Unbounded quantifiers via 2-categorical logic

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Why 2-categorical logic?

For the same reasons we study 1-categorical logic.

1. It tells us things about 2-categories.
   - Proofs about fibrations and stacks are simplified by the internal logic of their 2-categories, just as proofs about sheaves are simplified by the internal logic of a topos.
2. It tells us things about 2-logic.
   - 2-categorical models of 2-dimensional theories give insight into the structure of those theories, such as independence results and completeness theorems, just as 1-categorical models do for 1-dimensional theories.
   - “The true role of the axiom of choice in category theory.”

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Also:

3. It shows us how to extend 1-categorical logic to deal with unbounded quantifiers.

Outline

Categorical logic

Logic/Type theory	Category theory
syntax	semantics
$\vdash A : \mathrm{Type}$	$A \in \mathrm{ob}(\mathbf{C})$
$x : A \vdash f(x) : B$	$f : A \to B$
$x : A \vdash P(x) : \mathrm{Prop}$	$P \in \mathrm{Sub}(A)$
quantifiers $\exists, \forall$	adjoints to pullback $\exists_f \dashv f^* \dashv \forall_f$

However: in mathematics we frequently also use unbounded quantifiers, i.e. quantifiers over sets/types.

Universes

Standard solution:

Logic/Type theory	Category theory
a universe	a universal object $U = [\text{Type}]$
$(\forall A : \text{Type})\varphi(A)$	$\forall_\pi : \text{Sub}(X \times U) \to \text{Sub}(X)$

Remarks

- "Unbounded" quantifiers become quantifiers over the type of (small) types.
- E.g. in *algebraic set theory*, we have a *category of classes* with *small maps* which are representable by some universal object.

Why not universes?

- Not *intrinsic* to the category of (small) sets/types: need to construct a containing category of classes.
- Not *canonical*: a category of sets can be contained in many different categories of classes.
- Properties of a category of sets may not be reflected in a category of classes containing it (not *first-order*).
- Not *category-theoretic*: have to fix a notion of equality (rather than isomorphism) between sets.

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Goal

An *intrinsic, canonical, first-order, and category-theoretic* treatment of unbounded quantifiers, in which we can formulate axioms equivalent to set-theoretic axioms like replacement and full separation.

Adding quantifiers

Logic/Type theory	Category theory
propositional logic	Heyting algebra
$\downarrow$	$\downarrow$
predicate logic (“bounded” quantifiers)	categorical models (topoi, pretopoi)
$\downarrow$	$\downarrow$
“unbounded” quantifiers (quantifiers over Type)	??

Question: What is the *natural* object to fill the hole?

Sheaves, stacks, and 2-topoi

We do have a canonical way to do the first step on the categorical side.

(propositional logic)	Heyting algebra
	↓ topos of sheaves
(bounded quantifiers)	topos
	⇣
(unbounded quantifiers)	??

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(propositional logic)	Heyting algebra = 0-topos
	↓ topos of sheaves
(bounded quantifiers)	topos = 1-topos
	⇣ 2-topos of stacks
(unbounded quantifiers)	2-topos?

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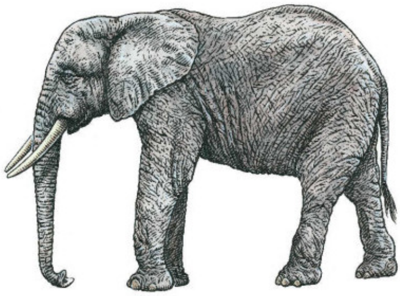
(unbounded quantifiers)	2-topos?

Next question: What is a 2-topos?

Outline

A topos is like an elephant

With apologies to André Joyal and Peter Johnstone



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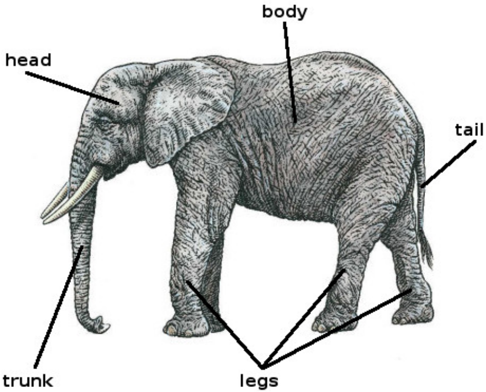

"It's all one animal."

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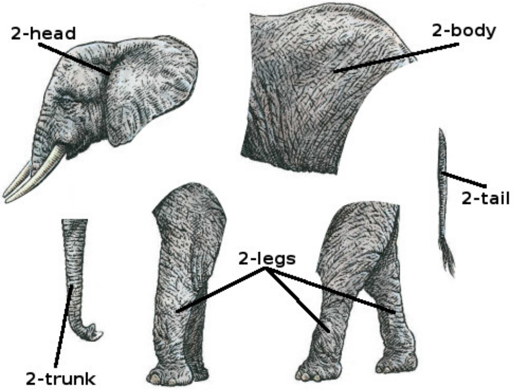

Is it necessarily all one animal?

Two parts of the elephant

- A topos is the category of sheaves on a site.
- A topos is a setting for intuitionistic higher-order logic.

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  - A category is a Grothendieck topos iff it is an infinitary pretopos with a small generating set (Giraud).
- A topos is a setting for intuitionistic higher-order logic.
  - A Heyting pretopos is a setting for intuitionistic first-order logic; a topos adds higher-order structure.

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Idea: Look first for a notion of 2-pretopos.

The 2-Giraud theorem

Theorem (Street, 1982)

A 2-category is the 2-category of stacks on a 2-site iff it is an infinitary 2-pretopos with a small strongly-generating set.

Such a category is called a Grothendieck 2-topos.

The 2-Giraud theorem

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So what is an infinitary 2-pretopos?

Regular 2-categories

- A regular 1-category has pullback-stable image factorizations, generalizing the (surjection, injection) factorizations in *Set*.
- A regular 2-category has pullback-stable image factorizations, generalizing the (essentially surjective, fully faithful) factorizations in *Cat*.

The main point

The subobjects in a 2-category are fully faithful inclusions.

Heyting 2-categories

- A coherent 2-category is regular and has stable unions of fully faithful morphisms.
- A Heyting 2-category is coherent and has right adjoints to pullback of ffs.

Observation

Any Grothendieck 2-topos is a Heyting 2-category.

This is all we need from the 2-Giraud theorem.

2-categorical logic

A Heyting 2-category has an internal first-order “2-logic.”

- Two sorts of terms: object-terms and arrow-terms.
- Represented by morphisms and 2-cells, respectively.

In the Grothendieck 2-topos of stacks on a site **S**, the internal 2-logic has a Kripke-Joyal description as a “stack” semantics over **S**.