Abstract homotopy theory

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March 6, 2012

Homotopy theory

Switching gears

Today will be almost all classical mathematics, in set theory or whatever foundation you prefer.

Slogan

Homotopy theory is the study of 1-categories whose objects are not just "set-like" but contain paths and higher paths.

Homotopies and equivalences

Question

What structure on a category **C** describes a "homotopy theory"?

We expect to have:

1. A notion of homotopy between morphisms, written $f \sim g$. This indicates we have paths $f(x) \rightsquigarrow g(x)$, varying nicely with x.

Given this, we can "homotopify" bijections:

Definition

A homotopy equivalence is $f \colon A \to B$ such that there exists $g \colon B \to A$ with $fg \sim 1_B$ and $gf \sim 1_A$.

Examples

- **C** $=$ topological spaces, homotopies $A \times [0,1] \to B$.
- **C** $=$ chain complexes, with chain homotopies.
- **C** $=$ categories, with natural isomorphisms.
- **C** $= \infty$-groupoids, with "natural equivalences"

∞-groupoids

Definition

Today, an ∞-groupoid means an algebraic structure:

1. Sets and "source, target" functions:

$$\cdots \rightrightarrows X_n \rightrightarrows \cdots \rightrightarrows X_2 \rightrightarrows X_1 \rightrightarrows X_0$$

   $X_0 =$ objects, $X_1 =$ paths or morphisms,
   $X_2 =$ 2-paths or 2-morphisms, ...

2. Composition/concatenation operations
   e.g. $p\colon x \rightsquigarrow y$ and $q\colon y \rightsquigarrow z$ yield $p@q\colon x \rightsquigarrow z$.
3. These operations are coherent up to all higher paths.

A topological space Z gives rise to an ∞-groupoid $\Pi_\infty(Z)$.

∞-functors

There are two ways to define morphisms of ∞-groupoids:

1. strict functors, which preserve all composition operations on the nose.
2. weak functors, which preserve operations only up to specified coherent equivalences.

Which should we use?

- We want to include weak functors in the theory.
- But the category of weak functors is ill-behaved: it lacks limits and colimits.
- The category of strict functors is well-behaved, but seems to miss important information.

Cofibrant objects

Theorem

If A is a free ∞-groupoid, then any weak functor $f\colon A \to B$ is equivalent to a strict one.

Proof.

Define $\widetilde{f}\colon A \to B$ as follows:

- $\widetilde{f}$ acts as f on the points of A.
- $\widetilde{f}$ acts as f on the generating paths in A. Strictness of $\widetilde{f}$ then uniquely determines it on the rest.
- $\widetilde{f}$ acts as f on the generating 2-paths in A. Strictness then uniquely determines it on the rest.
- ...

□

Cofibrant replacement

Theorem

Any ∞-groupoid is equivalent to a free one.

Proof.

Given A, define QA as follows:

- The objects of QA are those of A.
- The paths of QA are freely generated by those of A.
- The 2-paths of QA are freely generated by those of A.
- ...

We have $q\colon QA \to A$, with a homotopy inverse $A \to QA$ obtained by sending each path to itself. □

However: $QA \to A$ is a strict functor, but $A \to QA$ is not!

Weak equivalences

Definition

A left derivable category **C** is one equipped with:

- A class of objects called cofibrant.
- A class of morphisms called weak equivalences such that
  - if two of f, g, and gf are weak equivalences, so is the third.
- Every object A admits a weak equivalence $QA \xrightarrow{\sim} A$ from a cofibrant one.

Remarks

- A "weak morphism" $A \rightsquigarrow B$ in **C** is a morphism $QA \to B$.
- In good cases, a weak equivalence between cofibrant objects is a homotopy equivalence.

