The additivity and multiplicativity of fixed-point invariants

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Outline

Counting fixed points

Let M be a manifold, $f\colon M \to M$ continuous.

Question

What is a good invariant of f that tells us about its fixed points?

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$FP(f) =$ the number of fixed points of f.

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First try

$FP(f)$ = the number of fixed points of f.

Problems with this:

1. It's not very computable.
2. It's not invariant under deformations.

Counting with multiplicity

A better answer

Count them with multiplicity!

$$L(f) = \sum_{f(x)=x} \mathrm{ind}_f(x)$$

where $\mathrm{ind}_f(x)$ is the index of x (a fixed point of f).

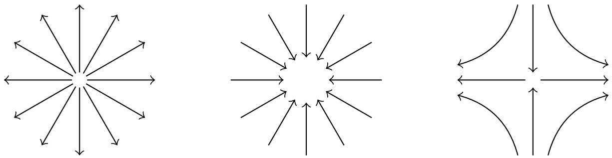
1. This is more computable (as we will see).
2. It is also invariant under deformations.

This is the total fixed point index or the Lefschetz number of f.

Indices of fixed points

The index is like the "determinant" of the local behavior of f near the fixed point.

index 1 index 1 index -1



Under a deformation:

- Two fixed points of index 1 can "merge" into one of index 2;
- Two fixed points of indices 1 and -1 can "annihilate";
- etc...

The Lefschetz fixed-point theorem

Theorem

If $L(f) \neq 0$, then f has a fixed point.

Proof.

Obvious! $\square$

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Obvious! $\square$

The work is in finding a definition of $L(f)$ that we can calculate without already knowing what the fixed points are.

Computing the Lefschetz number

One option is:

Theorem

$$L(f) = \sum_{i \in \mathbb{Z}} (-1)^i \operatorname{tr}(H_i(f))$$

where $H_i(f)$ is the map induced by f on i^{th} homology.

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Example

$L(\mathrm{id}_M) = \sum_i (-1)^i \dim(H_i(M)) = \chi(M)$, the Euler characteristic.

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$L(\mathrm{id}_M) = \sum_i (-1)^i \dim(H_i(M)) = \chi(M)$, the Euler characteristic.

Another option is to break down M into smaller pieces.

Trivial additivity and multiplicativity

Theorem

For $f: M \to M$ *and* $g: N \to N$*, we have* $f \amalg g: M \amalg N \to M \amalg N$*. Then*

$$L(f \amalg g) = L(f) + L(g).$$

Trivial additivity and multiplicativity

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$$L(f \amalg g) = L(f) + L(g).$$

Theorem

For $f\colon M \to M$ *and* $g\colon N \to N$*, we have* $f \times g\colon M \times N \to M \times N$*. Then*

$$L(f \times g) = L(f) \cdot L(g).$$

Nontrivial additivity

Let $A \subseteq M$, and $h \colon M \to M$ with $h(A) \subseteq A$. Define $f = h|_A \colon A \to A$, and induce $g \colon M/A \to M/A$.

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "f"'] & M \arrow[r] \arrow[d, "h"'] & M/A \arrow[d, "g"] \\
A \arrow[r] & M \arrow[r] & M/A
\end{tikzcd}$$

Theorem

$$L(h) = L(f) + L(g).$$

Nontrivial multiplicativity

Let $p\colon E \to B$ be a fiber bundle with fiber F, and let $f\colon E \to E$ be a map over $\overline{f}\colon B \to B$.

$$\begin{tikzcd} E \arrow[r, "f"] \arrow[d, "p"'] & E \arrow[d, "p"] \\ B \arrow[r, "\overline{f}"'] & B \end{tikzcd}$$

If b is a fixed point of $\overline{f}$, we have $f_b\colon F \to F$.

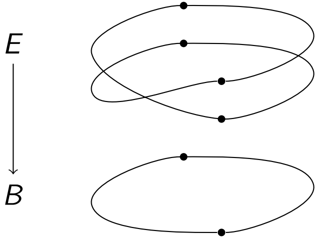
Theorem

If $E \to B$ is "orientable" and B is connected, then

$$L(f) = L(\overline{f}) \cdot L(f_b).$$

A nonorientable fibration

Let $E = B = S^1$, with $E \to B$ the double cover. Let $\overline{f}$ be reflection in the y-axis, and f some map over it.

