Introduction to type theory and homotopy theory

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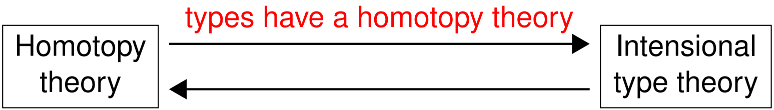
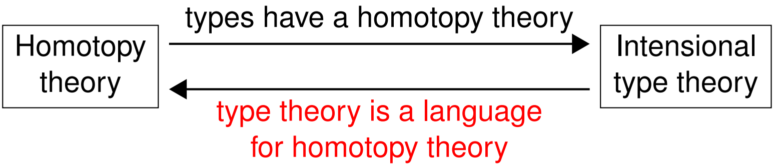
Homotopy type theory



- New perspectives on extensional vs. intensional
- Univalence: the correct identity types for a universe
- Homotopy levels: insight into proof-irrelevance
- Higher inductive types: quotients and free structures

Homotopy type theory



- Type theory is a formal system, like ZFC, but. . .
  - More computer-friendly
  - Naturally constructive
  - Can formalize homotopy theory more directly
- The same proof can apply to many homotopy theories (equivariant, parametrized, sheaves, . . .)

Outline

Typing judgments

Type theory consists of rules for manipulating judgments.
The most important judgment is a typing judgment:

$$x_1 : A_1,\ x_2 : A_2,\ \ldots x_n : A_n \ \vdash\ b : B$$

The turnstile $\vdash$ binds most loosely, followed by commas.
This should be read as:

In the context of variables x_1 of type A_1, x_2 of type A_2, ..., and x_n of type A_n, the expression b has type B.

Examples

$$\begin{aligned} &\vdash\ 0 : \mathbb{N} \\ x : \mathbb{N},\ y : \mathbb{N} &\vdash\ x + y : \mathbb{N} \\ f : \mathbb{R} \to \mathbb{R},\ x : \mathbb{R} &\vdash\ f(x) : \mathbb{R} \\ f : C^\infty(\mathbb{R}, \mathbb{R}),\ n : \mathbb{N} &\vdash\ f^{(n)} : C^\infty(\mathbb{R}, \mathbb{R}) \end{aligned}$$

Dependent types

We consider types A_i, B as also expressions, of type "Type".

Examples

$$
\begin{aligned}
&\vdash\ \mathbb{N} \colon \mathrm{Type}\\
A \colon \mathrm{Type},\ x \colon A &\vdash\ x \colon A\\
A \colon \mathrm{Type},\ B \colon A \to \mathrm{Type},\ x \colon A &\vdash\ B(x) \colon \mathrm{Type}\\
n \colon \mathbb{N} &\vdash\ \{k \colon \mathbb{N} \mid k < n\} \colon \mathrm{Type}\\
f \colon \mathbb{R} \to \mathbb{R} &\vdash\ \{x \colon \mathbb{R} \mid f(x) = 0\} \colon \mathrm{Type}
\end{aligned}
$$

A judgment $x \colon A \vdash B(x) \colon \mathrm{Type}$, or a term $B \colon A \to \mathrm{Type}$, is a dependent type (over A).

Type constructors

$$
\begin{aligned}
A\colon \text{Type}, B\colon \text{Type} &\vdash A \times B\colon \text{Type} \\
A\colon \text{Type}, B\colon \text{Type} &\vdash A \to B\colon \text{Type} \\
A\colon \text{Type}, B\colon \text{Type} &\vdash A + B\colon \text{Type} \\
A\colon \text{Type}, B\colon A \to \text{Type} &\vdash \prod_{x\colon A} B(x)\colon \text{Type} \\
A\colon \text{Type}, B\colon A \to \text{Type} &\vdash \sum_{x\colon A} B(x)\colon \text{Type}
\end{aligned}
$$

Each comes with rules for producing and using terms, e.g.

$$
\begin{aligned}
A\colon \text{Type}, B\colon \text{Type}, f\colon A \to B, x\colon A &\vdash f(x)\colon B \\
A\colon \text{Type}, B\colon \text{Type}, x\colon A &\vdash \mathrm{inl}(x)\colon A + B \\
A\colon \text{Type}, B\colon \text{Type}, y\colon B &\vdash \mathrm{inr}(y)\colon A + B
\end{aligned}
$$

Typing as programming

$$x_1 : A_1,\ x_2 : A_2,\ \ldots x_n : A_n \ \vdash\ b : B,$$

The term b can be viewed as a program, with inputs $x_1, \ldots, x_n$ of types $A_1, \ldots, A_n$ and output of type B, that can be computed.