Some left derivable categories

1. ∞-groupoids with strict functors
   - cofibrant = free
   - weak equivalences = strict functors that have homotopy inverse weak functors.
2. chain complexes
   - cofibrant = complex of projectives
   - weak equivalence = homology isomorphism
3. topological spaces
   - cofibrant = CW complex
   - weak equivalence = isomorphism on all higher homotopy groups π_n
4. topological spaces
   - cofibrant = everything
   - weak equivalence = homotopy equivalence

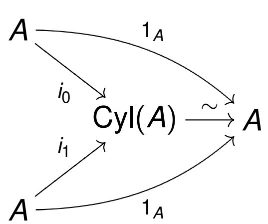
The homotopy theories of ∞-groupoids and topological spaces (CW complexes) are equivalent, via Π_∞.

Cylinder objects

What happened to our "notion of homotopy"?

Definition

A cylinder object for A is a diagram



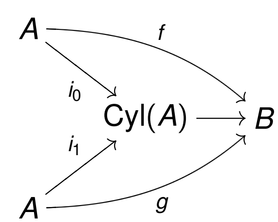
Examples

- In topological spaces, $A \times [0, 1]$.
- In chain complexes, $A \otimes (\mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z})$.
- In ∞-groupoids, $A \times I$, where I has two isomorphic objects.

Left homotopies

Definition

A left homotopy $f \sim g$ is a diagram



for some cylinder object of A.

Remark

For the previous cylinders, this gives the usual notions.

Duality

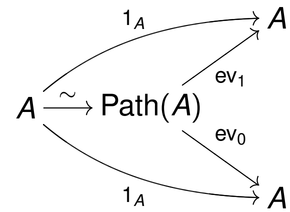
Definition

A right derivable category **C** is one equipped with:

- A class of objects called fibrant.
- A class of morphisms called weak equivalences, satisfying the 2-out-of-3 property.
- Every object A admits a weak equivalence $A \xrightarrow{\sim} RA$ to a fibrant one.

Path objects

A path object for A is a diagram

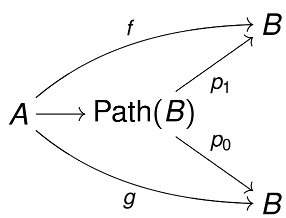


Example

In topological spaces, $\mathrm{Path}(A) = A^{[0,1]}$, with $\mathrm{ev}_0, \mathrm{ev}_1$ evaluation at the endpoints.

Right homotopies

A right homotopy $f \sim g$ is a diagram



Example

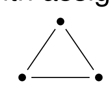
For the previous path object in topological spaces, this is again the usual notion.

Simplicial sets

Definition

A simplicial set X is a combinatorial structure of:

1. A set X_0 of vertices or 0-simplices
2. A set X_1 of paths or 1-simplices, each with assigned source and target vertices
3. A set X_2 of 2-simplices with assigned boundaries



4. . . .

- A simplicial set X has a geometric realization $|X|$, a topological space built out of topological simplices Δ^n according to the data of X.
- A topological spaces Z has a singular simplicial set $S_*(Z)$ whose n-simplices are maps $\Delta^n \to Z$.

Kan complexes

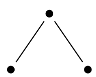
We can also think of a simplicial set as a model for an ∞-groupoid via

$$n\text{-simplices} = n\text{-paths}$$

but it doesn't have composition operations.

Definition

A simplicial set is fibrant (or: a Kan complex) if

1. Every "horn"  can be "filled" to a 2-simplex.
2. etc. ...

If Y is not fibrant, then there may not be enough maps $X \to Y$; some of the composite simplices that "should" be there in Y are missing.

Homotopy theory of simplicial sets

- Fibrant objects = Kan complexes
- Weak equivalences = maps that induce equivalences of geometric realization

This is also equivalent to ∞-groupoids.

Diagram categories

Question

If $\mathbf{C}$ has a homotopy theory, does a functor category $\mathbf{C}^D$ have one?

Fact

A natural transformation $\alpha\colon F \to G$ is a natural isomorphism iff each component α_x is an isomorphism.

