rightleftarrows \mathcal{P}_{rSt}. \]

2. In particular, for \( C \in \mathcal{P}_{rL} \), the suspension spectrum functor factors as
\[ \Sigma^\infty_+: C \to C_* \to \text{Mon}_{E_\infty}(C) \to \text{Grp}_{E_\infty}(C) \to \text{Sp}(C). \]
∞-CAT EgORICAL ‘DIRECT SUM’ K-THEORY

Classical picture:

\[ \mathcal{C} \] symmetric, monoidal category
\[ \tilde{\mathcal{C}} \] largest subgroupoid of \( \mathcal{C} \)
\[ |\tilde{\mathcal{C}}| \] associated \( \mathbb{E}_\infty \)-space
\[ K(\mathcal{C}) \] K-theory spectrum via group-completion

Some steps towards an \( \infty \)-categorical variant are:

1. The inclusion \( \mathcal{S} = \mathcal{G}_{\text{rp}} \to \mathcal{C}_{\text{at}} \) admits a right adjoint given by \( \mathcal{C} \mapsto \tilde{\mathcal{C}} \).

2. The \( \infty \)-categories \( \text{Sym MonCat}_{\infty} \) and \( \text{Mon}_{\mathbb{E}_\infty}(\mathcal{C}_{\text{at}}) \) are equivalent, compatibly with \( \infty \)-groupoids.
The algebraic K-theory $\text{SymMonCat}_\infty \to \text{Sp}$ is defined as the composition

$$\text{Mon}_{E_\infty}(\text{Cat}_\infty) \to \text{Mon}_{E_\infty}(S) \to \text{Grp}_{E_\infty}(S) \to \text{Sp}(S).$$

Will be discussed further by Thomas Nikolaus in the second part of this talk!

Thanks for your attention!