Categorical Aspects of Type Theory

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Motivation: To understand Martin-Löf type theory.

Conceptual mathematics $\rightarrow$ category theory.

Two questions:

- Is type theory soluble in category theory?
- Is category theory soluble in type theory?

I will not discuss the second question here.
Overview

Aspects of categorical logic:
- Locally cartesian closed categories
- Tribes

Homotopical logic:
- Weak factorization systems
- Homotopical algebra
- Pre-typoi
- Typoi
- Univalent typoi
Aspects of categorical logic

The basic principles of categorical logic was expressed in Lawvere’s paper *Adjointness in Foundation* (1969). I will use these principles implicitly.

- Locally cartesian closed categories
- Tribes
- Π-tribes
Terminal objects and terms

Recall that an object \( \top \) in a category \( \mathcal{C} \) is said to be \textbf{terminal} if for every object \( A \in \mathcal{C} \), there is a unique map \( A \to \top \).

If \( \top \) is a terminal object, then a map \( u : \top \to A \) is called

- a \textbf{global section} of the object \( A \), \( u \in \Gamma(A) \)
- an \textbf{element} of \( A \), \( u \in A \)
- a \textbf{constant} of sort \( A \), \( u \in A \)
- a \textbf{term} of \textbf{type} \( A \), \( u : A \).
Cartesian product

Recall that the **cartesian product** \( A \times B \) of two objects \( A \) and \( B \) in a category \( C \) is an object \( A \times B \) equipped with a pair of *projection*

\[
A \xleftarrow{p_1} A \times B \xrightarrow{p_2} B
\]

having the following universal property: for any object \( C \in C \) and any a pair of maps

\[
A \xleftarrow{f} C \xrightarrow{g} B,
\]

there is a unique map \( h = \langle f, g \rangle : C \to A \times B \) such that \( p_1 h = f \) and \( p_2 h = g \).
Cartesian category

The map \( h \mapsto (p_1 h, p_2 h) \) is a natural bijection between

\[
\begin{align*}
\text{the maps} & \quad C \to A \times B \\
\text{and the pairs of maps} & \quad C \to A, \ C \to B.
\end{align*}
\]

A category \( C \) is \textbf{cartesian} if it has (binary) cartesian product and a terminal object \( \top \).

Equivalently, a category \( C \) is cartesian if it has finite cartesian products.
Let $A$ and $B$ be two objects of a cartesian category $C$.

An object $[A, B]$ equipped with a map $\text{ev} : [A, B] \times A \to B$ is called the \textbf{exponential} of $B$ by $A$ if:

for every object $C \in C$ and every map $f : C \times A \to B$ there exists a unique map $\lceil f \rceil : C \to [A, B]$ such that

\[
\begin{array}{ccc}
C \times A & \xrightarrow{\lceil f \rceil \times A} & [A, B] \times A \\
\downarrow{f} & & \downarrow{\text{ev}} \\
B & & B
\end{array}
\]

We write $\lambda^A f := \lceil f \rceil$. 

Cartesian closed categories

The map \( f \mapsto \lambda^A f \) is a natural bijection between

\[
\text{the maps } C \times A \to B \\
\text{and the maps } C \to [A, B].
\]

A cartesian category \( C \) is said to be \textbf{closed} if the object \([A, B]\) exists for every pair of objects \( A, B \in C \).

A cartesian category \( C \) is closed if and only if the functor

\[
A \times (-) : C \to C
\]

has a right adjoint \([A, -]\) for every object \( A \in C \).
Examples of cartesian closed categories

- the category of sets \( \textbf{Set} \)
- the category of (small) catégories \( \textbf{Cat} \) (Lawvere)
- the category of groupoids \( \text{Grpd} \)
- Every category \( [\mathbb{C}, \textbf{Set}] \)
- The category of simplicial sets \( [\Delta^{op}, \textbf{Set}] \)
Every cartesian category $\mathcal{A}$ generates freely a cartesian closed category $CC[\mathcal{A}]$.

- the morphism in $CC[\mathcal{A}]$ are represented by lambda terms;
- lambda terms have a normal form;
- the category $CC[\mathcal{A}]$ is decidable if $\mathcal{A}$ is decidable.