Definition

A weak equivalence in $\mathbf{C}^D$ is a natural transformation such that each component is a weak equivalence in $\mathbf{C}$.

Homotopy natural transformations

Question

If $\alpha\colon F \to G$ has each component α_x a *homotopy* equivalence, is α a homotopy equivalence?

Let $\beta_x\colon Gx \to Fx$ be a homotopy inverse to α_x. Then for $f\colon x \to y$,

$$\begin{aligned} \beta_y \circ G(f) &\sim \beta_y \circ G(f) \circ \alpha_x \circ \beta_x \\ &= \beta_y \circ \alpha_y \circ F(f) \circ \beta_x \\ &\sim F(f) \circ \beta_x \end{aligned}$$

So β is only a natural transformation "up to homotopy".

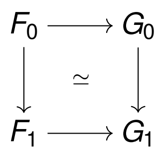
Conclusion: the "weak morphisms" of functors should include "homotopy-natural transformations".

Fibrant diagrams

Question

For what sort of functor G is every homotopy-natural transformation $F \to G$ equivalent to a strictly natural one?

Consider $D = (0 \to 1)$, so $\mathbf{C}^D$ is the category of arrows in $\mathbf{C}$.

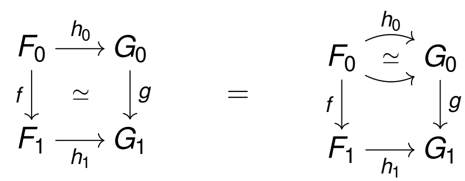


Fibrations

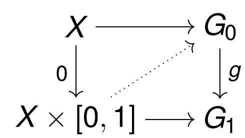
Theorem

For $g\colon G_0 \to G_1$ in spaces, the following are equivalent.

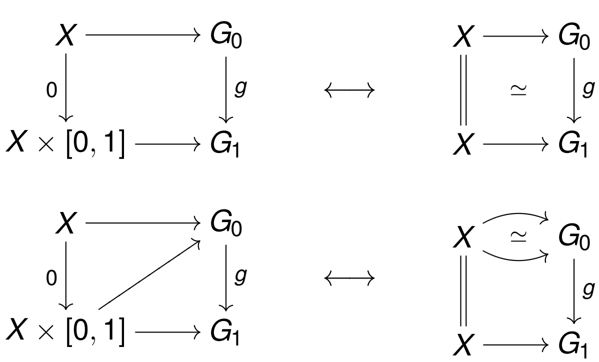
1. *Every homotopy commutative square into g is homotopic to a commutative one with the bottom map fixed:*



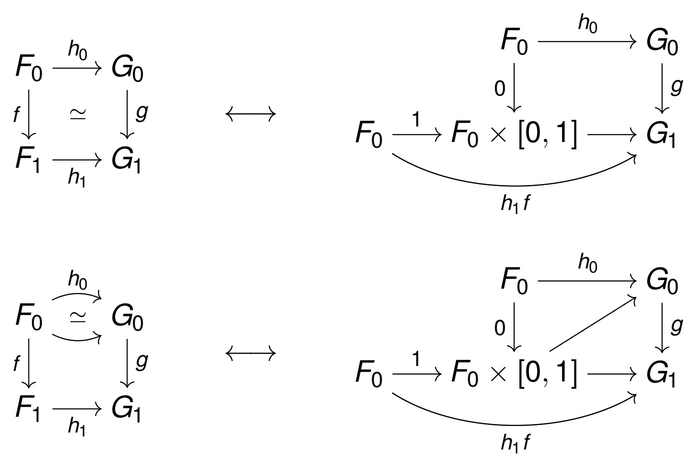
2. *g is a fibration:*



Proof: $1 \Rightarrow 2$



Proof: $2 \Rightarrow 1$



Fibrations and cofibrations

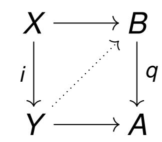
Conclusions

- Fibrations can be the fibrant objects in the category of arrows.
- Similarly, cofibrations (defined dually) can be the cofibrant objects.
- A “category with homotopy theory” should have notions of fibration and cofibration.
- And maybe more stuff, for diagrams $\mathbf{C}^D$ other than arrows?
- This is starting to look like a mess!