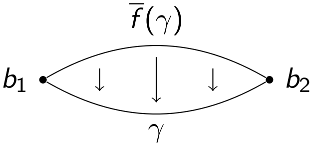


Then $L(f_b) = 0$ over one fixed point of $\overline{f}$, but $= 2$ over the other.

Fixed-point classes

Definition

Fixed points b_1 and b_2 of $\overline{f}$ are in the same fixed-point class if there is a path γ in B from b_1 and b_2, such that $\overline{f}(\gamma)$ can be deformed back to γ keeping the endpoints fixed.



Theorem

If b_1 and b_2 are in the same fixed-point class, then $L(f_{b_1}) = L(f_{b_2})$.

Nonorientable multiplicativity

Theorem (Ponto–S.)

$$L(f) = \sum_{\substack{\textit{fixed point} \\ \textit{classes } C}} \mathrm{ind}_{\overline{f}}(C) \cdot L(f_C)$$

- $L(f_C)$ means $L(f_b)$ for any $b \in C$
- $\mathrm{ind}_{\overline{f}}(C) = \sum_{b \in C} \mathrm{ind}_{\overline{f}}(b)$

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- $\mathrm{ind}_{\overline{f}}(C) = \sum_{b \in C} \mathrm{ind}_{\overline{f}}(b)$

Remark

If $E \to B$ is orientable and B is connected, all fixed-points are in the same class C. Thus $\mathrm{ind}_{\overline{f}}(C) = \sum_{f(b)=b} \mathrm{ind}_{\overline{f}}(b) = L(\overline{f})$, so

$$L(f) = L(\overline{f}) \cdot L(f_b)$$

Outline

Categorical traces

V a finite dim. vector space with basis $\{v_i\}$, $V^* = \hom(V, \Bbbk)$ its dual, $f \colon V \to V$ a linear map. The trace of f can be calculated by:

$$\Bbbk \xrightarrow{\eta} V \otimes V^* \xrightarrow{f \otimes \mathrm{id}} V \otimes V^* \xrightarrow{\cong} V^* \otimes V \xrightarrow{\epsilon} \Bbbk.$$

$$1 \longmapsto \textstyle\sum_i v_i \otimes v_i^* \longmapsto \sum_{ij} a_{ij} v_j \otimes v_i^* \longmapsto \sum_{ij} v_i^* \otimes a_{ij} v_j \longmapsto \sum_{ij} a_{ij} v_i^*(v_j)$$

$$= \textstyle\sum_i a_{ii}.$$

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$$= \textstyle\sum_i a_{ii}.$$

This is the definition of η.

ϵ is evaluation of covectors.

Geometrical traces

$M \hookrightarrow \mathbb{R}^p$ a smooth manifold, $T\nu_M$ the one-point compactification of its normal bundle, $f\colon M \to M$ continuous. Then

$$S^p \xrightarrow{\eta} M_+ \wedge T\nu \xrightarrow{f \wedge \mathrm{id}} M_+ \wedge T\nu \xrightarrow{\cong} T\nu \wedge M_+ \xrightarrow{\epsilon} S^p.$$

has degree equal to the Lefschetz number of f.

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"Pontryagin-Thom maps"

Symmetric monoidal categories

Definition

A symmetric monoidal category is a category equipped with

- A "tensor product" of objects $\otimes$;
- A "unit object" I;
- Natural isomorphisms $M \otimes (N \otimes P) \cong (M \otimes N) \otimes P$ and $M \otimes I \cong M \cong I \otimes M$;
- Satisfying certain axioms.

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Examples

1. Vector spaces with the usual tensor product, $I = \mathbb{Z}$.
2. "Pointed, stable" spaces with the smash product, $I = S^p$.

Spectra

By a “pointed, stable” space I really mean a spectrum, but here’s all you need to know.