Thus:

- Type theory helps analyze programming languages.
- Type theory can be understood and verified by a computer.

Typing as proving

Sometimes, $x_1 \colon P_1,\ x_2 \colon P_2,\ \ldots x_n \colon P_n \ \vdash\ q \colon Q$ means instead

Under hypotheses P_1, P_2, ..., P_n,
the conclusion Q is provable (with proof q).

Examples

$$
\begin{aligned}
f \colon P \to Q,\ x \colon P &\vdash f(x) \colon Q && \leftarrow \text{modus ponens} \\
x \colon P &\vdash \mathrm{inl}(x) \colon P \text{ or } Q \\
y \colon Q &\vdash \mathrm{inr}(y) \colon P \text{ or } Q \\
x \colon P,\ y \colon Q &\vdash \mathrm{inl}(x) \colon P \text{ or } Q && \leftarrow \text{two different (?) proofs} \\
x \colon P,\ y \colon Q &\vdash \mathrm{inr}(y) \colon P \text{ or } Q && \leftarrow \text{of the same thing}
\end{aligned}
$$

Propositions as types

a.k.a. proofs as terms, or the Curry-Howard correspondence

Types	$\longleftrightarrow$	Propositions
$A \times B$	$\longleftrightarrow$	P and Q
$A + B$	$\longleftrightarrow$	P or Q
$A \to B$	$\longleftrightarrow$	P implies Q
$\prod_{x:\,A} B(x)$	$\longleftrightarrow$	$(\forall x: A)P(x)$
$\sum_{x:\,A} B(x)$	$\longleftrightarrow$	$(\exists x: A)P(x)$

Predicate logic

Also, $x_1 : A_1,\ x_2 : A_2,\ y_1 : P_1,\ y_2 : P_2 \ \vdash\ q : Q$ can mean

In the context of variables x_1 of type A_1 and x_2 of type A_2,
and under hypotheses P_1 and P_2,
the conclusion Q is provable (with proof q).

Examples

$$n : \mathbb{N},\ e : \mathrm{even}(n) \ \vdash\ s(e) : \mathrm{odd}(n+1)$$

$$x : \mathbb{R},\ y : \mathbb{R} \ \vdash\ \mathrm{comm}(x,y) : (x + y = y + x)$$

$$x : \mathbb{R},\ \mathrm{nz} : \neg(x = 0) \ \vdash\ \mathrm{inv}(x) : (\exists y : \mathbb{R})(x \cdot y = 1)$$

Uses of type theory

Since type theory includes both programming and logic, it is a natural context in which to prove things about programs.

$$x : \text{input} \vdash \text{de}(x) : (\text{decrypt}(\text{encrypt}(x)) = x)$$

$$x : \text{user},\ y : \text{moonPhase} \vdash p(x, y) : \neg\, \text{crashes}(\text{Windows})$$

We can also develop mathematics in type theory, which can then be formally verified by a computer.

$$g : \text{Graph},\ p : \text{Planar}(g) \vdash c : \text{Coloring}(g, 4) \quad ✔$$

$$x : \mathbb{N},\ y : \mathbb{N},\ z : \mathbb{N},\ n : \mathbb{N},\ f : (x^n + y^n = z^n) \vdash w : (n \leq 2 \text{ or } z \leq 1)$$