Lifting properties

Definition

Given $i\colon X \to Y$ and $q\colon B \to A$ in a category, we say $i \boxslash q$ if any commutative square



admits a dotted filler.

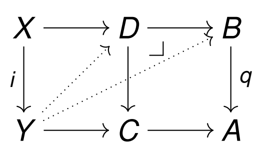
- $\mathcal{I}^{\boxslash} = \{\, q \mid i \boxslash q \quad \forall i \in \mathcal{I}\,\}$
- ${}^{\boxslash}\mathcal{Q} = \{\, i \mid i \boxslash q \quad \forall q \in \mathcal{Q}\,\}$
- fibrations = $\{X \to X \times [0,1]\}^{\boxslash}$

Closure properties of lifting properties

Lemma

If $i \boxslash q$, then $i \boxslash$ (any pullback of q).

Proof.



□

- Similarly, (any pushout of i) $\boxslash q$.
- Also closed under *retracts*.

Weak factorization systems

Definition

A weak factorization system in a category is $(\mathcal{I}, \mathcal{Q})$ such that

1. $\mathcal{I} = {}^{\boxslash}\mathcal{Q}$ and $\mathcal{Q} = \mathcal{I}^{\boxslash}$.
2. Every morphism factors as $q \circ i$ for some $q \in \mathcal{Q}$ and $i \in \mathcal{I}$.

Examples

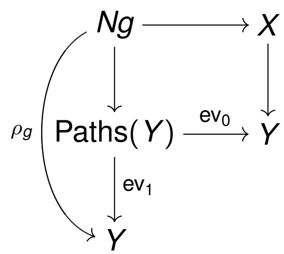
- in sets, $\mathcal{I} =$ surjections, $\mathcal{Q} =$ injections.
- in sets, $\mathcal{I} =$ injections, $\mathcal{Q} =$ surjections.

Note: $\left({}^{\boxslash}(\mathcal{I}^{\boxslash}), \mathcal{I}^{\boxslash}\right)$ always satisfies condition 1.

- $\mathcal{Q} =$ fibrations $= \{X \to X \times [0,1]\}^{\boxslash}$
  $\mathcal{I} = {}^{\boxslash}\mathcal{Q}$
  Factorization?

The mapping path space

Definition
Given $g\colon X \to Y$ in topological spaces, its mapping path space is the pullback



$$Ng = \{\, (x, y, \alpha) \mid x \in X, y \in Y, \alpha\colon g(x) \rightsquigarrow y \,\}$$

and $\rho_g(x, y, \alpha) = y$.

Acyclic cofibrations

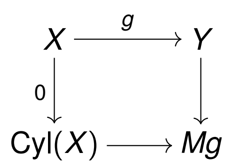
Facts
- $\rho_g\colon Ng \to Y$ is a fibration.
- The map $\lambda_g\colon X \to Ng$ defined by $x \mapsto (x, g(x), c_{g(x)})$ is a cofibration and a homotopy equivalence.
- The composite $X \xrightarrow{\lambda_g} Ng \xrightarrow{\rho_g} Y$ is g.
- ${}^{\boxslash}$(fibrations) = (cofibrations) $\cap$ (homotopy equivalences)

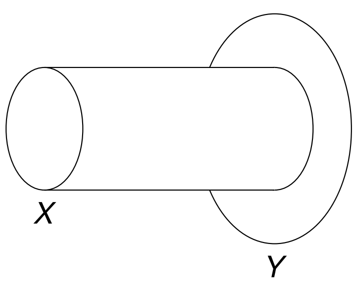
Definition
An acyclic cofibration is a cofibration that is also a homotopy equivalence.