Lambek and Scott: *Higher categorical logic*. 
Slice categories

Recall that the slice category $\mathcal{C}/A$ has for objects the pairs $(X, p)$, where $p$ is a map $X \to A$ in $\mathcal{C}$. The map $p : X \to A$ is called the structure map of $(X, p)$.

A morphism $(X, p) \to (Y, q)$ in $\mathcal{C}/A$ is a map $u : X \to Y$ in $\mathcal{C}$ such that $qu = p$,

\[
\begin{array}{ccc}
X & \xrightarrow{u} & Y \\
p \downarrow & & \downarrow q \\
A & & A
\end{array}
\]
Push-forward

To every map $f : A \to B$ in a category $C$ we can associate a **push-forward** functor

$$f_! : C/A \to C/B$$

by putting $f_!(X, p) = (X, fp)$ for every map $p : X \to A$,

\[
\begin{array}{ccc}
X & \xrightarrow{\cong} & X \\
\downarrow p & & \downarrow fp \\
A & \xrightarrow{f} & B.
\end{array}
\]
Recall that the fiber product of two maps $X \to A$ and $Y \to A$ in a category $\mathcal{C}$ is their cartesian product $X \times_A Y$ as objects of the category $\mathcal{C}/A$.

The square is also called a pullback square.
Base changes

In a category with finite limits $\mathcal{C}$ the push-forward functor $f_! : \mathcal{C}/A \to \mathcal{C}/B$ has a right adjoint

$$f^* : \mathcal{C}/B \to \mathcal{C}/A$$

for any map $f : A \to B$. The functor $f^*$ takes a map $p : X \to B$ to the map $p_1 : A \times_B X \to A$ in a pullback square

$$
\begin{array}{ccc}
A \times_B X & \xrightarrow{p_2} & X \\
p_1 \downarrow & & \downarrow p \\
A & \xrightarrow{f} & B.
\end{array}
$$

The map $p_1$ is said to be the **base change** of the map $p : X \to B$ along the map $f : A \to B$. 
Locally cartesian closed categories

A category with finite limits $\mathcal{C}$ is said to be **locally cartesian closed** (lcc) if the category $\mathcal{C}/A$ is cartesian closed for every object $A \in \mathcal{C}$.

A category with finite limits $\mathcal{C}$ is lcc if and only if the base change functor $f^{\ast} : \mathcal{C}/B \to \mathcal{C}/A$ has a right adjoint

$$
  f_{\ast} : \mathcal{C}/A \to \mathcal{C}/B
$$

for every map $f : A \to B$ in $\mathcal{C}$. 
If $X = (X, p) \in C/A$, then $f_*(X) \in C/B$ is called the \textbf{internal product} of $X$ \textit{along} $f : A \to B$, and denoted

$$\Pi_f(X) := f_*(X).$$

If $Y = (Y, q) \in C/B$, there is a natural bijection between

| the maps $Y \to \Pi_f(X)$ in $C/B$ | and the maps $f^*(Y) \to X$ in $C/A$ |

Locally cartesian closed categories(2)
Locally cartesian closed categories

Examples of lcc categories

- the category $\text{Set}$
- every category $[\mathcal{C}, \text{Set}]$
- every Grothendieck topos
- every elementary topos

Non-examples

The category $\text{Cat}$

The category $\text{Grpd}$
Logical functors

**Definition**
A functor $F : \mathcal{C} \to \mathcal{D}$ between lcc categories is **logical** if it preserves

- finite limits;
- internal products.

The last condition means that the comparison map

$$F\Pi f(X) \to \Pi_{F(f)}(FX)$$

is an isomorphism for any pair of maps $X \to A$ and $f : A \to B$ in $\mathcal{C}$. 
If $\mathcal{C}$ is a locally cartesian closed category, then

- the Yoneda functor $y : \mathcal{C} \to \hat{\mathcal{C}} = [\mathcal{C}^{\text{op}}, \text{Set}]$ is logical;

- the base change functor $f^* : \mathcal{C}/B \to \mathcal{C}/A$ is a logical for every map $f : A \to B$ in $\mathcal{C}$. 
Let $i : \mathcal{C} \to \mathcal{C}/A$ be the base change functor. By definition, $i(X) = (A \times X, p_1)$ for every $X \in \mathcal{C}$.

