Large categories and quantifiers in topos theory

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January 26, 2021
Cambridge Category Theory Seminar

Outline

The problem of large objects in topos theory

$$\begin{array}{ccc}
\text{elementary topos} & & \text{internal logic} \\
\text{(objects} \sim \text{sets)} & \longrightarrow & \text{objects} = \text{types} \\
\downarrow & & \downarrow \\
\text{indexed categories} & & \\
\sim \text{large categories} & \longrightarrow & ?? \\
(+\text{ unbounded quantifiers, } \ldots) & &
\end{array}$$

Idea

Embed an elementary topos $\mathcal{E}$ in a larger category $\widehat{\mathcal{E}}$, whose internal logic includes "large objects" like indexed categories and can quantify over all objects of $\mathcal{E}$.

(Throughout, we assume $\mathcal{E}$ is small.)

Three possibilities for $\widehat{\mathcal{E}}$

1. The topos $\mathrm{Sh}(\mathcal{E})$ of sheaves for the coherent topology on $\mathcal{E}$.
    - An $\mathcal{E}$-indexed category (stack) can be represented by many internal categories in $\mathrm{Sh}(\mathcal{E})$, only weakly equivalent.
    - Not all indexed functors represented by internal ones in $\mathrm{Sh}(\mathcal{E})$.
    - In general, introduces spurious notions of equality of objects.

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2. The 2-category $\mathsf{Ps}(\mathcal{E}^{\mathrm{op}}, \mathsf{Cat})$ of $\mathcal{E}$-indexed categories (pseudofunctors $\mathcal{E}^{\mathrm{op}} \to \mathsf{Cat}$).
   - Constructions like opposites aren't internal.
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   - Constructions like opposites aren't internal.
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3. A fibration structure on the category $[\mathcal{E}^{\mathrm{op}}, \mathsf{Gpd}]$ of (strict) presheaves of groupoids.
   - Strictifications unique up to strong equivalence.
   - Includes all functors.
   - No equality of objects stricter than isomorphism.
   - Can define opposites, etc., for internal categories.
   - Internal logic is Martin-Löf type theory with "homotopy".

Outline

High-level idea

The 2-categorical structure of Gpd can be encoded in its underlying 1-category by path objects:

$$X \underset{g}{\overset{f}{\underset{\Downarrow}{\rightrightarrows}}} Y \qquad \longleftrightarrow \qquad \begin{array}{ccc} & & Y^{\mathcal{I}} \\ & \nearrow & \downarrow \\ X & \xrightarrow[(f,g)]{} & Y \times Y \end{array}$$

where $\mathcal{I} = (\bullet \cong \bullet)$ is the free-living isomorphism.
Moreover, $Y^{\mathcal{I}} \to Y \times Y$ is characterized as the replacement of the diagonal $Y \to Y \times Y$ by a fibration.

We want a similar fibrational encoding of $\mathsf{Ps}(\mathcal{E}^{\mathrm{op}}, \mathsf{Gpd})$, or its subcategory $\mathsf{St}(\mathcal{E}^{\mathrm{op}}, \mathsf{Gpd})$ of stacks.

Coflexible presheaves

$[\mathcal{E}^{\mathrm{op}}, \mathsf{Gpd}]$ = strict functors and strict natural transformations.
$\mathsf{Ps}(\mathcal{E}^{\mathrm{op}}, \mathsf{Gpd})$ = pseudofunctors and pseudonatural transformations.

Lemma

The inclusion $[\mathcal{E}^{\mathrm{op}}, \mathsf{Gpd}] \hookrightarrow \mathsf{Ps}(\mathcal{E}^{\mathrm{op}}, \mathsf{Gpd})$ *has a right adjoint* $\mathfrak{R}$*.*

Definition

$X \in [\mathcal{E}^{\mathrm{op}}, \mathsf{Gpd}]$ is coflexible if the map $X \to \mathfrak{R}X$ has a retraction.