1. Any space M becomes pointed with a disjoint basepoint, M_+.
2. Pointed spaces have a “smash product” $\wedge$.
3. We have $(M \times N)_+ \cong M_+ \wedge N_+$.
4. “Stabilizing” allows us to smash by high-dimensional spheres S^p “without changing anything”.

Duality and trace

Definition

In a symmetric monoidal category, a dualizable object is M with M^* and maps

$$I \xrightarrow{\eta} M \otimes M^* \qquad M^* \otimes M \xrightarrow{\varepsilon} I$$

satisfying axioms.

Definition

If M is dualizable, the trace of $f\colon M \to M$ is

$$\begin{tikzcd} I \arrow[r, "\eta"] \arrow[rrrr, bend right, "\mathrm{tr}(f)"'] & M \otimes M^* \arrow[r, "f\otimes 1"] & M \otimes M^* \arrow[r, "\cong"] & M^* \otimes M \arrow[r, "\varepsilon"] & I \end{tikzcd}$$

Categorical trivial multiplicativity

Theorem (Easy)

If M and N are dualizable and $M \xrightarrow{f} M$, $N \xrightarrow{g} N$, then

$$\begin{tikzcd} I \arrow[r, "\mathrm{tr}(f)"] \arrow[rr, bend right, "\mathrm{tr}(f \otimes g)"'] & I \arrow[r, "\mathrm{tr}(g)"] & I \end{tikzcd}$$

Examples

1. For vector spaces, I is the ground field $\mathbb{k}$, and composition of linear maps $\mathbb{k} \to \mathbb{k}$ is multiplication.
2. For pointed stable spaces, I is a big sphere S^p, and composition of maps $S^p \to S^p$ multiplies their degrees.

Categorical trivial additivity

Theorem (Easy)

In a suitably "additive" context, we have

$$\begin{tikzcd} I \arrow[r, "\begin{bmatrix}1\\1\end{bmatrix}"] \arrow[rr, bend right, "\mathrm{tr}(f\oplus g)"'] & I \oplus I \arrow[r, "{[\mathrm{tr}(f),\mathrm{tr}(g)]}"] & I \end{tikzcd}$$

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Examples

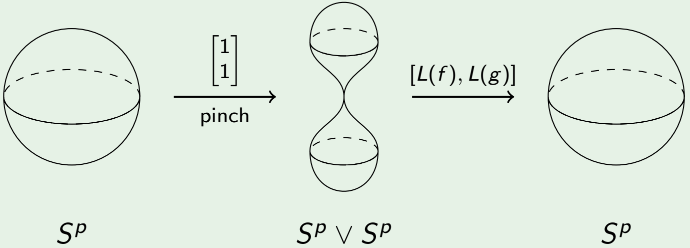
1. For vector spaces, this is matrix multiplication:

$$[\mathrm{tr}(f), \mathrm{tr}(g)] \cdot \begin{bmatrix}1\\1\end{bmatrix} = \mathrm{tr}(f) + \mathrm{tr}(g)$$

Categorical trivial additivity

Examples

2. For manifolds, we have



Categorical nontrivial additivity

Theorem (Not so easy) (May)

In a suitably "stable" context, for

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & M \arrow[r] \arrow[d, "h"'] & M/A \arrow[d, "g"] \\ A \arrow[r] & M \arrow[r] & M/A \end{tikzcd}$$

we have

$$\begin{tikzcd} I \arrow[r, "\begin{bmatrix} -1 \\ 1 \end{bmatrix}"] \arrow[rr, bend right, "\mathrm{tr}(g)"'] & I \oplus I \arrow[r, "{[\mathrm{tr}(f), \mathrm{tr}(h)]}"] & I \end{tikzcd}$$

Arrow-theoretic nonorientable multiplicativity

$$L(f) = \sum_{\substack{\text{fixed point} \\ \text{classes } C}} \mathrm{ind}_{\overline{f}}(C) \cdot L(f_C)$$