**Theorem**

The functor $i : \mathcal{C} \to \mathcal{C}/A$ is logical and $\mathcal{C}/A$ is obtained from $\mathcal{C}$ by adding freely a term $x_A : i(A)$.

More precisely, $i(\top) = (A, 1_A)$ and $i(A) = (A \times A, p_1)$.

The diagonal $A \to A \times A$ is a map $x_A : i(\top) \to i(A)$. 
For any logical functor $F : C \rightarrow \mathcal{E}$ with values in a lcc category $\mathcal{E}$ and any term $a : F(A)$, there exists a logical functor $F' : C/A \rightarrow D$ and a natural isomorphism $\alpha : F \simeq F' \circ i$

such that $\alpha_A(a) = F'(x_A)$.

Moreover, the pair $(F', \alpha)$ is unique up to a unique iso of pairs. Thus, $C/A = C[x_A]$ and the term $x_A : i(A)$ is generic.
A class of maps $\mathcal{F}$ in a category $C$ is said to be **closed under base changes** if

\[ X \to B \text{ in } \mathcal{F} \implies A \times_B X \to A \text{ exists and belong to } \mathcal{F} \]

for any map $f : A \to B$ in $C$
Definition

Let $\mathcal{C}$ be a category with terminal object $\top$. We say that a class of maps $\mathcal{F} \subseteq \mathcal{C}$ is a tribe structure if the following conditions are satisfied:

- every isomorphism belongs to $\mathcal{F}$;
- $\mathcal{F}$ is closed under composition and base changes;
- the map $X \to \top$ belongs to $\mathcal{F}$ for every object $X \in \mathcal{C}$.

We shall say that the pair $(\mathcal{C}, \mathcal{F})$ is a tribe.

A map in $\mathcal{F}$ is a family or a fibration of the tribe.
The fiber of a fibration $p : X \to A$ at a point $a : \top \to A$ is the object $X(a)$ defined by the pullback square

$$
\begin{array}{ccc}
X(a) & \to & X \\
\downarrow & & \downarrow \text{p} \\
\top & \to & A.
\end{array}
$$

A fibration $p : X \to A$ is an internal family $(X(a) : a \in A)$ of objects parametrized by the codomain of $p$. 
The full subcategory of $\mathcal{C}/A$ whose objects are the fibrations $X \to A$ is denoted $\mathcal{C}(A)$.

The category $\mathcal{C}(A)$ has the structure of a tribe where a morphism $f : (X, p) \to (Y, q)$ in $\mathcal{C}(A)$ is a \textit{fibration} if $f : X \to Y$ is a fibration in $\mathcal{C}$.

And object of $\mathcal{C}(A)$ is a type which \textit{depends} on the type $A$.

If $u : A \to B$, then the base change functor $u^* : \mathcal{C}(B) \to \mathcal{C}(A)$ is an operation of \textit{change of parameters}: we have

$$u^*(Y)(a) = Y(u(a))$$

for every every fibration $Y \to B$ and every term $a : A$. 
Tribes (4)

Definition
A **morphism of tribes** \( F : \mathcal{C} \to \mathcal{D} \) is a functor which

- takes fibrations to fibrations;
- preserves base changes of fibrations;
- preserves terminal objects.

For example, the base change functor

\[
u^* : \mathcal{C}(B) \to \mathcal{C}(A)\]

is a morphism of tribes for any map \( u : A \to B \) in a tribe \( \mathcal{C} \).
The base change functor $i : \mathcal{C} \to \mathcal{C}(A)$ is a morphism of tribes.

**Theorem**

*The tribe $\mathcal{C}(A)$ is obtained from $\mathcal{C}$ by adding freely a term $x_A : i(A)$.*

The term $x_A : i(A)$ is **generic**.
Types and contexts

An object \( p : E \to A \) of \( C(A) \) is a **type** \( E(x) \) in **context** \( x : A \).

