# Logic, homotopy levels, and equivalences

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# Outline

# Intensional type theory

From now on, we work in a type theory with

1. Dependent products
2. Inductive type families (including identity types)
3. At least one “universe” Type

Basically: the fragment of Coq’s type theory that ignores coinductive types and the sort Prop.

# Interlude

(Coq)

# Function extensionality

We also assume:

Axiom (Function extensionality)

$$f, g\colon \prod_{x\colon A} B(x) \vdash \left(\prod_{x\colon A}(f(x) = g(x))\right) \longrightarrow (f = g)$$

- Not provable in plain type theory.
- True in set theory: “If two functions are pointwise equal, then they are equal as functions.”
- True in homotopy theory: “If two functions are homotopic, they are connected by a path in the space of functions.”

# The eta rule

Define $\eta(f) := (\lambda x.f(x))$. Then for any $a$,

$$\eta(f)(a) = (\lambda x.f(x))(a) \to_\beta f(a).$$

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Hence, function extensionality implies

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This is a proof of a proposition, i.e. a term in the type $(f = \eta(f))$. It would be stronger to assert a computation rule $f \to_\eta \eta(f)$ or $\eta(f) \to_\eta f$; the upcoming Coq v8.4 will do the latter.

# Paths

We think of terms $p : (x = y)$ as paths $x \rightsquigarrow y$.

- Reflexivity becomes the constant path $\mathsf{refl}_x : x \rightsquigarrow x$.
- Transitivity becomes the concatenation $p \mathbin{@} q : x \rightsquigarrow z$ of paths $x \overset{p}{\rightsquigarrow} y \overset{q}{\rightsquigarrow} z$.
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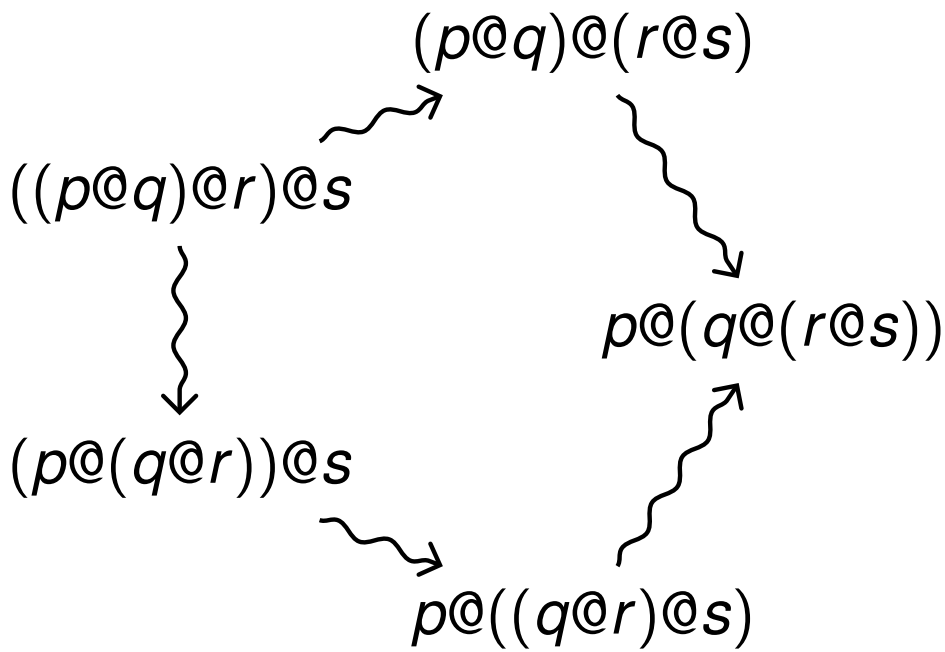
But now there is more to say.

- Concatenation is associative: $(p@q)@r = p@(q@r)$.
- Better: there is a path $\alpha_{p,q,r}\colon (p@q)@r \rightsquigarrow p@(q@r)$.

