Traces in monoidal categories and bicategories

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What is “applied category theory”?

- *Mathematics* is the study of patterns, usually arising from observations about the real world.
- *Applied mathematics* is the study of those patterns that are typically useful in domains outside of mathematics.
- *Category theory* is one way to study patterns arising from observations about mathematics itself.
- *Applied category theory* is the study of those categorical patterns that are typically useful in fields of mathematics outside of category theory.

This talk is about one particular such categorical pattern: trace.

Outline

1. Traces in symmetric monoidal categories
   - Two examples
   - Symmetric monoidal traces: the general case
   - Application: the Lefschetz fixed-point theorem
   - String diagrams for monoidal categories
2. Traces in bicategories
   - Noncommutative traces
   - Bicategories, shadows, and traces
   - String diagrams for bicategories with shadows

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The trace of a linear map—classical version

Let V be a finite-dimensional vector space over a field k, and $f\colon V \to V$ a linear map.

Definition

The trace of f is the sum

$$\mathrm{tr}(f) = a_{11} + a_{22} + \cdots + a_{nn}$$

where

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}$$

is the matrix of f with respect to some basis of V.

The trace of a linear map—categorical version

Let V^* be the dual vector space of V, let $\varepsilon\colon V^* \otimes V \to k$ be the evaluation map, and let $\eta\colon k \to V \otimes V^*$ be defined by

$$\eta(1) = \sum_i v_i \otimes v_i^*$$

for some basis $\{v_i\}$ of V, with dual basis $\{v_i^*\}$ for V^*.

Definition

The trace of f is the composite

$$k \xrightarrow{\eta} V \otimes V^* \xrightarrow{f \otimes \mathrm{id}} V \otimes V^* \xrightarrow{\cong} V^* \otimes V \xrightarrow{\varepsilon} k.$$

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$$1 \overset{\eta}{\longmapsto} \textstyle\sum_i v_i \otimes v_i^* \overset{f\otimes\mathrm{id}}{\longmapsto} \sum_{ij} a_{ij} v_j \otimes v_i^* \overset{\cong}{\longmapsto} \sum_{ij} v_i^* \otimes a_{ij} v_j \overset{\varepsilon}{\longmapsto} \sum_{ij} a_{ij} v_i^*(v_j)$$

$$= \textstyle\sum_i a_{ii}.$$

The fixed-point index of a map—classical version

Let M be a closed smooth n-manifold and $f\colon M \to M$ a map with a discrete (hence finite) set of fixed points.

- If m is a fixed point, then there is a small $(n-1)$-sphere S_m around m which is approximately mapped to itself by f.
- Recall that any self-map $g\colon S^{n-1} \to S^{n-1}$ of a sphere has a degree $\deg(g) \in \mathbb{Z}$.

Definition

The fixed-point index of $f\colon M \to M$ is the sum

$$\sum_{f(m)=m} \deg\left(f|_{S_m}\right)$$

over all fixed points of f.

The fixed-point index of a map—categorical version

Let M be embedded in $\mathbb{R}^p$. Let $T\nu$ be the Thom space of the normal bundle of the embedding. Let $\eta\colon S^p \to M_+ \wedge T\nu$ be the composite of the Pontryagin-Thom map for ν with the Thom diagonal. Let $\varepsilon\colon T\nu \wedge M_+ \to S^p$ be the Pontryagin-Thom map for the diagonal followed by projection to S^p.

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Compare:

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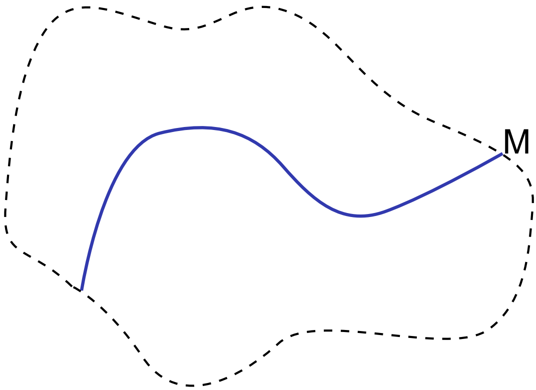
More fundamentally, what do all these words mean?

The Thom space of the normal bundle

To define $T\nu$:

1. Choose a tubular neighborhood of the embedding $M \hookrightarrow \mathbb{R}^p$.
2. Collapse everything outside this neighborhood to a basepoint.