Type theorists write

\[
\vdash x : A \vdash E(x) : type
\]

where \( E(x) \) is the general fiber of the map \( p : E \to A \),

\[
\begin{array}{ccc}
E(x) & \to & E \\
\downarrow & & \downarrow p \\
\top & \xrightarrow{x} & A
\end{array}
\]

The object \( E \) is the **total space** of the fibration \( p : E \to A \),

\[
E = \sum_{x:A} E(x).
\]
Terms and types

A term $t(x)$ of type $E(x)$ is a section $t$ of the map $p : E \to A$.

Type theorists write

\[
x : A \vdash t(x) : E(x)
\]

Topologists write

\[
\begin{array}{c}
  E \\
  \downarrow p \\
  A \\
  \uparrow t
\end{array}
\]
Push-forward and sum

To every fibration $f : A \to B$ in tribe $C$ we can associate a *push-forward* functor

$$f_! : C(A) \to C(B)$$

by putting $f_!(E, p) = (E, fp)$,

$$
\begin{array}{ccc}
E & \xrightarrow{p} & E \\
\downarrow & & \downarrow \\
A & \xrightarrow{f} & B
\end{array}
$$

Formally,

$$f_!(E)(b) = \sum_{f(a) = b} E(a)$$

for every fibration $E \to A$ and every $b : B$. 
The functor $f_! : \mathcal{C}(A) \to \mathcal{C}(B)$ is left adjoint to the functor $f^*$. 

For very $X \in \mathcal{C}(A)$ and $Y \in \mathcal{C}(B)$, there is a natural bijection between

\[
\begin{array}{ccc}
\text{the maps} & f_!(X) \to Y & \text{in } \mathcal{C}(B) \\
\text{and the maps} & X \to f^*(Y) & \text{in } \mathcal{C}(A)
\end{array}
\]
Sum formation

\[
\begin{align*}
\Gamma, \; x : A \vdash E(x) : \text{type} \\
\Gamma \vdash \sum_{x : A} E(x) : \text{type}
\end{align*}
\]

\[
\begin{array}{c}
E \\
\downarrow (p, q)
\end{array} \quad \begin{array}{c}
\downarrow p
\end{array}
\]

\[
\begin{array}{c}
\Gamma \times A \\
p_1 \\
\downarrow \Gamma
\end{array}
\]
**Definition**

We shall sat that a tribe $C$ is $\Pi$-**closed**, or that it is a $\Pi$-**tribe**, if every fibration $E \to A$ has a product along every fibration $f : A \to B$,

\[
\begin{array}{ccc}
E & \xrightarrow{\Pi_f(E)} & \downarrow \\
\downarrow & & \downarrow \\
A & \xrightarrow{f} & B
\end{array}
\]

and the structure map $\Pi_f(E) \to B$ is a fibration.

The object $\Pi_f(E)$ is a **product** of $E = (E, p)$ along $f$. Formally,

\[
\Pi_f(E)(b) = \prod_{f(a) = b} E(a)
\]

for every $b \in B$. 

\[\pi\text{-}tribes\]
It follows that the base change functor $f^* : \mathcal{C}(B) \to \mathcal{C}(A)$ has a right adjoint

$$f_* = \Pi_f : \mathcal{C}(A) \to \mathcal{C}(B)$$

for every fibration $f : A \to B$.

For very $X \in \mathcal{C}(A)$ and $Y \in \mathcal{C}(B)$, there is a natural bijection between

<table>
<thead>
<tr>
<th>the maps</th>
<th>$Y \to \Pi_f(X)$</th>
<th>in $\mathcal{C}(B)$</th>
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<tbody>
<tr>
<td>and the maps</td>
<td>$f^*(Y) \to X$</td>
<td>in $\mathcal{C}(A)$</td>
</tr>
</tbody>
</table>
Product formation

\[
\Gamma, \ x : A \vdash E(x) : type \\
\Gamma \vdash \prod_{x:A} E(x) : type
\]
If $C$ is a $\Pi$-tribe, then so is the tribe $C(A)$ for every $A \in C$.

A $\Pi$-tribe is cartesian closed:

$$B^A = \Pi_A B = \prod_{a:A} B$$

The category $C(A)$ is cartesian closed for every $A \in C$. 