Type-theoretic foundations

Set theory

Logic

$\wedge, \vee, \Rightarrow, \neg, \forall, \exists$

Sets

$\times, +, \rightarrow, \prod, \sum$

$x \in A$ is a proposition

Type theory

Types

$\times, +, \rightarrow, \prod, \sum$

Logic

$\wedge, \vee, \Rightarrow, \neg, \forall, \exists$

$x : A$ is a typing judgment

Type theory and category theory

Type theory can be a syntax for describing objects and morphisms in a category.

$$
\begin{aligned}
A\colon \text{Type},\ B\colon \text{Type} \quad &\longleftrightarrow \quad \text{Objects}\\
x\colon A \vdash b\colon B \quad &\longleftrightarrow \quad \text{Morphism } A \to B\\
x_1 : A_1,\ x_2 : A_2 \vdash b\colon B \quad &\longleftrightarrow \quad \text{Morphism } A_1 \times A_2 \to B\\
A \times B\colon \text{Type} \quad &\longleftrightarrow \quad \text{Categorical product}\\
A + B\colon \text{Type} \quad &\longleftrightarrow \quad \text{Categorical coproduct}\\
A \to B\colon \text{Type} \quad &\longleftrightarrow \quad \text{Categorical exponential}\\
&\vdots
\end{aligned}
$$

Anything proven in type theory will hold in any such category.

Dependent types, categorically

Recall a dependent type is $x\colon A \vdash B(x)\colon \mathrm{Type}$ or $B\colon A \to \mathrm{Type}$. There are two ways to interpret this in a category:

- As a morphism $A \xrightarrow{B} U$, where U is a "universe" like "the set of all small sets".
- As a morphism $p\colon |B| \to A$, where the type/object $B(x)$ is the "fiber" of p over x.

These are related by a pullback square:

$$\begin{array}{ccc} |B| & \longrightarrow & \widetilde{U} \\ \downarrow & & \downarrow \\ A & \longrightarrow & U \end{array}$$

Type theories and category theories

Simply typed λ-calculus	$\longleftrightarrow$	Cartesian closed category
Dependent type theory	$\longleftrightarrow$	Locally c.c. category
First-order predicate logic	$\longleftrightarrow$	Boolean/Heyting category
Geometric logic	$\longleftrightarrow$	Grothendieck topos
Higher-order logic	$\longleftrightarrow$	Elementary topos
??	$\longleftrightarrow$	Homotopical category

Not too surprising

There is a type theory that goes in ??.

Surprising (to me)

That type theory was already invented by type theorists, long before anyone realized it had to do with homotopy!

Sets, revisited

Ignoring considerations of size, a set is…

- …a collection of elements
- …together with a notion of when two elements are equal
- …which is transitive, symmetric, and reflexive.

"To define a set we prescribe… what we… must do in order to construct an element of the set, and what we must do to show that two elements are equal"
– Errett Bishop, "Foundations of Constructive Analysis"

Groupoids

A groupoid is. . .

- . . . a collection of "elements" or "points"
- . . . with, for all points x and y, a set $\hom(x, y)$ of "isomorphisms" or "paths" from x to y
- . . . which can be composed, inverted, and have identities

Equivalently: a category in which all morphisms are invertible.

Examples

- Elements = sets, isomorphisms = bijections
- Elements = any set X, isomorphisms = only identities
- Elements = $\star$, isomorphisms = any group G

NB: Two points connected by a path are regarded as the same.

∞-groupoids

An ∞-groupoid is

- …a collection of points
- …for all points x and y, a collection $\hom(x, y)$ of paths from x to y
- …for all paths f and g from x to y, a collection $\hom(f, g)$ of 2-paths from f to g
- …
- with composition, inversion, identities, …

Examples

- Any set, with only identity n-paths for $n > 0$
- Any groupoid, with only identity n-paths for $n > 1$
- Points = ∞-groupoids, …(more later)

The fundamental ∞-groupoid

A topological space is a set together with a notion of "closeness", "continuity", or "deformation".

The fundamental ∞-groupoid of a topological space X has

- As points, the points of X.
- As paths, the continuous paths $[0, 1] \to X$.
- As 2-paths, continuous deformations between paths.
- ...

We denote this by $\Pi_\infty(X)$.