# 2-paths

The “associator” $\alpha_{p,q,r}$ is coherent:



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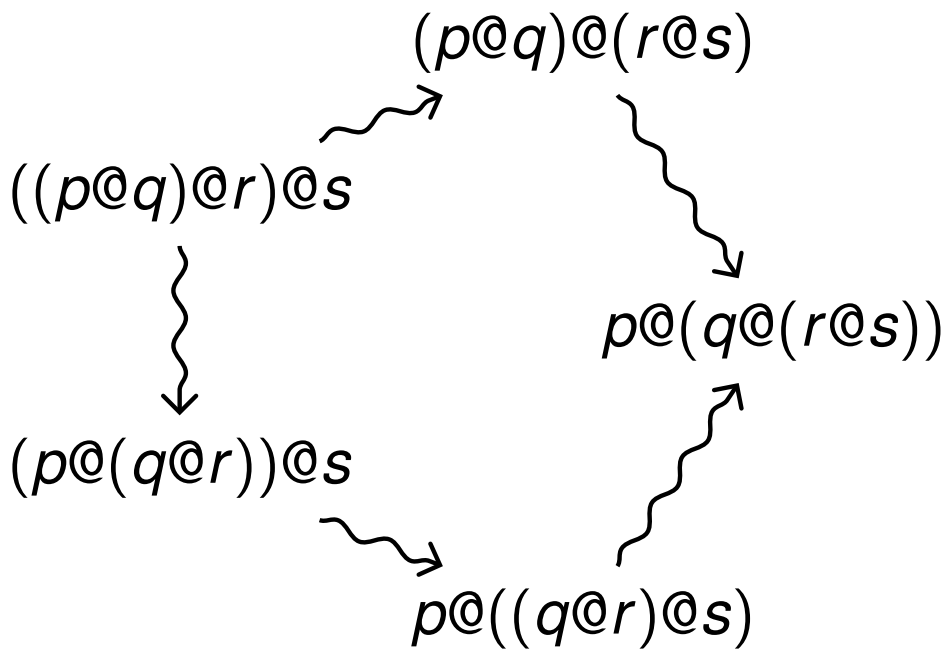


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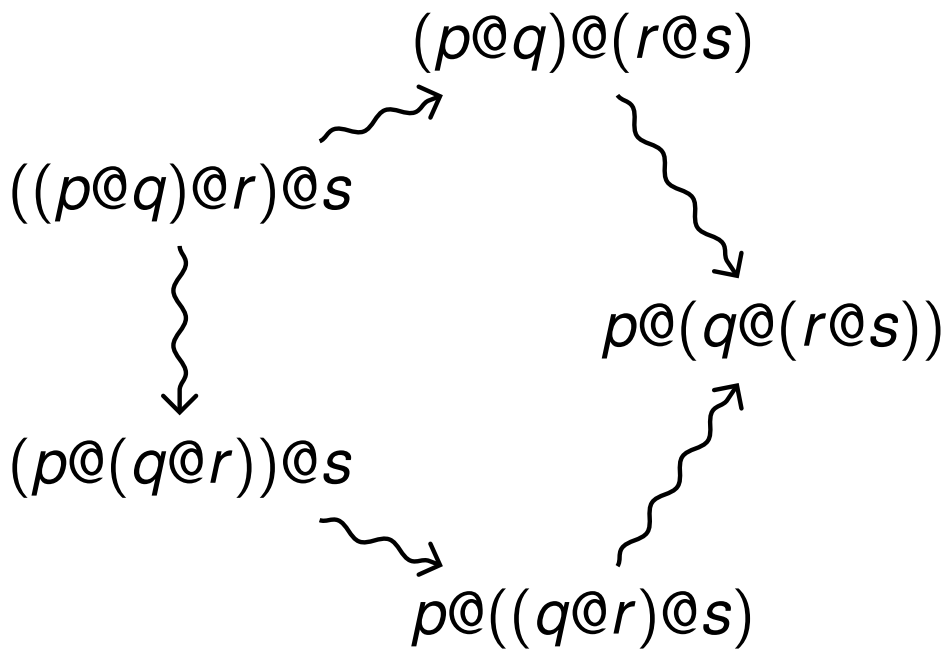


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# $\infty$-groupoids

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An $\infty$-groupoid is a collection of points, together with paths between points, 2-paths between paths, and so on, with all possible coherent and consistent concatenation operations.

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- Basically this means that any reasonable fact about paths and higher paths is true.
- This is a theorem *in type theory*. (But not completely formalized internally, due to issues with infinity.)
- Separately (later), $\infty$-groupoids defined *in set theory* are a model of type theory.

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BUT: We cannot prove any of this in our current type theory. Later, we'll extend the theory; for now, these are *intended* examples.

# Some non-homotopy types

## "Definition"

An h-set (or just a set) is a type that contains no nontrivial $k$-paths for any $k \geq 1$.

## Examples

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- list($A$), if $A$ is an h-set.
- All datatypes arising in everyday programming.
- Any type equivalent to an h-set is an h-set.