Examples of Π-tribes

- Every locally cartesian closed category is a Π-tribe.
- The category of small groupoids $\text{Grpd}$ is a Π-tribe, where a fibration is a Grothendieck fibration.
- The category of Kan complexes is a Π-tribe, where a fibration is a Kan fibration.
Morphisms of Π-tribes

Definition
A morphism of Π-tribes $F : \mathcal{C} \to \mathcal{D}$ is a functor which preserves
▶ terminal objects, fibrations and base changes of fibrations;
▶ the internal product $\Pi_f(X)$.

The base change functor $u^* : \mathcal{C}(B) \to \mathcal{C}(A)$ is a morphism of
Π-tribes for any map $u : A \to B$ in a Π-tribe $\mathcal{C}$.

The Yoneda functor $y : \mathcal{C} \to \hat{\mathcal{C}} = [\mathcal{C}^{op}, \textbf{Set}]$ is a morphism of
Π-tribes for any Π-tribe $\mathcal{C}$.
Homotopical logic

- Weak factorization systems
- Quillen model categories
- Pre-typoi
- Typoi
- Univalent typoi
Weak factorisation systems

The relation $u \pitchfork f$ for two maps $u : A \to B$ and $f : X \to Y$ in a category $C$ means that every commutative square

\[
\begin{array}{ccc}
A & \xrightarrow{a} & X \\
\downarrow{u} & & \downarrow{f} \\
B & \xrightarrow{b} & Y
\end{array}
\]

has a diagonal filler $d : B \to X$, $du = a$ and $fd = b$.

The map $u$ is said to have the **left lifting property** with $f$, and the map $f$ to have the **right lifting property** with respect to $u$. 
Weak factorisation systems (2)

For a class of maps $S \subseteq C$, let us put

\[
S^\llcorner = \{ f \in C : \forall u \in S \ u \llcorner f \} \\
\llcorner S = \{ u \in C : \forall f \in S \ u \llcorner f \}
\]

Definition

A pair $(\mathcal{L}, \mathcal{R})$ of classes of maps in a category $C$ is said to be a **weak factorization system** if the following two conditions are satisfied

- $\mathcal{R} = \mathcal{L}^\llcorner$ and $\mathcal{L} = \llcorner \mathcal{R}$
- every map $f : A \to B$ in $C$ admits a factorization $f = pu : A \to E \to B$ with $u \in \mathcal{L}$ and $p \in \mathcal{R}$. 
Recall that a class $\mathcal{W}$ of maps in a category $\mathcal{E}$ is said to have the \textbf{3-for-2 property} (3 apples for the price of two!) if two sides a commutative triangle

\[
\begin{array}{ccc}
A & \xrightarrow{u} & B \\
\downarrow_{uv} & & \downarrow_{v} \\
C & & 
\end{array}
\]

belongs to $\mathcal{W}$, then so is the third.
Homotopical algebra(2)

Quillen (1967)

Definition
A model structure on a category $\mathcal{E}$ consists on three class of maps $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ respectively called the cofibrations, the weak equivalences and the fibrations, such that:

- $\mathcal{W}$ has the 3-for-2 property;
- the pair $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ is a weak factorisation system;
- the pair $(\mathcal{C}, \mathcal{F} \cap \mathcal{W})$ is a weak factorisation system.

A model category is a category equipped with a model structure.

A map in $\mathcal{W}$ is said to be acyclic.
A **path object** for a fibrant object $X$ in a model category $\mathcal{E}$ is a factorisation of the diagonal $\Delta: X \to X \times X$ as a weak equivalence $\sigma: X \to PX$ followed by a fibration $(\partial_0, \partial_1): PX \to X \times X$.

![Diagram](https://example.com/diagram.png)

The path object is **perfect** if $\sigma$ is an acyclic cofibration.
Identity type

For every type $A$ there is another type

$$x : A, y : A \vdash ld_A(x, y) : type$$

called the identity type of $A$ and a term

$$x : A \vdash r(x) : ld_A(x, x)$$

called the reflexivity term.

A term $p : ld_A(a, b)$ is a proof that $a = b$.