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Can be expressed as a composition:

$$\begin{tikzcd}[column sep=huge]
S^n \arrow[r, "\begin{bmatrix} \vdots \\ \mathrm{ind}_{\overline{f}}(C) \\ \vdots \end{bmatrix}"] \arrow[rr, bend right, "L(f)"'] & ?? \arrow[r, "{[\ldots, L(f_C), \ldots]}"] & S^n
\end{tikzcd}$$

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\end{tikzcd}$$

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$$\begin{tikzcd}
S^n \arrow[r, "\begin{bmatrix} \vdots \\ \mathrm{ind}_{\overline{f}}(C) \\ \vdots \end{bmatrix}"] \arrow[rr, bend right, "L(f)"'] & \Lambda^{\overline{f}} B_+ \wedge S^n \arrow[r, "{[\dots, L(f_C), \dots]}"] & S^n
\end{tikzcd}$$

The twisted free loop space

For $\overline{f}\colon B \to B$, we define

$$\Lambda^{\overline{f}} B = \left\{ \gamma\colon [0,1] \to B \;\middle|\; \gamma(0) = \overline{f}(\gamma(1)) \right\}$$

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$$\Lambda^{\bar{f}}B = \left\{\gamma\colon [0,1] \to B \;\middle|\; \gamma(0) = \bar{f}(\gamma(1))\right\}$$

- If b is a fixed point, then the constant path c_b lies in $\Lambda^{\bar{f}}B$.
- Fixed points b_1, b_2 are in the same class exactly when c_{b_1} and c_{b_2} lie in the same path-component of $\Lambda^{\bar{f}}B$.

Outline

The Hattori-Stallings trace

R a noncommutative ring, $M \cong R^n$ a finitely generated free right R-module, $f\colon M \to M$ an R-map.

- Can give f a “matrix” (a_{ij}), each $a_{ij} \in R$.
- Define $\mathrm{tr}(f) = \sum_i a_{ii}$?

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Definition

For an R-R-bimodule N, its shadow is $\langle\!\langle N \rangle\!\rangle = N \Big/ \langle r \cdot n = n \cdot r \rangle$.

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Definition

The Hattori-Stallings trace of f is the image of $\sum_i a_{ii}$ in $\langle\!\langle R \rangle\!\rangle$.

Our f.g. free right module M has a dual $M^* = \hom_R(M, R)$, a left R-module.

$$\mathbb{Z} \xrightarrow{\eta} M \otimes_R M^* \xrightarrow{f \otimes \mathrm{id}} M \otimes_R M^* \xrightarrow{??} M^* \otimes_{\mathbb{Z}} M \xrightarrow{\epsilon} R.$$

Categorical Hattori-Stallings

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$$\mathbb{Z} \xrightarrow{\eta} M \otimes_R M^* \xrightarrow{f \otimes \mathrm{id}} M \otimes_R M^* \xrightarrow{\cong} \langle\langle M^* \otimes_{\mathbb{Z}} M \rangle\rangle \xrightarrow{\epsilon} \langle\langle R \rangle\rangle.$$

sends $1 \in \mathbb{Z}$ to the Hattori-Stallings trace $\mathrm{tr}(f) \in \langle\langle R \rangle\rangle$.

Generalization #1: twisted traces

$\phi\colon R \to R$ a ring homomorphism, $f\colon M \to M$ a ϕ-equivariant map:

$$f(m \cdot r) = f(m) \cdot \phi(r).$$

Then $\operatorname{tr}(f)$ must take values in $\langle\!\langle R_\phi \rangle\!\rangle = R \Big/ \langle r \cdot s = s \cdot \phi(r) \rangle$.

$$\mathbb{Z} \xrightarrow{\eta} M \otimes_R M^* \xrightarrow{f \otimes \mathrm{id}} M \otimes_{R_\phi} M^* \xrightarrow{\cong} \langle\!\langle M^* \otimes_{\mathbb{Z}} M_\phi \rangle\!\rangle \xrightarrow{\epsilon} \langle\!\langle R_\phi \rangle\!\rangle.$$

Generalization #2: traces for bimodules

M an R-S-bimodule which is f.g. free as an S-module.
Then M^* is an S-R-bimodule.