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- Any two paths in unit are the same.
- A path in $\prod_{x:A} B$ is a "pointwise path" (using function extensionality).
- A path in $\sum_{x:A} B$ is. . . what?

# Transporting along paths

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A path $(x, u) = (y, v)$ in $\sum_{x:A} B(x)$ consists of

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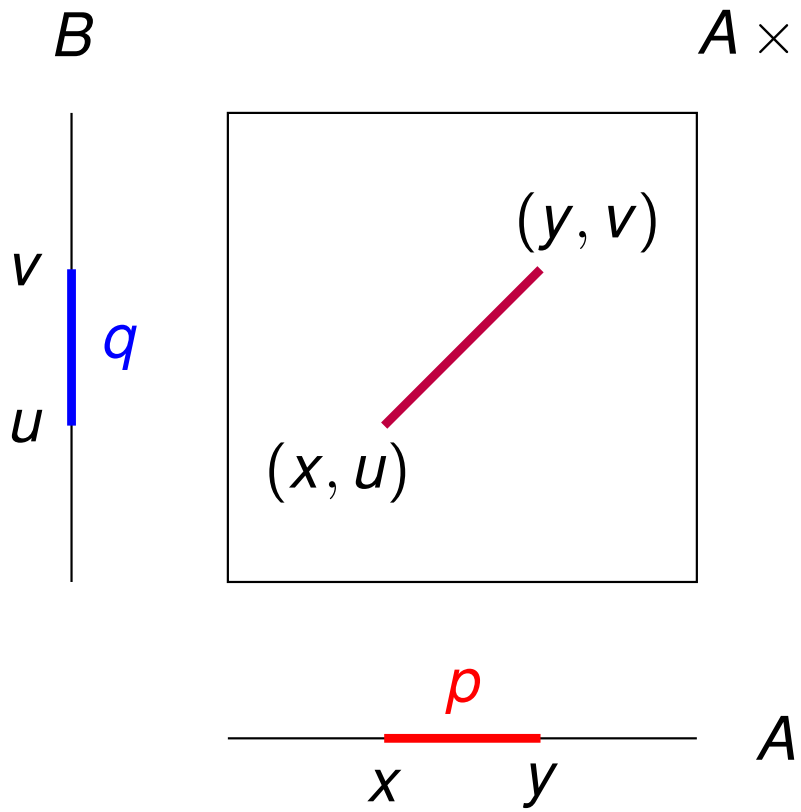
Note: If $B$ is independent of $x$, then $\mathrm{transport}(p, u) = u$.