The term $r(x) : ld_A(x, x)$ is the proof that $x = x$. 
Equivalently, for every $A \in \mathcal{C}$ there is a diagram

$$
\begin{array}{ccc}
A & \xrightarrow{\Delta} & A \times A \\
\downarrow & & \downarrow \\
\downarrow & & \Downarrow \\
A & \xrightarrow{r} & Id_A \\
\end{array}
$$

with $(s, t) \in \mathcal{F}$. 

Identity type(2)
The *J*-rule of type theory

If \( p : X \to \text{Id}_A \) is a fibration, then every commutative square

\[
\begin{array}{ccc}
A & \xrightarrow{u} & X \\
\downarrow r & & \downarrow p \\
\text{Id}_A & = & \text{Id}_A
\end{array}
\]

has a diagonal filler \( d = J(u) \),

\[
\begin{array}{ccc}
A & \xrightarrow{u} & X \\
\downarrow r & & \downarrow p \\
\text{Id}_A & = & \text{Id}_A
\end{array}
\]
Homotopical algebra and type theory(1)

Theorem (Awodey-Warren):

Martin-Löf type theory can be interpreted in a model category:

- types are interpreted as fibrant objects;
- display maps are interpreted as fibrations;
- the identity type \( \text{Id}_A \rightarrow A \times A \) is a path object for \( A \);
- the reflexivity term \( r : A \rightarrow \text{Id}_A \) is an acyclic cofibration.
Let $C(\mathbb{T})$ be the syntactic category of Martin-Löf type theory.

Let $\mathcal{F}$ be the class of display maps in $C(\mathbb{T})$.

**Theorem (Gambino-Garner):**

Every map $f : A \to B$ in $C(\mathbb{T})$ admits a factorization $f = pu : A \to E \to B$ with $u \in \cap \mathcal{F}$ and $p \in \mathcal{F}$.
**Pre-typoi**

We say that a map in a tribe $\mathcal{C} = (\mathcal{C}, \mathcal{F})$ is **anodyne** if it belongs to the class $\pitchfork \mathcal{F}$.

**Definition**

We say that a tribe $\mathcal{C}$ is a **pre-typos** if the following two conditions are satisfied

- the base change of an anodyne map along a fibration is anodyne;
- every map $f : A \to B$ admits a factorization $f = pu : A \to E \to B$ with $u$ an anodyne map and $p$ a fibration.

(★) Named after a joke by Steve Awodey. Do you have a better name?
Examples

- The category \textbf{Grpd};
- The category of Kan complexes;
- The syntactic category of type theory.
Path objects in a pre-typos

If $X$ is an object of a typos $\mathcal{C}$, then a perfect* path object for $X$ is a factorisation

$$\langle \partial_0, \partial_1 \rangle \sigma : X \to PX \to X \times X$$

of the diagonal $X \to X \times X$ as an anodyne map $\sigma : X \to PX$ followed by a fibration $\langle \partial_0, \partial_1 \rangle : PX \to X \times X$.

(*) The general notion of path objects will be introduced later.
Paths and equality

The map $\langle \partial_0, \partial_1 \rangle : PX \to X \times X$ of a path object for $X$ is a fibration. Its fiber $PX(x, y)$ at $(x, y) \in X \times X$ is the object of paths $p : x \rightsquigarrow y$. We may write

$$\Gamma \vdash h : f \rightsquigarrow g$$

to indicate that $h : \Gamma \to PX$ is a homotopy between two maps $f, g : \Gamma \to X$.

Type theorists write instead

$$\Gamma \vdash h : \text{Id}_X(f, g)$$

and regard $h$ as a proof that $f = g$. Weird?
Homotopy relation

A **homotopy** between two maps \( f, g : X \rightarrow Y \) in a typos \( C \) is a map \( h : X \rightarrow PY \) such that \( \partial_0 h = f \) and \( \partial_1 h = g \).

![Diagram showing homotopy relation]

We write \( H : f \sim g \) or \( f \sim g \).
Homotopy equivalences

Theorem
The homotopy relation $f \sim g$ is a congruence on the arrows of the category $\mathcal{C}$.