$\Pi_\infty(\star)$

Let $\star$ be the one-point topological space. Then $\Pi_\infty(\star)$ has:

- One point.
- One path from that point to itself.
- One 2-path from that path to itself.
- $\ldots$

This is the same as the ∞-groupoid arising from the set $\{\star\}$.

$\Pi_\infty(\mathbb{R})$

For $\mathbb{R}$ the real numbers, $\Pi_\infty(\mathbb{R})$ has:

- The real numbers $x \in \mathbb{R}$ as points...
  - ...but any two of them are connected by a path, so there is essentially only one point.
- Paths between them...
  - ...but any two such paths are related by a deformation.
- ...

$\Pi_\infty(\mathbb{R})$ has essentially only one of all these things.
Thus it is equivalent to $\Pi_\infty(\star)$. The same is true for $\Pi_\infty(\mathbb{R}^n)$.

An ∞-groupoid that is equivalent to $\Pi_\infty(\star)$ is called contractible.

$\Pi_\infty(S^1)$

Let $S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$. Then $\Pi_\infty(S^1)$ has:

- The points $(x, y) \in S^1$ as points. . .
  - . . . but any two of them are connected by a path, so there might as well only be one point, say $(1, 0)$.
- Lots of paths from $(1, 0)$ to itself. Two such paths α and β are inter-deformable exactly when they wind around the circle an equal number of times.
  - Thus there are essentially $\mathbb{Z}$-many paths from $(1, 0)$ to itself.
- For $n > 1$, every n-path is trivial.

This is (arguably) the simplest groupoid that is not a set.

$\Pi_\infty(S^2)$

Let $S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$. Then $\Pi_\infty(S^2)$ has:

- Essentially only one point, say $(1, 0, 0)$.
- Essentially only the constant path from $(1, 0, 0)$ to itself.
- Essentially $\mathbb{Z}$-many 2-paths from the constant path to itself (how many times the deformation wraps over the surface).
- Essentially $\mathbb{Z}$-many 3-paths from any 2-path to itself.
- Essentially two 4-paths from any 3-path to itself.
- Essentially two 5-paths from any 4-path to itself.
- Essentially twelve 6-paths from any 5-path to itself.
- . . . essentially 336 14-paths from any 13-path to itself . . .
- . . .

Computing these homotopy groups of spheres for all S^n is a big part of classical homotopy theory.

Presenting ∞-groupoids

There are many ways to define ∞-groupoids. All are "equivalent", but most are technical. Instead, we can use:

Fact

Every ∞-groupoid is equivalent to $\Pi_\infty(X)$ for some X.

Fact

For nice X and Y, we have $\Pi_\infty(X) \simeq \Pi_\infty(Y)$ if and only if X and Y are homotopy equivalent (next slide).

Thus it suffices to think about topological spaces up to homotopy equivalence.

Homotopy

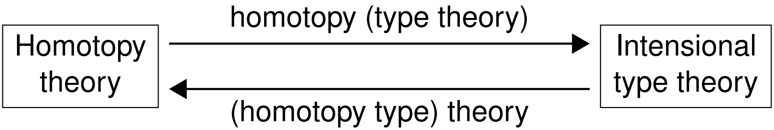
A homotopy $f \sim g$ between continuous maps $f, g\colon X \to Y$ is

- A path from f to g in the space Y^X of continuous functions (with a suitable topology)
- OR: a map $H\colon X \times [0,1] \to Y$ such that $H(x,0) = f(x)$ and $H(x,1) = g(x)$.
- OR: a map $\overline{H}\colon X \to Y^{[0,1]}$ such that $\overline{H}(x)(0) = f(x)$ and $\overline{H}(x)(1) = g(x)$.

A map $f\colon X \to Y$ is a homotopy equivalence if there exists $g\colon Y \to X$ and homotopies $g \circ f \sim \mathrm{id}_X$ and $f \circ g \sim \mathrm{id}_Y$.

Homotopy types

"Topological spaces up to homotopy equivalence" were studied long before "∞-groupoids". They were called homotopy types.