# Paths in cartesian products



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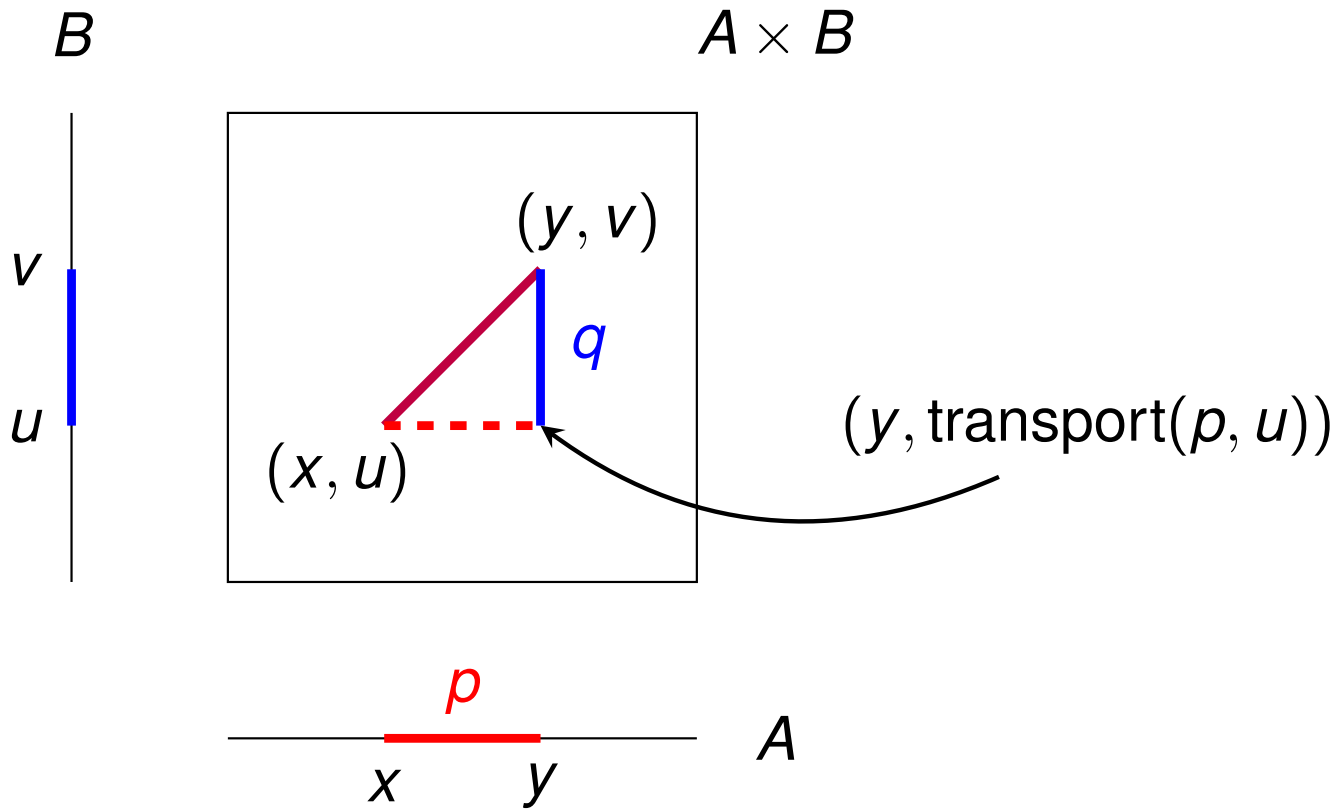
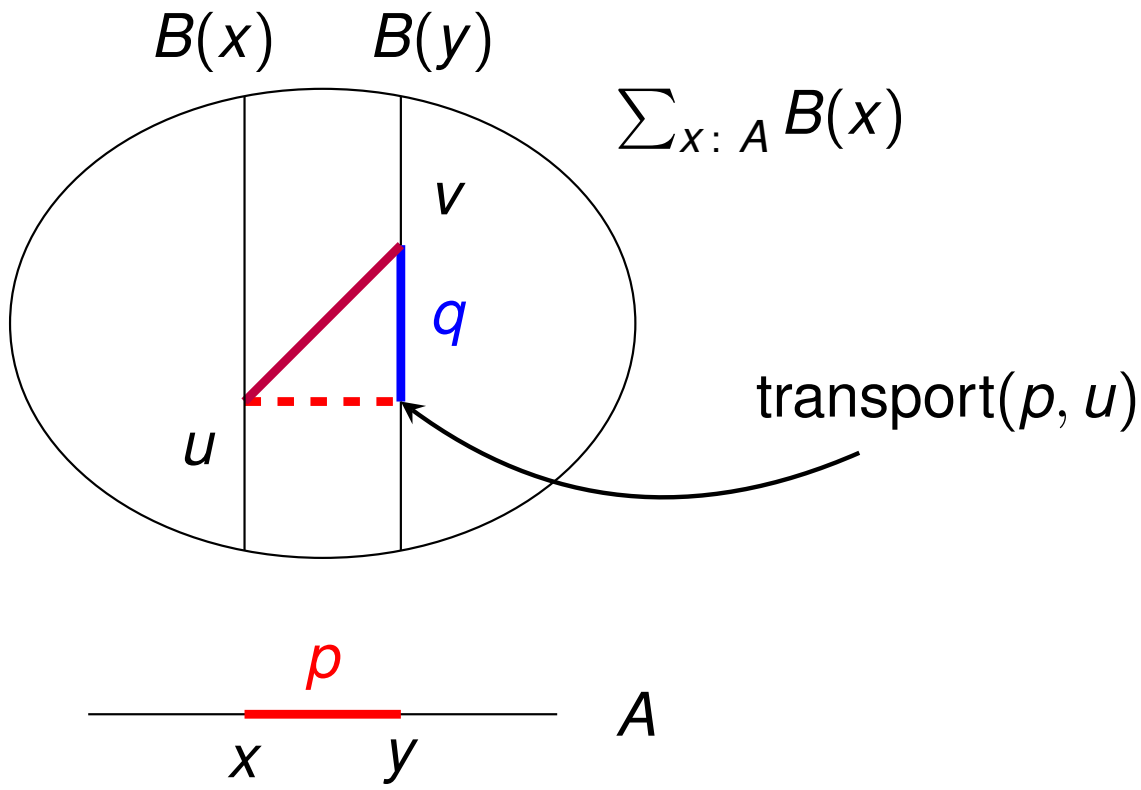


# Paths in dependent sums



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In homotopy type theory, the basic activity is constructing terms belonging to types.