The homotopy category $Ho(\mathcal{C})$ is the quotient category $\mathcal{C}/\sim$.

A map $f : X \to Y$ in $\mathcal{C}$ is a homotopy equivalence if it is invertible in $Ho(\mathcal{C})$.

For example, every anodyne map is a homotopy equivalence.

An object $X \in \mathcal{C}$ is contractible if the map $X \to \top$ is a homotopy equivalence.
General path objects

If $X$ is an object of a typos $C$, then a (general) **path object** for $X$ is a factorisation

$$\langle \partial_0, \partial_1 \rangle \sigma : X \to PX \to X \times X$$

of the diagonal $X \to X \times X$ as a *homotopy equivalence* $\sigma$ followed by a fibration $\langle \partial_0, \partial_1 \rangle$.

The path object is **perfect** if $\sigma : X \to PX$ is anodyne.
Mapping path object

A mapping path object of a map \( f : A \to B \) is a factorisation

\[
\langle q_0, q_1 \rangle u : A \to M(f) \to A \times B
\]

of the map \( \langle 1_A, f \rangle : A \to A \times B \) as a homotopy equivalence \( u \) followed by a fibration \( \langle q_0, q_1 \rangle \),

\[
\begin{array}{ccc}
A & \xrightarrow{u} & M(f) \\
\downarrow{1_A} & & \downarrow{q_0} \\
A & \xrightarrow{q_1} & B
\end{array}
\]

The mapping path object is perfect if \( u \) is anodyne.
Homotopy fiber

A mapping path object of a map $f : A \to B$ can be constructed by the following diagram with a pull-back square

Thus, $M(f) = \{(p, x, y) | x : A, y : B, \ p : f(x) \sim y\}$. The fiber of the projection $M(f) \to B$ at $b \in B$ is the **homotopy fiber** of $f$.

$$M(f)(b) = \{(p, x) | x : A, \ p : f(x) \sim y\}$$
Let $C$ be a pre-typos. If $X \in C$, then the fibration $\langle \partial_0, \partial_1 \rangle : PX \to X \times X$ is an object $P(X)$ of $C(X \times X)$.

**Definition**
We say that an object $X \in C$ is a (-1)-**type** if $P(X)$ is contractible in the pre-typos $C(X \times X)$.

An object $X \in C$ is a (-1)-type if and only if the map $X \to X \times X$ is a homotopy equivalence.

A (-1)-type is like a *truth value*.
**Definition**

If $n \geq 0$, then an object $X \in C$ is said to be a $n$-type if $P(X)$ is a $(n-1)$-type in $C(X \times X)$.

A 0-type is like a set.

A 1-type is like a groupoid.

A 1-type is like a 2-groupoid.
Morphisms of pre-typoi

Definition

A morphism of pre-typoi \( F : \mathcal{C} \to \mathcal{D} \) is a functor which preserves

- terminal objects, fibrations and base changes of fibrations;
- the homotopy relation.

For example, the base change functor \( u^* : \mathcal{C}(B) \to \mathcal{C}(A) \) is a morphism of pre-typoi for any map \( u : A \to B \) of a pre-typos \( \mathcal{C} \).
Definition

A pre-typos $C$ is called a *typos* if it is a $\Pi$-tribe and the product functor $\Pi_f : C(A) \to C(B)$ preserves the homotopy relation for every fibration $f : A \to B$.

If $C$ is a typos, then so is the tribe $C(A)$ for any object $A \in C$.

(*) Do you have a better name?
Examples

**Theorem**

*(Hoffman and Streicher)* The category of groupoids $\text{Grpd}$ has the structure of a typos in which the fibrations are the Grothendieck fibrations.

**Theorem**

*(Awodey-Warren-Voevodsky)* The category of Kan complexes has the structure of a typos in which the fibrations are the Kan fibrations.

**Theorem**

*(Gambino-Garner)* The syntactic category of type theory has the structure of a typos in which the fibrations are constructed from the display maps.
From typoi to hyperdoctrines

If $u : A \to B$ is a map in a typos $C$, then the functor

$$\text{Ho}(u^*) : \text{Ho}(\mathcal{C}(B)) \to \text{Ho}(\mathcal{C}(A))$$

has a both a left adjoint and a right adjoint.