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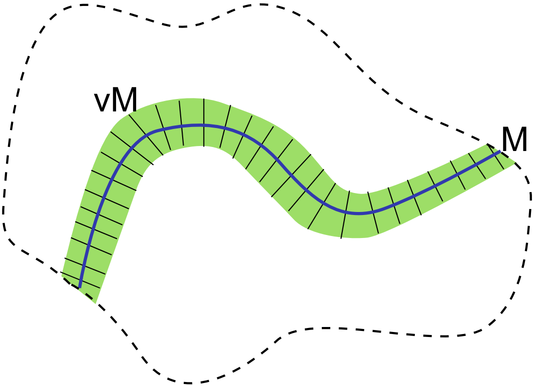

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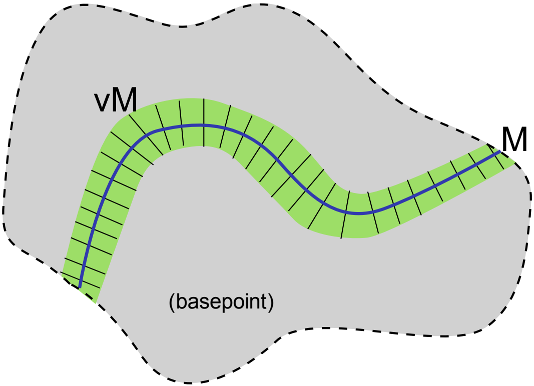
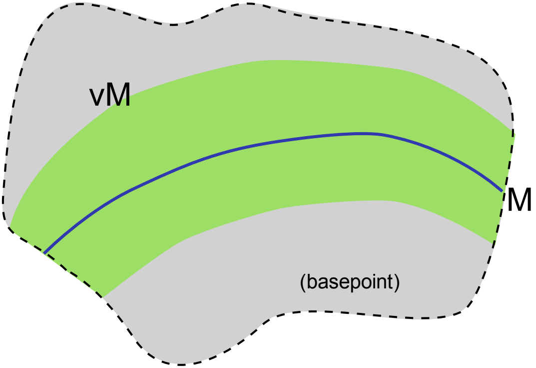


The topological unit

To define $\eta \colon S^p \to M_+ \wedge T\nu$:

1. Identify S^p with the one-point compactification of $\mathbb{R}^p$.
2. Points outside the tubular neighborhood go to the basepoint.
3. Points inside go to themselves, paired with their projection to M.



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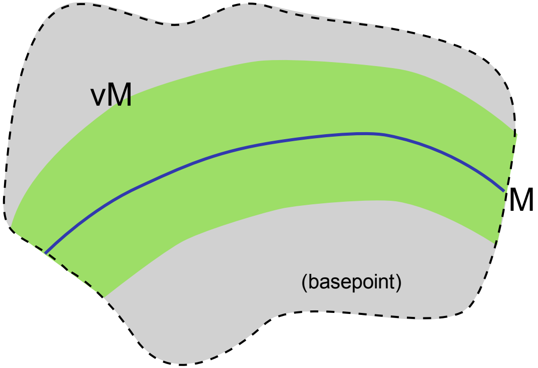

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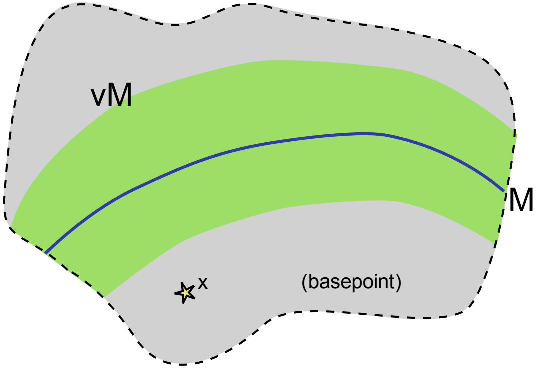

$\eta(x) = \text{(basepoint)}$

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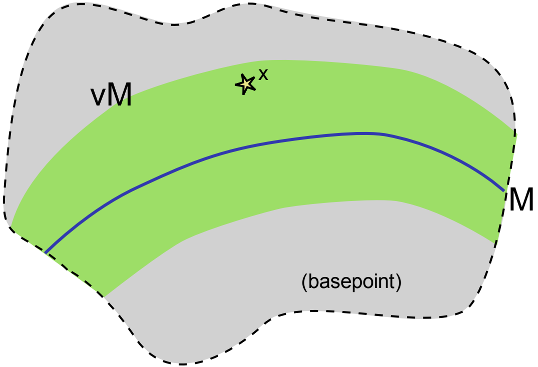