Theorem

Every pseudonatural transformation between coflexible presheaves is isomorphic to a strict natural transformation.

Injective fibrations

For $f : X \to Y$ in $[\mathcal{E}^{\mathrm{op}}, \mathsf{Gpd}]$, define $\mathfrak{R}_Y X$ as the pullback

$$\begin{array}{ccccc} X & \dashrightarrow & \mathfrak{R}_Y X & \longrightarrow & \mathfrak{R} X \\ & \overset{f}{\searrow} & \downarrow & \lrcorner & \downarrow {\scriptstyle \mathfrak{R} f} \\ & & Y & \longrightarrow & \mathfrak{R} Y. \end{array}$$

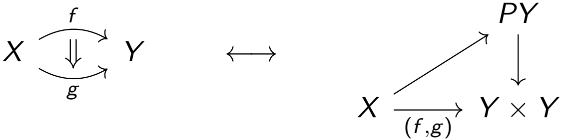
Definition

f is an injective fibration if

1. Each $f_U : X_U \to Y_U$ is a fibration of groupoids, and
2. The map $X \to \mathfrak{R}_Y X$ has a retraction over Y.

The fibration category of indexed groupoids

The category Coflex($\mathcal{E}^{\mathrm{op}}$, Gpd) of coflexible presheaves, with injective fibrations, encodes the 2-categorical structure of Ps($\mathcal{E}^{\mathrm{op}}$, Gpd):



where $PY \to Y \times Y$ is the replacement of the diagonal $Y \to Y \times Y$ by an injective fibration.

Stacks

Definition

A (pseudo)functor $X : \mathcal{E}^{\mathrm{op}} \to \mathsf{Gpd}$ is a stack (for the coherent topology) if

1. $X(0)$ is equivalent to 1.
2. Each map $X(U \sqcup V) \to X(U) \times X(V)$ is an equivalence.
3. For any equivalence relation R on $U \in \mathcal{E}$, the following diagram is a bicategorical limit:

$$X(U/R) \longrightarrow X(U) \rightrightarrows X(R) \rightrightarrows X(R \times_U R)$$

Definition

Let $\widehat{\mathcal{E}}$ denote the category of coflexible strict presheaves that are stacks, with the injective fibration structure.

Outline

Groupoid type theory

The internal language of the fibration category $\widehat{\mathcal{E}}$ is Martin-Löf dependent type theory.

objects (coflexible stacks)	$\sim$	types
morphisms $A \to B$	$\sim$	terms $x : A \vdash f(x) : B$
injective fibrations $B \twoheadrightarrow A$	$\sim$	dependent types $x : A \vdash B(x)$
composite of fibrations	$\sim$	dependent sum type $\sum_{x:A} B(x)$
pushforward of fibrations	$\sim$	dependent function type $\prod_{x:A} B(x)$
$PA \twoheadrightarrow A \times A$	$\sim$	identity type $x : A, y : A \vdash \mathrm{Id}_A(x, y)$

Idea

Dependent types $x : A \vdash B(x)$ generalize predicates $x : A \vdash \varphi(x)$, with $\sum, \prod, \mathrm{Id}$ "generalizing" $\exists, \forall, =$.

However, for a general groupoid-like object A, the identity type $\mathrm{Id}_A(x, y)$ represents the "hom-set" $A(x, y)$.

Propositions and logic

Definition

A type A is a proposition if it has at most one element, i.e.,

$$\prod_{x:A}\prod_{y:A}\mathsf{Id}_A(x,y).$$

Semantically, "pointwise either empty or contractible".

- If each type $B(x)$ is a proposition, then so is $\prod_{x:A} B(x)$, so we can call it $\forall_{x:A} B(x)$.
- But $\sum_{x:A} B(x)$ is not; it's more like $\{\, x : A \mid B(x)\,\}$.