Theorem
(acyclic cofibrations, fibrations) is a weak factorization system.

Mapping cylinders

Given $g\colon X \to Y$, its mapping cylinder is the pushout





Acyclic fibrations

Facts
- $X \to Mg$ is a cofibration
- $Mg \to Y$ is a fibration and a homotopy equivalence.
- The composite $X \to Mg \to Y$ is g.
- (cofibrations)${}^{\boxslash}$ = (fibrations) $\cap$ (homotopy equivalences)

Definition
An acyclic fibration is a fibration that is also a homotopy equivalence.

Theorem
(cofibrations, acyclic fibrations) is a weak factorization system.

Model categories

Definition (Quillen)

A model category is a category **C** with limits and colimits and three classes of maps:

- $\mathcal{C}$ = cofibrations
- $\mathcal{F}$ = fibrations
- $\mathcal{W}$ = weak equivalences

such that

1. $\mathcal{W}$ has the 2-out-of-3 property.
2. $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ and $(\mathcal{C}, \mathcal{F} \cap \mathcal{W})$ are weak factorization systems.

Not all "categories with homotopy" are model categories, but very many are. When it exists, a model category is a very convenient framework.

Homotopy theory in a model category

- X is cofibrant if $\emptyset \to X$ is a cofibration.
- Y is fibrant if $Y \to 1$ is a fibration.
- For any X, we have $\emptyset \to QX \to X$ with QX cofibrant and $QX \to X$ an acyclic fibration (hence a weak equivalence).
- For any Y, we have $Y \to RY \to 1$ with RX fibrant and $Y \to RY$ an acyclic cofibration.
- Any X has a very good cylinder object

$$X + X \xrightarrow{\text{cof.}} \mathrm{Cyl}(X) \xrightarrow{\text{acyc. fib.}} X$$

- Any Y has a very good path object

$$Y \xrightarrow{\text{acyc. cof.}} \mathrm{Paths}(Y) \xrightarrow{\text{fib.}} Y \times Y$$

Homotopy theory in a model category

Let X and Y be fibrant-and-cofibrant in a model category **C**.

- Left and right homotopy agree for maps $X \to Y$.
- Homotopy is an equivalence relation on maps $X \to Y$.
- A map $X \to Y$ is a weak equivalence iff it is a homotopy equivalence.
- In good cases, every functor category $\mathbf{C}^D$ is also a model category.

Some model categories

Example

Topological spaces, with

- Fibrations as before
- Cofibrations defined dually
- Weak equivalences = homotopy equivalences

Example

Topological spaces, with

- Fibrations as before
- Cofibrations = homotopy equivalent to rel. cell complexes
- Weak equivalences = maps inducing isos on all π_n

The second one is equivalent to ∞-groupoids.

Some model categories

Example

Chain complexes, with

- Fibrations = degreewise-split surjections
- Cofibrations = degreewise-split injections
- Weak equivalences = chain homotopy equivalences

Example

Chain complexes, with

- Fibrations = degreewise surjections
- Cofibrations = degreewise-split injections with projective cokernel
- Weak equivalences = maps inducing isos on all H_n

Some model categories

Example

Small categories (or groupoids), with

- Fibrations = functors that lift isomorphisms
- Cofibrations = injective on objects
- Weak equivalences = equivalences of categories

Example

Any category, with

- Fibrations = all maps
- Cofibrations = all maps
- Weak equivalences = isomorphisms

Simplicial sets

Example

Simplicial sets, with

- Fibrations = Kan fibrations
- Cofibrations = monomorphisms
- Weak equivalences = geometric realization equivalences.

This is unreasonably well-behaved in many ways.

Display map categories

Recall

A display map category is a category with

- A terminal object.
- A subclass of its morphisms called the display maps, denoted $B \twoheadrightarrow A$ or $B \rightarrowtriangle A$.
- Any pullback of a display map exists and is a display map.