$$\langle\!\langle R \rangle\!\rangle \xrightarrow{\eta} \langle\!\langle M \otimes_S M^* \rangle\!\rangle \xrightarrow{f \otimes \mathrm{id}} \langle\!\langle M \otimes_S M^* \rangle\!\rangle \xrightarrow{\cong} \langle\!\langle M^* \otimes_R M \rangle\!\rangle \xrightarrow{\epsilon} \langle\!\langle S \rangle\!\rangle.$$

sends each $r \in \langle\!\langle R \rangle\!\rangle$ to the H-S trace of the S-module map

$$m \mapsto r \cdot f(m)$$

Generalization #3: twisted traces for bimodules

M an R-S-bimodule which is f.g. free as an S-module.
Let $\phi\colon R \to R$, $\psi\colon S \to S$ be ring homomorphisms, and let $f\colon M \to M$ be ϕ-ψ-equivariant:

$$f(r \cdot m \cdot s) = \phi(r) \cdot f(m) \cdot \psi(s).$$

Then the trace

$$\langle\!\langle R_\phi \rangle\!\rangle \xrightarrow{\eta} \langle\!\langle M \otimes_{S_\psi} M^*_\phi \rangle\!\rangle \xrightarrow{f \otimes \mathrm{id}} \langle\!\langle M \otimes_{S_\psi} M^*_\phi \rangle\!\rangle \xrightarrow{\cong} \langle\!\langle M^* \otimes_{R_\phi} M_\psi \rangle\!\rangle \xrightarrow{\epsilon} \langle\!\langle S_\psi \rangle\!\rangle.$$

is a map $\langle\!\langle R_\phi \rangle\!\rangle \to \langle\!\langle S_\psi \rangle\!\rangle$.

Bicategories

Definition

A bicategory is a structure $\mathscr{B}$ with

- "Objects" or "0-cells" $A, B, C, \ldots$;
- "Hom-categories" $\mathscr{B}(A, B), \ldots$;
- "Composition" or "tensor product" functors

$$\mathscr{B}(A, B) \times \mathscr{B}(B, C) \xrightarrow{\odot} \mathscr{B}(A, C)$$

- "Unit" objects $U_A \in \mathscr{B}(A, A)$;
- Natural isomorphisms $M \odot (N \odot P) \cong (M \odot N) \odot P$ and $M \odot U_B \cong M \cong U_A \odot M$;
- Satisfying certain axioms.

Example

Objects $=$ rings, $\mathscr{B}(R, S) =$ the category of R-S-bimodules.

Shadows

Definition (Ponto)

A shadow on a bicategory $\mathscr{B}$ is a collection of functors

$$\langle\!\langle - \rangle\!\rangle \colon \mathscr{B}(A, A) \to \mathscr{T}$$

together with

- Isomorphisms $\langle\!\langle M \odot N \rangle\!\rangle \cong \langle\!\langle N \odot M \rangle\!\rangle$;
- Satisfying certain axioms.

Traces with shadows

Definition

An object $M \in \mathscr{B}(A,B)$ is dualizable if there is $M^* \in \mathscr{B}(B,A)$ and maps

$$U_A \xrightarrow{\eta} M \odot M^* \qquad M^* \odot M \xrightarrow{\varepsilon} U_B$$

satisfying axioms.

Definition

If M is dualizable, the trace of $f\colon M \to M$ is

$$\begin{tikzcd}
\langle\langle U_A \rangle\rangle \arrow[r, "\eta"] \arrow[rrrr, bend right, "\mathrm{tr}(f)"'] & \langle\langle M \odot M^* \rangle\rangle \arrow[r, "f \otimes 1"] & \langle\langle M \odot M^* \rangle\rangle \arrow[r, "\cong"] & \langle\langle M^* \odot M \rangle\rangle \arrow[r, "\varepsilon"] & \langle\langle U_B \rangle\rangle
\end{tikzcd}$$