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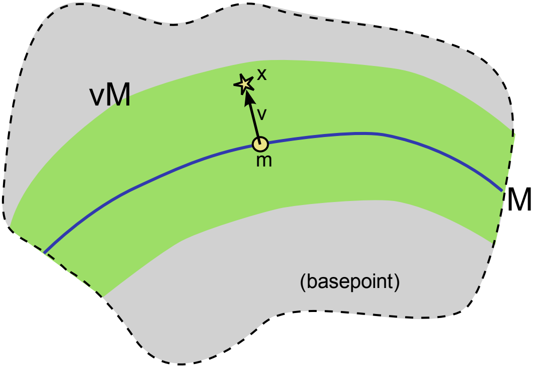
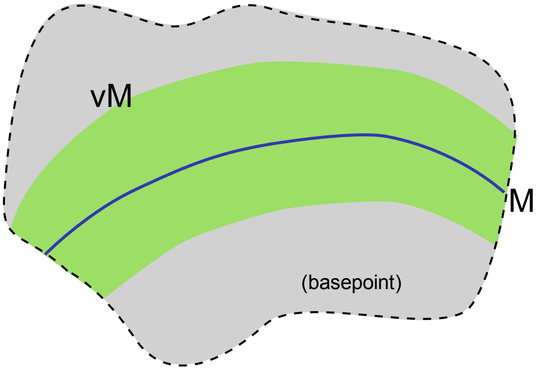


$$\eta(x) = (m, v)$$

The topological counit

To define $\varepsilon\colon T\nu \wedge M_+ \to S^p$:

1. If m and v are far apart, then (m, v) goes to the basepoint.
2. If they are close together, then they are added ($T_pM \oplus \nu_pM \cong \mathbb{R}^p$).



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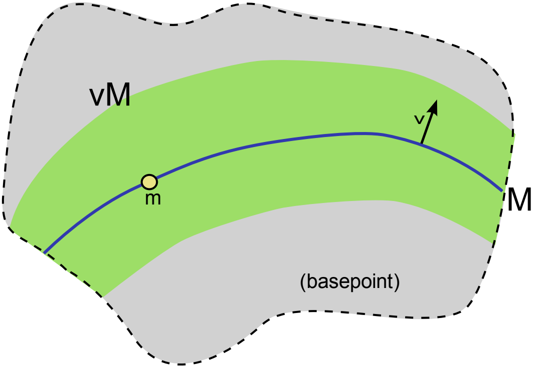

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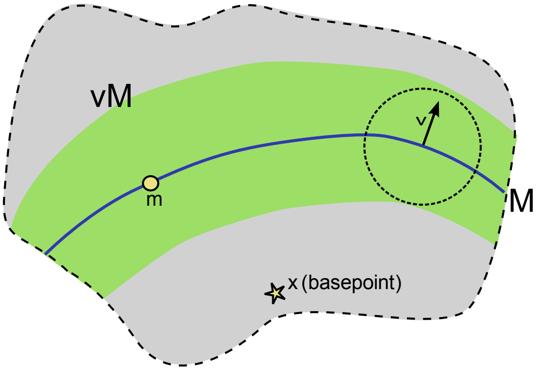

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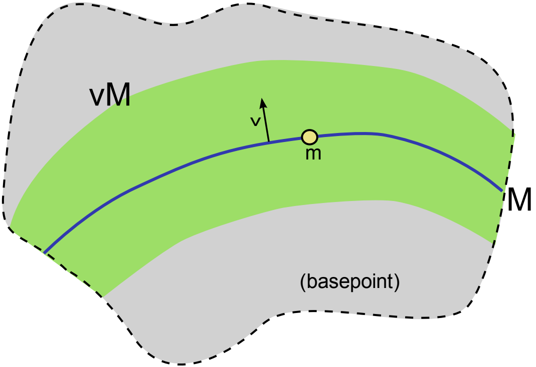