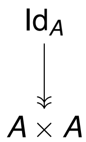
Note: The right class of any weak factorization system can be a class of display maps.

Identity types in d.m. categories

The dependent identity type

$$x\colon A, y\colon A \vdash (x = y)\colon \mathsf{Type}$$

must be a display map

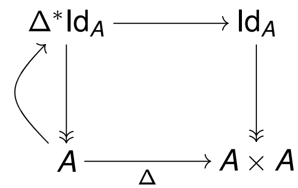


Identity types in d.m. categories

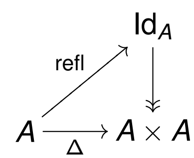
The reflexivity constructor

$$x\colon A \vdash \mathsf{refl}(x)\colon (x = x)$$

must be a section

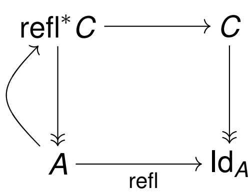


or equivalently a lifting

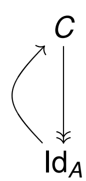


Identity types in d.m. categories

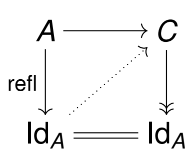
The eliminator says given a dependent type with a section



there exists a compatible section

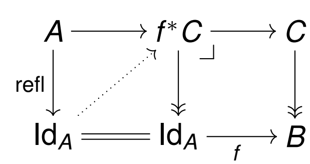


In other words, we have the lifting property



Identity types in d.m. categories

In fact, refl ⧄ all display maps.



Conclusion

Identity types factor $\Delta\colon A \to A \times A$ as

$$A \xrightarrow{\mathsf{refl}} \mathsf{Id}_A \overset{q}{\twoheadrightarrow} A \times A$$

where q is a display map and refl ⧄ (display maps).

General factorizations

Theorem (Gambino–Garner)

In a display map category that models identity types, any morphism $g\colon A \to B$ factors as

$$A \xrightarrow{i} Ng \xrightarrow{q} B$$

where q is a display map, and $i \boxslash$ all display maps.

$$Ng = [\![y\colon B, x\colon A, p\colon (g(x) = y)]\!]$$

is the type-theoretic mapping path space.

Corollary

- $\mathcal{I} = {}^{\boxslash}$(display maps)
- $\mathcal{Q} = \mathcal{I}^{\boxslash}$

is a weak factorization system.

Modeling identity types

Theorem (Awodey–Warren,Garner–van den Berg)

In a display map category, if

$$\left({}^{\boxslash}(\textit{display maps}), ({}^{\boxslash}(\textit{display maps}))^{\boxslash}\right)$$

is a well-behaved weak factorization system, then the category models identity types.

products	$\longleftrightarrow$	categorical products
disjoint unions	$\longleftrightarrow$	categorical coproducts
	⋮	
identity types	$\longleftrightarrow$	weak factorization systems

Type theory of homotopy theory

The model category of simplicial sets is well-behaved.

Conclusion

We can prove things about ordinary homotopy theory by reasoning inside homotopy type theory.
(The complete theory of simplicial sets).

Interpreted in ordinary homotopy theory,

- Function extensionality holds.
- The univalence axiom holds (Voevodsky).
- A space A is n-truncated just when $\pi_k(A) = 0$ for $k > n$.
- An equivalence is a classical (weak) homotopy equivalence.

Other homotopy theories

To model homotopy type theory, we need a category that

1. has finite limits and colimits (for $\times$, $+$, etc.)
2. has a well-behaved WFS (for identity types),
3. is compatibly locally cartesian closed (for Π),
4. has a univalent universe (for coherence and univalence)

These requirements basically restrict us to $(\infty, 1)$-toposes.

Unfortunately, no one has yet found *sufficiently coherent* univalent universes in any $(\infty, 1)$-topos other than simplicial sets (i.e. ∞-groupoids).