  By $\exists_{x:A} B(x)$ we instead mean $\|\sum_{x:A} B(x)\|$, where $\|\cdot\|$ is the propositional truncation: the reflection into propositions.
- Similarly, if A and B are propositions, so are the function-type $A \to B$ (hence it is $A \Rightarrow B$) and $A \times B$ (hence it is $A \wedge B$), but by $A \vee B$ we mean $\|A \sqcup B\|$.

The universe of sets

Definition

A type A is discrete if each $\mathsf{Id}_A(x, y)$ is a proposition.

A discrete stack is equivalent to a sheaf. Internally, discrete types are often called "sets", but we will use that for something else:

Definition

Let $\mathscr{U} = \Re\mathscr{E}$, where $\mathscr{E} \in \mathsf{Ps}(\mathcal{E}^{\mathrm{op}}, \mathsf{Gpd})$ is defined by setting $\mathscr{E}(X) =$ the maximal subgroupoid of $\mathcal{E}/X$.

There is a canonical fibration $\mathscr{U}_\bullet \to \mathscr{U}$, so any element of $\mathscr{U}$ "is" a type by pullback. A type is a set if it is isomorphic to one in $\mathscr{U}$.

The sets in the empty context are the representable sheaves. (Those in other contexts are "representable natural transformations", i.e., $\mathscr{U}$ is a "classifier of representables".)

Univalence

$\widehat{\mathcal{E}}$ also satisfies the following axiom, called universe extensionality (Hofmann–Streicher) and univalence (Voevodsky).

Axiom

For sets $A, B : \mathscr{U}$ we have (canonically)

$$\mathsf{Id}_{\mathscr{U}}(A, B) \cong \mathsf{Iso}(A, B).$$

Here the type $\mathsf{Iso}(A, B)$ of isomorphisms is by definition

$$\sum_{(f:A\to B)} \sum_{(g:B\to A)} \mathsf{Id}_{A\to A}(g \circ f, 1_A) \times \mathsf{Id}_{B\to B}(f \circ g, 1_B).$$

Univalence means that we cannot distinguish between objects more finely than up to isomorphism.

Outline

Small and large logic

- All propositions are discrete, but not all propositions are sets.
- There is a set Ω (the representable stack on the subobject classifier of $\mathcal{E}$) that classifies the set-propositions.
- Ω is closed under $\wedge, \vee, \Rightarrow$ and under $\exists, \forall$ over sets, which coincide with the usual internal logic of $\mathcal{E}$.
- The internal logic of $\widehat{\mathcal{E}}$ thus extends this to include non-set propositions like $\forall_{(X:\mathscr{U})} \varphi(X)$.

Large propositions are sieves

Given $A \in \widehat{\mathcal{E}}$, taking isomorphism classes of each groupoid A_X we obtain a presheaf, whose sheafification is called $\pi_0(A) \in \mathsf{Sh}(\mathcal{E})$.

Lemma

Dependent propositions (predicates) $x : A \vdash B(x)$ are equivalent to subsheaves of $\pi_0(A)$.

Examples

- If $A = よX$ is representable, then B is a sieve on $X \in \mathcal{E}$ that's closed under the coherent topology.
- If $A = \mathscr{U}$, then B is a set of isomorphism classes in each slice $\mathcal{E}/X$, closed under pullback and under coherent descent. (Uses univalence!)

"Truth is invariant under isomorphism."

Kripke-Joyal stack semantics

The definitions of $\exists, \forall, \wedge, \vee, \Rightarrow$ in $\widehat{\mathcal{E}}$ yield the usual Kripke-Joyal clauses for truth:

- $x : \text{よ}X \vdash B(x) \wedge C(x)$ is the sieve of morphisms $Y \to X$ belonging to both B and C.
- $x : \text{よ}X \vdash B(x) \vee C(x)$ is the sieve of morphisms $Y \to X$ such that $Y = W \cup Z$, where $(W \to X) \in B$ and $(Z \to X) \in C$.
- $x : \text{よ}X \vdash B(x) \Rightarrow C(x)$ is the sieve of $Y \to X$ such that for all $Z \to Y$, if $(Z \to X) \in B$ then $(Z \to X) \in C$.
- $x : \text{よ}X \vdash \forall_{(y:B)} C(x, y)$ is the sieve of $Y \to X$ such that for any $Z \to Y$ with $W \in B(Z)$, we have $(Z \to X, W) \in C$.
- $x : \text{よ}X \vdash \exists_{(y:B)} C(x, y)$ is the sieve of $Y \to X$ such that there is an epi $Z \twoheadrightarrow Y$ and $W \in B(Z)$ such that $(Z \to X, W) \in C$.

Taking $B = \mathscr{U}$ in the last two clauses, we obtain an interpretation of unbounded quantifiers directly in terms of $\mathcal{E}$ itself.

Example 1: Projective objects

Definition

P is projective if for any object E, every epi $E \twoheadrightarrow P$ has a section.

For $P \in \mathcal{E}$, the interpretation of "P is projective" in $\widehat{\mathcal{E}}$ (with "for any object E" meaning $\forall_{E:\mathscr{U}}$") becomes in $\mathcal{E}$:

> *The sieve of $Y \to 1$ such that for any $Z \to Y$ and epi $E \twoheadrightarrow Z \times P$, there exists an epi $W \twoheadrightarrow Z$ such that E has a section when pulled back to $W \times P$.*

Theorem

This is precisely the sieve of $Y \to 1$ such that $Y \times P$ is internally projective in $\mathcal{E}/Y$, i.e., exponentiating by it preserves epis.

Example 2: Constructing membership-based set theory

Assume $\mathcal{E}$ has a NNO.

Definition

Let $\mathscr{V}$ be the type of well-founded accessible pointed graphs, i.e.,

$$\mathscr{V} := \sum_{(X:\mathscr{U})} \sum_{(R:X\times X\to\Omega)} \sum_{(\star:X)} \mathrm{acc}(X, R, \star) \times \mathrm{wf}(X, R)$$

Theorem

- *$\mathscr{V}$ is discrete (though not a set).*
- *Internal to $\widehat{\mathcal{E}}$, we can prove that $\mathscr{V}$ is a model of Intuitionistic Bounded Zermelo set theory.*

Example 3: Strong set-theoretical axioms

Recall that in ZF set theory,

- The separation axiom schema says that for any set A and formula $\varphi(x)$, there is a set $\{ x \in A \mid \varphi(x) \}$.
- The replacement axiom schema says that for any set A and formula $\varphi(x, y)$ such that $\forall_{x \in A} \exists!_y \, \varphi(x, y)$, there is a set $\{ y \mid \exists_{x \in A} \, \varphi(x, y) \}$.
- The collection axiom schema is a constructively necessary strengthening of replacement.

Elementary topos theory is equiconsistent with Bounded Zermelo set theory, which lacks all of these axioms.

Example 3: Strong set-theoretical axioms

Using $\widehat{\mathcal{E}}$, we can express a topos-theoretic analogue of separation:

Definition

$\mathcal{E}$ satisfies second-order separation if Ω is closed under the quantifiers $\exists_{(X:\mathscr{U})}$ and $\forall_{(X:\mathscr{U})}$ in the internal logic of $\widehat{\mathcal{E}}$.

Definition

$\mathcal{E}$ satisfies first-order separation, or is autological, if any proposition built from $\wedge, \vee, \Rightarrow$, quantifiers $\exists, \forall$ over sets, and $\exists_{(X:\mathscr{U})}$ and $\forall_{(X:\mathscr{U})}$, is itself a set.

Using Kripke-Joyal stack semantics, autology can be expressed as a first-order axiom schema for $\mathcal{E}$.

Comparing elementary toposes to set theory

Theorem

Any Grothendieck or realizability topos over Intuitionistic ZF — and in particular, the category of sets in IZF — is autological.