Twisted traces

Definition

If M is dualizable, the trace of $f\colon P \odot M \to M \odot Q$ is

$$\begin{tikzcd} \langle\!\langle P \rangle\!\rangle \arrow[r, "\eta"] \arrow[rrrr, bend right, "\mathrm{tr}(f)"'] & \langle\!\langle P\, M\, M^* \rangle\!\rangle \arrow[r, "f \otimes 1"] & \langle\!\langle M\, Q\, M^* \rangle\!\rangle \arrow[r, "\cong"] & \langle\!\langle Q\, M^*\, M \rangle\!\rangle \arrow[r, "\varepsilon"] & \langle\!\langle Q \rangle\!\rangle \end{tikzcd}$$

(Omitting the $\odot$ symbols for space reasons.)

Bicategorical multiplicativity

Theorem (Easy)

If M and N are dualizable in a bicategory, and $f\colon P \odot M \to M \odot Q$ and $g\colon Q \odot N \to N \odot R$, then

$$\begin{tikzcd} \langle\!\langle P \rangle\!\rangle \arrow[r, "\mathrm{tr}(f)"] \arrow[rr, bend right, "\mathrm{tr}(f \otimes g)"'] & \langle\!\langle Q \rangle\!\rangle \arrow[r, "\mathrm{tr}(g)"] & \langle\!\langle R \rangle\!\rangle \end{tikzcd}$$

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Corollary

For f a ϕ-ψ-equivariant map and g a ψ-χ-equivariant map,

$$\begin{tikzcd} \langle\!\langle R_\phi \rangle\!\rangle \arrow[r, "\mathrm{tr}(f)"] \arrow[rr, bend right, "\mathrm{tr}(f\otimes g)"'] & \langle\!\langle S_\psi \rangle\!\rangle \arrow[r, "\mathrm{tr}(g)"] & \langle\!\langle T_\chi \rangle\!\rangle \end{tikzcd}$$

Outline

A different bicategory

We work in a bicategory of spaces and (pointed, stable) fibrations.

$$\begin{array}{rcl} \text{rings } R,\ S & \longleftrightarrow & \text{spaces } A,\ B \\ R\text{-}S\text{-bimodules} & \longleftrightarrow & \text{fibrations } E \to A \times B \end{array}$$

A different bicategory

We work in a bicategory of spaces and (pointed, stable) fibrations.

$$\begin{array}{rcl} \text{rings } R,\ S & \longleftrightarrow & \text{spaces } A,\ B \\ R\text{-}S\text{-bimodules} & \longleftrightarrow & \text{fibrations } E \to A \times B \\ \text{the integers } \mathbb{Z} & \longleftrightarrow & \text{the one-point space } \star \\ \text{right } R\text{-module} & \longleftrightarrow & \text{fibration } E \to B \\ \text{left } R\text{-module} & \longleftrightarrow & \text{fibration } E \to B \end{array}$$

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(This bicategory of spectra was constructed by May-Sigurdsson.)

The refined fiberwise Lefschetz number

Let $f\colon E \to E$ be a map over $\overline{f}\colon B \to B$.

$$\begin{tikzcd} E \arrow[r, "f"] \arrow[d, "p"'] & E \arrow[d, "p"] \\ B \arrow[r, "\overline{f}"'] & B \end{tikzcd}$$

We can regard $E \to B$ as a "left B-module", and f as an "$\overline{f}$-equivariant map." It is dualizable if the fibers of p are manifolds, and its trace is

$$\Lambda^{\overline{f}} B_+ \wedge S^p \xrightarrow{[\ldots, L(f_C), \ldots]} S^p$$

Definition

This is the refined fiberwise Lefschetz number of f.

The Reidemeister trace

We can also regard $\mathrm{id}_B\colon B \to B$ as a "right B-module", and $\overline{f}$ itself as an "$\overline{f}$-equivariant map". It is dualizable if B is a manifold, and its trace is

$$S^p \xrightarrow{\begin{bmatrix} \vdots \\ \mathrm{ind}_{\overline{f}}(C) \\ \vdots \end{bmatrix}} \Lambda^{\overline{f}} B_+ \wedge S^p$$

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(The Reidemeister trace is of independent interest; it refines $L(\overline{f})$ and supports a converse to the Lefschetz fixed point theorem.)