The ∞-groupoid of ∞-groupoids

For spaces X and Y, let $\mathrm{Equiv}(X, Y) \subseteq Y^X$ be the subspace consisting of the homotopy equivalences.

Definition

The ∞-groupoid of ∞-groupoids has

- As points, topological spaces
- As paths, homotopy equivalences (= points of $\mathrm{Equiv}(X, Y)$)
- As 2-paths, paths in $\mathrm{Equiv}(X, Y)$
- As 3-paths, 2-paths in $\mathrm{Equiv}(X, Y)$
- ...

Categories with homotopies

Recall:

$$?? \quad \longleftrightarrow \quad \text{Homotopical category}$$

What structure on the category of topological spaces encapsulates its homotopy theory?

- The interval $[0, 1]$.
- The class of homotopy equivalences.
- For each X, there is a cylinder $\mathrm{Cyl}(X) := X \times [0, 1]$.
- For each Y, there is a path space $\mathrm{Paths}(Y) := Y^{[0,1]}$.

These can all work — and they work best when combined!
But Paths(Y) matches the type theory best.

Path objects

A category has path objects if each object Y has a factorization of the diagonal

$$\begin{array}{ccc} Y & \longrightarrow & \mathrm{Paths}(Y) \\ & {\scriptstyle\Delta}\searrow & \downarrow \\ & & Y \times Y \end{array}$$

satisfying certain axioms.

Examples

- Topological spaces, with $\mathrm{Paths}(Y) := Y^{[0,1]}$.
- Chain complexes, with $\mathrm{Paths}(Y) := \underline{\mathrm{Hom}}(\mathbb{I}, Y)$
- Any category, with $\mathrm{Paths}(Y) := Y$ (trivial homotopy theory)

Equality

In logic, formulas are built from atomic formulas using connectives and quantifiers:

$$\wedge, \vee, \Rightarrow, \neg, \top, \bot, \forall, \exists$$

The most basic atomic formula is equality ($x = y$).

In type theory:

$$\begin{array}{rcl} \text{propositions} & \longleftrightarrow & \text{types} \\ \text{connectives and quantifiers} & \longleftrightarrow & \text{type constructors} \\ \text{equality} & \longleftrightarrow & ?? \end{array}$$

Identity types

We may include a type constructor

$$A\colon \text{Type},\ x\colon A,\ y\colon A \ \vdash\ (x = y)\colon \text{Type}$$

This is an equality type or identity type.

There is a clean, concise, and computational way to obtain the rules for identity types, as an "inductive family" (Martin-Löf).

But this method doesn't imply the rule

$$x\colon A,\ y\colon A,\ p\colon (x = y),\ q\colon (x = y)\ \vdash ?\colon (p = q)$$

If we add this additional rule, we have extensional type theory; otherwise it is intensional.

Identity types are path objects

The dependent type $x : A, \, y : A \vdash (x = y) :$ Type must be interpreted by:

$$\begin{array}{ccc} A & \longrightarrow & |x = y| \\ & \searrow & \downarrow \\ & & A \times A \end{array}$$

Theorem (Awodey, Gambino, Garner, van den Berg, Lumsdaine, Warren)

Path objects exactly model identity types.

The trivial path objects $Y \to Y \times Y$ model extensional identity types. Others are intensional.

Homotopy theory in type theory

We can define internally in type theory:

- When a type is contractible.
- When a type is "set-like", or "groupoid-like", etc.
- When a function is a homotopy equivalence.
- The type of homotopy equivalences $\mathrm{Equiv}(X, Y)$
- Loop spaces, homotopy groups, fibration sequences, ...

The univalence axiom

Recall that a type is a term of type Type.

$$\vdash \; A \colon \mathrm{Type}$$

Thus, we have identity types:

$$A \colon \mathrm{Type},\; B \colon \mathrm{Type} \;\vdash\; (A = B) \colon \mathrm{Type}$$

What should this be? The standard rules don't determine it.

The univalence axiom (Voevodsky)

For types A and B, the identity type $(A = B)$ is homotopy equivalent to the type $\mathrm{Equiv}(A, B)$ of homotopy equivalences between A and B.