1. Defining a type = constructing a term in Type
2. Defining a function = constructing a term in $A \to B$
3. ...
4. Stating a theorem = constructing a type that is an h-prop
5. Proving a theorem = constructing a term in an h-prop

# H-props

## Definition

An h-proposition (or h-prop) is an h-set that is a *subsingleton* (any two points are equal).

- In classical logic, an h-prop is "either empty or contractible".
- These are the "truth values" for embedding logic in homotopy type theory.

# Constructing h-props

Recall: propositions are built from "proposition constructors":

| Types | $\longleftrightarrow$ | Propositions |
|---|---|---|
| $A \times B$ | $\longleftrightarrow$ | $P$ and $Q$ |
| $A + B$ | $\longleftrightarrow$ | $P$ or $Q$ |
| $A \to B$ | $\longleftrightarrow$ | $P$ implies $Q$ |
| unit | $\longleftrightarrow$ | $\top$ (true) |
| $\emptyset$ | $\longleftrightarrow$ | $\bot$ (false) |
| $\prod_{x:A} B(x)$ | $\longleftrightarrow$ | $(\forall x : A) P(x)$ |
| $\sum_{x:A} B(x)$ | $\longleftrightarrow$ | $(\exists x : A) P(x)$ |

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BUT: not all of these type constructors preserve h-props.

# Constructing h-props

The following are h-props:

- unit and $\emptyset$ (true and false)
- $A \times B$, if $A$ and $B$ are h-props (and)
- $A \to B$, if $A$ and $B$ are h-props (implies)
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These are not:

- $A + B$, even if $A$ and $B$ are h-props (or)
- $\sum_{x:A} B(x)$, even if each $B(x)$ is an h-prop (there exists)
- $(x = y)$ for $x, y : A$, unless $A$ is an h-set.

# Supports

In set theory, subsingletons are a reflective subcategory of sets, and even of $\infty$-groupoids.

## Definition

The support of $A$, denoted $\mathrm{supp}(A)$, is a subsingleton that contains a point precisely when $A$ does.

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Eventually, we'll need a type constructor that does this. But let's see how far we can get without it. (This will also tell us how to formulate that type constructor.)

# Internalizing h-props

Let's try to internalize "$A$ is an h-prop":

1. for all $x, y : A$, there exists a path $x \rightsquigarrow y$
2. for all $x, y : A$ and paths $p, q : x \rightsquigarrow y$, there exists a 2-path $p \rightsquigarrow q$.
3. for all $x, y : A$, paths $p, q : x \rightsquigarrow y$, and 2-paths $r, s : p \rightsquigarrow q$, there exists a 3-path $r \rightsquigarrow s$.
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is the h-prop "for all $x, y : A$ there exists a path from $x$ to $y$"

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is the type of functions which assign to any pair $x, y : A$ a path from $x$ to $y$, "varying continuously" with $x$ and $y$.

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Such a function implies the former h-prop, but also more...

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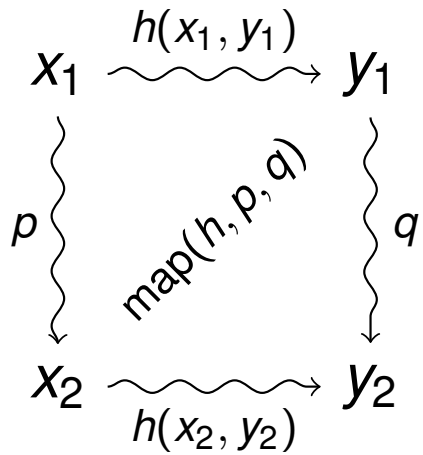

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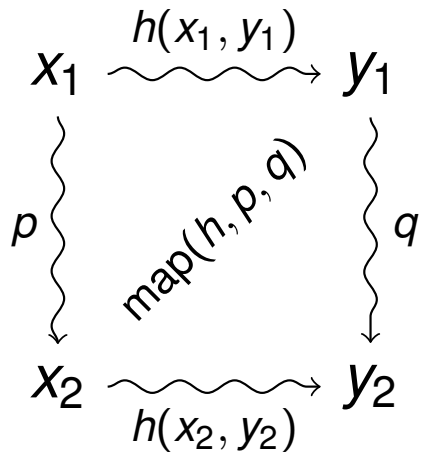