Theorem

For any elementary topos $\mathcal{E}$ with NNO, the model $\mathscr{V}$ satisfies the full collection schema. If $\mathcal{E}$ is autological, then $\mathscr{V}$ satisfies the full separation schema, hence is a model of IZF.

$$\begin{array}{ccc} BZ & \longleftrightarrow & \text{elementary topos} \\ \downarrow & & \downarrow \\ ZF & \longleftrightarrow & \text{autological topos} \end{array}$$

Outline

Categories in groupoid type theory

Definition (Hofmann–Streicher, Ahrens–Kapulkin–S.)

In groupoid type theory, a category consists of

- A type A_0 of objects.
- A family $(x : A_0), (y : A_0) \vdash A(x, y)$ of morphism types.
- Each type $A(x, y)$ is discrete.
- A family of identities $x : A_0 \vdash 1_x : A(x, x)$.
- Composition maps $(x : A_0), (y : A_0), (z : A_0) \vdash A(y, z) \times A(x, y) \to A(x, z)$.
- Composition is associative and unital.

It is univalent if each $\mathsf{Id}_{A_0}(x, y) \cong \mathsf{Iso}(x, y)$ canonically.

In a univalent category, we cannot distinguish objects more finely than up to isomorphism. Thus, anything we can say about them inside type theory is categorically invariant.

Examples of categories

Semantically, a univalent category is a Cat-valued stack.

Example

The category of sets, with $\mathsf{Set}_0 :\equiv \mathscr{U}$, is univalent (by the univalence axiom). It corresponds to the self-indexing of $\mathcal{E}$.

Similarly, any category of structured sets is also univalent, and corresponds to an appropriate indexed category.

Example

A small category, in which A_0 and each $A(x, y)$ are sets, is not usually univalent. It corresponds to an internal category in $\mathcal{E}$.

Any category also has a univalent completion, which corresponds semantically to stackification.

The magic of univalence

In fact, univalent categories are better than classical categories defined in set theory! Consider the statement

A fully faithful and essentially surjective functor is an equivalence of categories.

- In ZF set theory, this is equivalent to the axiom of choice.
- In particular, it is false for internal categories in most toposes (including $\mathsf{Sh}(\mathcal{E})$), leading to notions like "weak equivalence" and "anafunctor".
- But for univalent categories, it is just true!

Properties of categories

We can develop category theory "naively" inside type theory. For univalent categories, the obvious definitions of properties such as

- locally small (each $A(x, y)$ is a set)
- finite limits and colimits
- small (set-indexed) limits and colimits
- generating sets
- well-poweredness
- other comprehension/definability properties
- . . .

all correspond semantically in $\widehat{\mathcal{E}}$ to the usual "indexed" versions of these properties.

Theorems of category theory

Traditionally, theorems of category theory like

- The Adjoint Functor Theorem
- Giraud's Theorem
- Diaconescu's Theorem

have to be proven separately in "indexed" versions, manually translating families of objects into objects of fibers.

But if the usual (constructive) proofs are written in the internal type theory of $\widehat{\mathcal{E}}$ (which is generally easy), they yield the indexed versions automatically.

Internal categorical set theory

Assume $\mathcal{E}$ has a NNO.

Theorem

In the internal logic of $\widehat{\mathcal{E}}$:

- *The univalent category* Set *is a model of "Intuitionistic ETCS": a constructively well-pointed topos with NNO.*
- Set *always satisfies a categorical "collection axiom schema"*
- *If $\mathcal{E}$ is autological,* Set *satisfies a categorical "separation axiom schema".*

If we construct a membership-based set theory from Set in the usual way, still internal to $\widehat{\mathcal{E}}$, we obtain the model $\mathscr{V}$ from earlier.

Thanks!

- S., "Stack semantics and the comparison of material and structural set theories". arXiv:1004.3802
- S., "Comparing material and structural set theories", APAL 170(4), 2019, arXiv:1808.05204
- S., "All $(\infty, 1)$-toposes have strict univalent universes", arXiv:1904.07004