Nonorientable multiplicativity

The "tensor product" of the "right B-module" $\mathrm{id}_B \colon B \to B$ and the "left B-module" $E \to B$ is the space E.

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\begin{tikzcd}[column sep=huge]
S^p \arrow[r, "\begin{bmatrix} \vdots \\ \mathrm{ind}_{\overline{f}}(C) \\ \vdots \end{bmatrix}"] \arrow[rr, bend right, "L(f)"'] & \Lambda^{\overline{f}} B_+ \wedge S^p \arrow[r, "{[\ldots, L(f_C), \ldots]}"] & S^p
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Theorem

$$L(f) = \sum_{\substack{\textit{fixed point} \\ \textit{classes } C}} \mathrm{ind}_{\overline{f}}(C) \cdot L(f_C)$$

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Also generalizes to compute $R(f)$ in terms of $R(f_C)$.

An easy route to nontrivial additivity

Work in yet another bicategory.

$$\begin{array}{rcl} \text{rings } R,\ S & \longleftrightarrow & \text{small categories } A,\ B \\ R\text{-}S\text{-bimodules} & \longleftrightarrow & \text{functors } B^{op} \times A \to \text{Spaces} \\ \text{ring map } \phi\colon R \to S & \longleftrightarrow & \text{functor } \phi\colon A \to B \end{array}$$

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Definition

The shadow of a category A is the disjoint union of all endomorphisms in A, modulo "conjugacy".

$$\langle\!\langle A \rangle\!\rangle = \left(\coprod_{a \in A} \hom_A(a, a) \right) \Big/ (\alpha\beta \sim \beta\alpha)$$

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In particular, $\langle\!\langle A \rangle\!\rangle$ contains a class for each identity morphism $[1_a]$, with $[1_a] = [1_b]$ if and only if $a \cong b$.

- Let $B = (a \to b)$. Then (stabilizing)

$$\langle\!\langle B \rangle\!\rangle = \left\{[1_a], [1_b]\right\}_+ \wedge S^p \;\cong\; S^p \vee S^p.$$

The refined Lefschetz number of a diagram

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- An $h\colon M \to M$ with $f = h|_A$ gives a "module map" $M \to M$.
- M is dualizable if M and A are manifolds, and the trace of h is

$$S^p \vee S^p \xrightarrow{[L(f), L(h)]} S^p$$

The Reidemeister trace of a category

With $B = (a \to b)$ as before, we have a functor $N\colon B^{op} \to \text{Spaces}$:

$$N(a) = *$$
$$N(b) = S^p$$

This is dualizable as a "right B-module", and its trace is

$$S^p \xrightarrow{\begin{bmatrix} -1 \\ 1 \end{bmatrix}} S^p \vee S^p$$

Nontrivial additivity

The "tensor product" of the "right B-module" $A \hookrightarrow M$ and the "left B-module N is the quotient M/A.

$$\begin{tikzcd} S^p \arrow[r, "\begin{bmatrix}-1\\1\end{bmatrix}"] \arrow[rr, bend right, "L(g)"'] & S^p \vee S^p \arrow[r, "{[L(f),L(h)]}"] & S^p \end{tikzcd}$$

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\end{tikzcd}
$$

Theorem

$$L(g) = L(h) - L(f)$$

Generalizes to colimits of other diagram shapes.

Example

- B a small "homotopically finite" category.
- The trivial left B-module has trace $[1, 1, \ldots, 1]$.
- The trivial right B-module has a trace that assigns a "weight" k_a to each isomorphism class of objects $[1_a]$.
- The tensor product of these modules is $|NB|$, and so

$$\chi(|NB|) = \sum_a k_a$$

is the Euler characteristic of B (Leinster).

Conclusion

1. Addition and multiplication are both certain special cases of linear combinations.
2. "Sums" and "products" of spaces are both certain kinds of tensor product.
3. Additivity and multiplicativity formulas for trace-like invariants follow formally from these identifications.