The functor

$$A \mapsto \text{Ho}(\mathcal{C}(A))$$

is a hyper-doctrine in the sense of Lawvere!
Morphisms of typoi

Definition
A morphism of typoi $F : C \to D$ is a functor which preserves
  ▶ terminal objects, fibrations and base changes of fibrations;
  ▶ the internal products $\Pi_f(X)$;
  ▶ the homotopy relation.

For example, the base change functor $u^* : C(B) \to C(A)$ is a
morphism of typoi for any map $u : A \to B$ in a typos $C$. 
Let $A$ be an object of a typos $\mathcal{C}$.

If $PA \to A \times A$ is a path object for $A$, then the object

$$T_{-1}(A) = \prod_{x:A} \prod_{y:A} PA(x, y)$$

is the internal statement that $A$ is a $(-1)$-type.

A term $p : T_{-1}(A)$ is a proof that $A$ is a $(-1)$-type.
The object
\[ \text{Cont}(A) = A \times T_{-1}(A) \]
is the internal statement that $A$ is contractible.

A term $p : A \times T_{-1}(A)$ is a proof that $A$ is contractible.

If $n \geq 0$, then the object
\[ T_n(A) = \prod_{x:A} \prod_{y:A} T_{n-1}(PA(x, y)) \]
is the internal statement that $A$ is a $n$-type.

A term $p : T_n(A)$ is a proof that $A$ is a $n$-type.
Internal equivalences

A fibration $p : X \to B$ is a homotopy equivalence if and only if $\Pi_B(X, p)$ is contractible.

Thus, $\text{Cont}(\Pi_B(X, p))$ is the internal statement that $p : X \to A$ is a homotopy equivalence.

A general map $f : A \to B$ is a homotopy equivalence if and only if the fibration $q_1 : M(f) \to B$ is a homotopy equivalence.

Thus, $\text{Cont}(\Pi_B(M(f), q_1))$ is the internal statement that $f : A \to B$ is a homotopy equivalence.
Classifying equivalences

For any pair of objects $X$ and $Y$ of a typos, there is an object $Eq(X, Y)$ classifying the homotopy equivalences $X \rightarrow Y$.

For every fibration $X \rightarrow A$, there is a category object

$$(s, t) : Eq_A(X) \rightarrow A \times A$$

where

$$Eq_A(X)(a, b) = Eq(X(a), X(b))$$

for $a : A$ and $b : A$. 
Univalent fibrations

Definition
We say that a fibration $X \to A$ is univalent if the unit map $u : A \to Eq_A(X)$ is an equivalence.

A fibration $X \to A$ is univalent if and only if the factorization

$$
\begin{array}{ccc}
Eq_A(X) & \to & A \\
\downarrow & & \downarrow \\
\downarrow & & \downarrow \\
A & \to & A \times A \\
(1_A,1_A) & \nearrow & (s,t)
\end{array}
$$

is a path object for $A$. 
Small fibrations and universes

A typos $\mathcal{C}$ may contain a sub-typos of small fibrations.

A small fibration $q : U' \to U$ is universal if for every small fibration $p : X \to A$ there exists a cartesian square:

$$
\begin{array}{ccc}
X & \xrightarrow{\chi'} & U' \\
\downarrow{p} & & \downarrow{q} \\
A & \xrightarrow{\chi} & U.
\end{array}
$$

The map $\chi$ is classifying $(X, p)$.

A universe is the codomain of a universal small fibration $U' \to U$.

Martin-Löf axiom: There is a universe $U$. 
Univalent typoi

We would like to say that the pair \((\chi, \chi')\) classifying a fibration \(p : X \to A\) is homotopy unique.

Voevodsky axiom: *The universal fibration \(U' \to U\) is univalent.*

Theorem (Voevodsky)
The category of Kan complexes \(\textbf{Kan}\) has the structure of a univalent typos in which the fibrations are the Kan fibrations.
Conclusions

Homotopy type theory is soluble in category theory
Bibliography

THE BOOK OF INFORMAL TYPE THEORY