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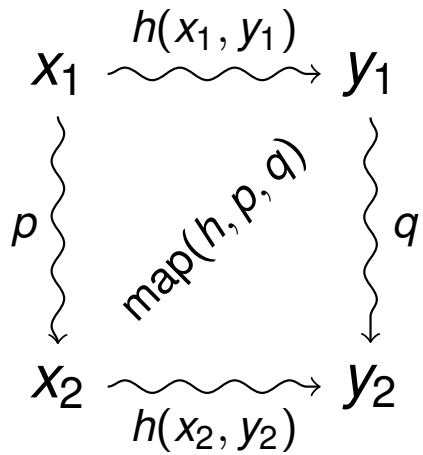


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These are all things we expect to exist anyway in an h-prop!

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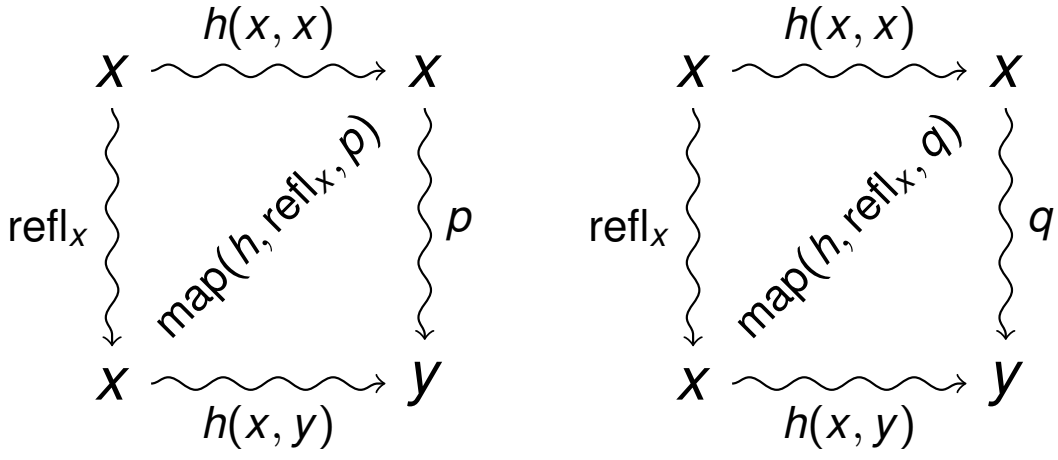

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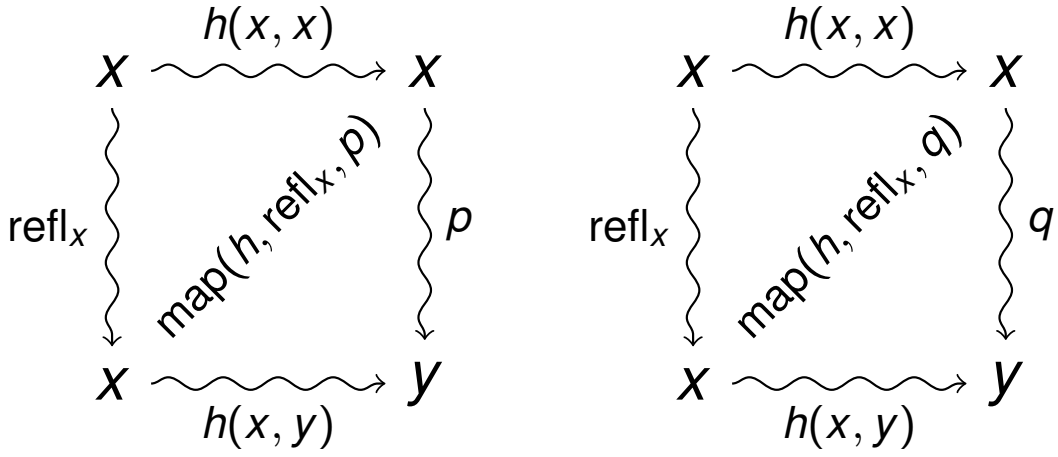