The meaning of univalence

Univalence means that

Under the interpretation of types as ∞-groupoids, the type Type corresponds to the ∞-groupoid of ∞-groupoids.

Some consequences of univalence:

- If there is a homotopy equivalence $f\colon A \to B$, then there is a path $p\colon (A = B)$.
- The subtype of set-like types in Type corresponds to the groupoid of sets.
- Dependent functions are strongly extensional (Voevodsky).

A simple use of univalence

In a "set-like" type, any path is deformable to the identity.

Theorem

Type *is not set-like. Hence, not all types are set-like.*

Proof.

1. Let bool $:= \{\top, \bot\}$ be a two-element type.
2. The "flip" map $f\colon \mathrm{bool} \to \mathrm{bool}$, defined by $f(\bot) := \top$ and $f(\top) := \bot$, is a homotopy equivalence that is not homotopic to the identity.
3. Hence, by univalence, it gives a path $p_f\colon (\mathrm{bool} = \mathrm{bool})$ in Type that is not deformable to the identity.

□

Modeling univalence

Theorem (Voevodsky)

There is a model of type theory in ∞-groupoids, for which the univalence axiom holds.

- This is nontrivial (only) because of strictness and coherence issues.
- Many other categories (called "$(\infty, 1)$-toposes") contain objects like "the ∞-groupoid of ∞-groupoids", but we don't yet know how to overcome the technical issues there.

Inductive types

A large class of type constructors are inductive types.

Example

$\mathbb{N}$ is inductively generated by $0 \colon \mathbb{N}$ and $s \colon \mathbb{N} \to \mathbb{N}$.

- $\Leftrightarrow$ Every $n \colon \mathbb{N}$ is generated in a unique way from 0 and s.
- $\Leftrightarrow$ We can define functions by recursion on $\mathbb{N}$.
- $\Leftrightarrow$ We can prove theorems by induction on $\mathbb{N}$.

Example

The disjoint union $A + B$ is inductively generated by $\mathrm{inl} \colon A \to A + B$ and $\mathrm{inr} \colon B \to A + B$.

- $\Leftrightarrow$ Every $x \colon A + B$ arises from exactly one of inl and inr.
- $\Leftrightarrow$ We can define functions by cases on $A + B$.
- $\Leftrightarrow$ We can prove theorems by cases on $A + B$.

Higher inductive types

We can extend this to homotopy types (Lumsdaine–S.).

Example

S^1 is inductively generated by $b \colon S^1$ and a path $\ell \colon (b = b)$.

- $\Leftrightarrow$ Every point, path, or higher path of S^1 is "generated uniquely" from b and ℓ.
- $\Leftrightarrow$ We can define functions $S^1 \to A$ "recursively" or "by cases", by giving a point $f_b \colon A$ and a path $f_\ell \colon (f_b = f_b)$.
- $\Leftrightarrow$ We can prove theorems by "induction" on S^1.

Example

S^2 is inductively generated by $c \colon S^2$ and a 2-path $\sigma \colon (\mathrm{id}_c = \mathrm{id}_c)$.

$\Pi_\infty(S^1)$

Recall S^1 is inductively generated by $b\colon S^1$ and $\ell\colon (b = b)$.

Theorem (S.)

Assuming the univalence axiom, the identity type $(b = b)$ *is equivalent to* $\mathbb{Z}$*.*

- Thus, paths from *b* to *b* are classified by integers, and there are no nontrivial higher paths.
- In homotopy-theoretic language, this implies $\pi_1(S^1) \cong \mathbb{Z}$.
- The proof is completely written inside of type theory, and has been fully verified by the computer proof assistant Coq.

Looking ahead: this quarter

1. Some type theory, precisely
2. Programming type theory in the proof assistant Coq
3. Some homotopy theory, precisely
4. Programming homotopy type theory in Coq
5. Categorical models of homotopy type theory
6. ...

Suggested Homework

1. Install Coq: `http://coq.inria.fr`
2. Learn a bit of functional programming.
3. Learn a bit of category theory.