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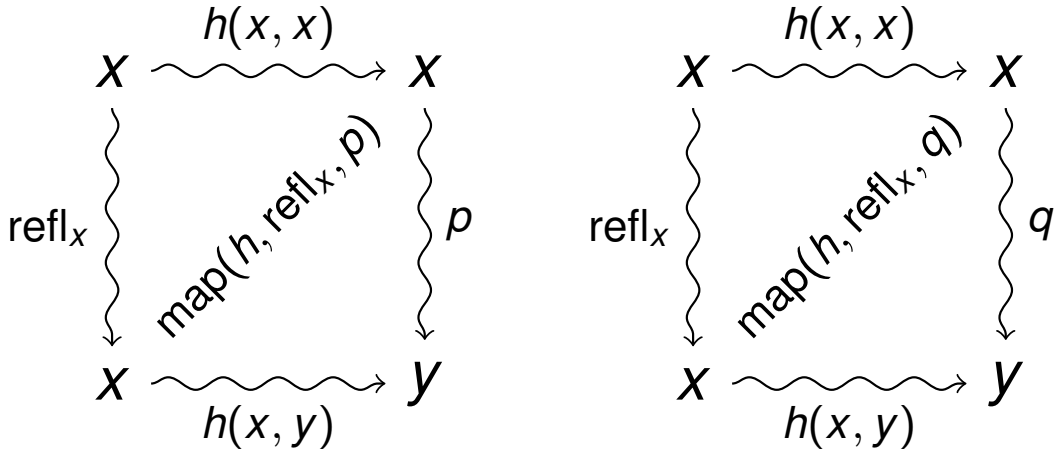


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$$\mathsf{isProp}(A) :\equiv \mathsf{supp}\left(\prod_{x:A}\prod_{y:A}(x = y)\right).$$

But amazingly, $\prod_{x:A}\prod_{y:A}(x = y)$ is already an h-prop, even though $(x = y)$ is not!

## Definition

$$\mathsf{isProp}(A) :\equiv \prod_{x:A}\prod_{y:A}(x = y)$$

## Theorem

*For any A, we can construct a term in*

$$\mathsf{isProp}(\mathsf{isProp}(A)).$$

# Outline

# Homotopy levels

"Definition"

A type is *n*-truncated if it has no nontrivial $k$-paths for any $k > n$.

- Sets are the 0-truncated types.
- $S^1$ is 1-truncated.
- The type of sets (that is, the type whose points are sets) is 1-truncated.
- The type of $n$-truncated types is $(n+1)$-truncated.
- $S^2$, and the type Type of all types, are not $n$-truncated for any $n$.

# Negative thinking

## Observations

- A $k$-path in $A$ is a $(k-1)$-path in $(x = y)$ for some $x, y : A$.
- Thus $A$ is $n$-truncated $\iff$ $(x = y)$ is $(n-1)$-truncated for all $x, y : A$.

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- Thus $A$ is $n$-truncated $\iff$ $(x = y)$ is $(n-1)$-truncated for all $x, y : A$.

We've seen that if $A$ is 0-truncated, then $(x = y)$ is an h-prop. Thus it makes sense to define

## Definition

A type is $(-1)$-truncated if it is an h-prop.

# Internalizing truncation

By induction, starting with $n = (-1)$:

## Definition

A type $A$ is

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```
Fixpoint isTrunc (n:nat) (A:Type) : Type :=
  match n with
    | -1   => isProp A
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  end.
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What can we say about $(x = y)$ if $A$ is an h-prop?

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A type is contractible, or $(-2)$-truncated, if it is an inhabited h-prop.

(After this point, it's "turtles all the way down": $(-3)$-truncated is the same as $(-2)$-truncated.)

# Alternative contractibility

Suppose $A$ is contractible; thus we have $a : A$ and

$$h : \mathsf{isProp}(A) :\equiv \prod_{x:A} \prod_{y:A} (x = y).$$

Then

$$(a, h(a)) : \sum_{x:A} \prod_{y:A} (x = y).$$

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Conversely, if

$$(a, k) : \sum_{x:A} \prod_{y:A} (x = y)$$

then $a : A$ and

$$\lambda x^A y^A.\big(!k(x) \mathbin{@} k(y)\big) : \mathsf{isProp}(A).$$

# Alternative contractibility

It turns out that

$$\mathrm{isContr}(A) :\equiv \sum_{x:A} \prod_{y:A} (x = y)$$

is also always an h-prop. So we can start the induction at $-2$:

## Definition

A type $A$ is

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This is what we usually do in practice.

## Definition

A type has h-level $n$ if it is $(n-2)$-truncated.

# Outline

# Homotopy equivalences

## Definition

A function $f : A \to B$ is a homotopy equivalence if there exists $g : B \to A$ and homotopies $g \circ f \sim \mathrm{id}_A$ and $f \circ g \sim \mathrm{id}_B$.

$$\mathrm{isEquiv}(f) := \mathrm{supp}\left(\sum_{g : B \to A} \Big((g \circ f = \mathrm{id}_A) \times (f \circ g = \mathrm{id}_B)\Big)\right)$$

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This would not be an h-prop without supp. Can we avoid it?

# Back to bijections

A function $f\colon A \to B$ between sets is a bijection if

1. There exists $g\colon B \to A$ such that $g \circ f = \mathrm{id}_A$ and $f \circ g = \mathrm{id}_B$.

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2. OR: For each $b \in B$, the set $f^{-1}(b)$ is a singleton.
3. OR: There exists $g\colon B \to A$ such that $g \circ f = \mathrm{id}_A$ and also $h\colon B \to A$ such that $f \circ h = \mathrm{id}_B$.

# Better equivalences

## Definition

The homotopy fiber of $f : A \to B$ at $b : B$ is

$$\mathsf{hfiber}(f, b) :\equiv \sum_{x : A} (f(x) = b).$$

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$$\mathsf{hfiber}(f, b) :\equiv \sum_{x : A} (f(x) = b).$$

## Definition (Voevodsky)

$f$ is an equivalence if each $\mathsf{hfiber}(f, b)$ is contractible:

$$\mathsf{isEquiv}(f) :\equiv \prod_{b : B} \mathsf{isContr}(\mathsf{hfiber}(f, b))$$

This is an h-prop.

# H-isomorphisms

## Definition (Joyal)

$f : A \to B$ is an h-isomorphism if we have $g : B \to A$ and a homotopy $g \circ f \sim \mathrm{id}_A$, and also $h : B \to A$ and a homotopy $f \circ h \sim \mathrm{id}_B$.

$$\mathrm{isHIso}(f) := \left( \sum_{g :\, B \to A} (g \circ f = \mathrm{id}_A) \right) \times \left( \sum_{h :\, B \to A} (f \circ h = \mathrm{id}_B) \right)$$

This is also an h-prop.

# Adjoint equivalences

Given a homotopy equivalence, we can also ask for more coherence from $r : (g \circ f = \mathrm{id}_A)$ and $s : (f \circ g = \mathrm{id}_B)$.

(1a) For all $b : B$, we have $u(b) : (r(g(b)) = \mathrm{map}(g, s(b)))$.

(1b) For all $a : A$, we have $v(a) : (\mathrm{map}(f, r(a)) = s(f(a)))$.

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This gives an h-prop if we stop between any ($n$a) and ($n$b).

## Definition

$f$ is an adjoint equivalence if we have $g$, $r$, $s$, and $u$.

$$\mathrm{isAdjEquiv}(f) := \sum_{g : B \to A} \sum_{r : \ldots} \sum_{s : \ldots} \Big( r(g(b)) = \mathrm{map}(g, s(b)) \Big)$$

# All equivalences are the same

## Theorem

*The following are equivalent:*

1. *$f$ is a homotopy equivalence.*
2. *$f$ is a (Voevodsky) equivalence.*
3. *$f$ is a (Joyal) h-isomorphism.*
4. *$f$ is an adjoint equivalence.*

The last three are supp-free h-props, so we have equivalences

$$\mathsf{isEquiv}(f) \simeq \mathsf{isHIso}(f) \simeq \mathsf{isAdjEquiv}(f)$$

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## Definition

The type of equivalences between $A, B : \mathsf{Type}$ is

$$\mathsf{Equiv}(A, B) :\equiv \sum_{f : A \to B} \mathsf{isEquiv}(f).$$

# Outline

# The univalence axiom

For $A, B : \mathrm{Type}$, we have

$$\mathrm{pathToEquiv}_{A,B} : (A = B) \to \mathrm{Equiv}(A, B)$$

defined by induction on paths.
Note: $(A = B)$ is a path-type of "Type".

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defined by induction on paths.
Note: $(A = B)$ is a path-type of "Type".

## Axiom (Univalence)

For all $A, B$, the function $\mathrm{pathToEquiv}_{A,B}$ is an equivalence.

$$\prod_{A : \mathrm{Type}} \prod_{B : \mathrm{Type}} \mathrm{isEquiv}(\mathrm{pathToEquiv}_{A,B})$$

In particular, every equivalence yields a path between types.

# Remarks about univalence

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3. In set-theoretic models, univalence should correspond to "object classifiers" in "$(\infty, 1)$-toposes" (Rezk, Lurie)
4. So far, only a few actual models known (coherence issues).
5. Many other uses.