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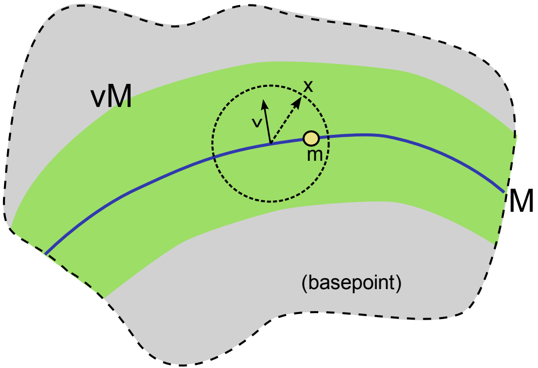

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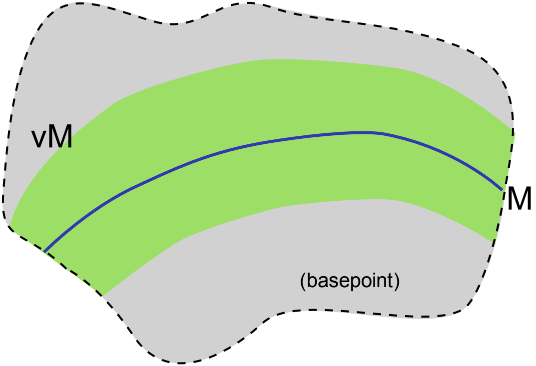
The categorical fixed-point index

$$S^p \xrightarrow{\eta} M_+ \wedge T\nu \xrightarrow{f \otimes \mathrm{id}} M_+ \wedge T\nu \xrightarrow{\cong} T\nu \wedge M_+ \xrightarrow{\varepsilon} S^p.$$



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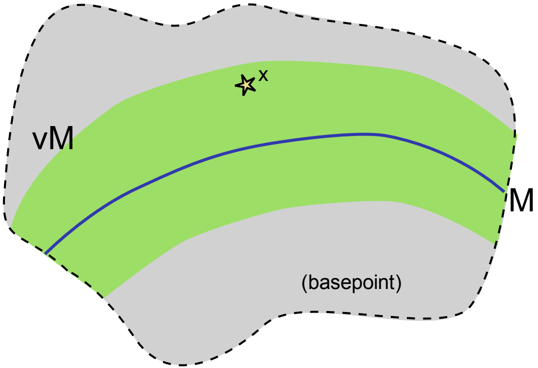

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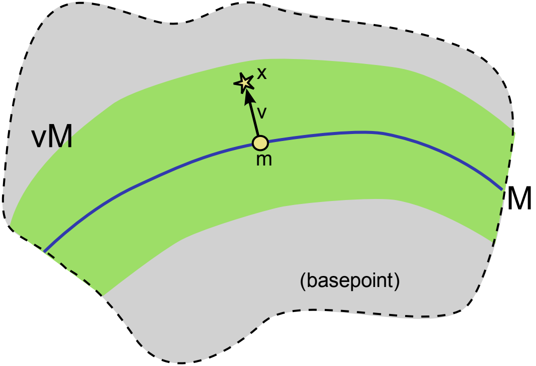

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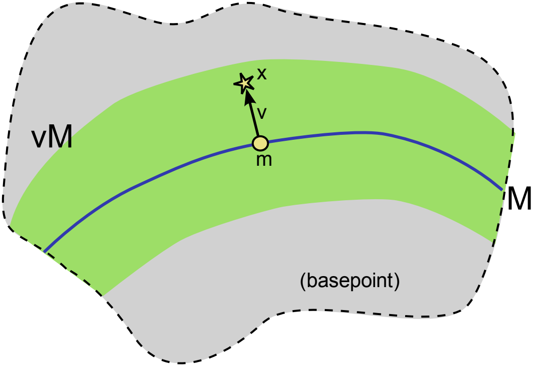

Two cases:

1. $f(m)$ is far from m.
2. $f(m)$ is close to m.

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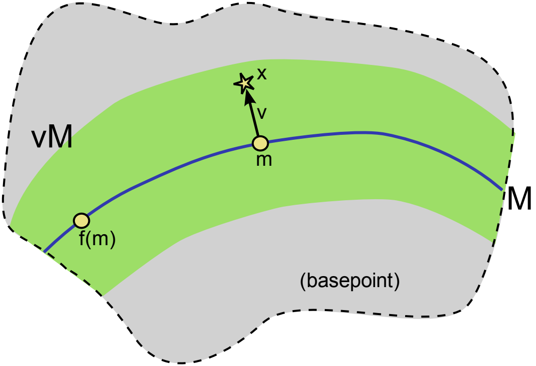

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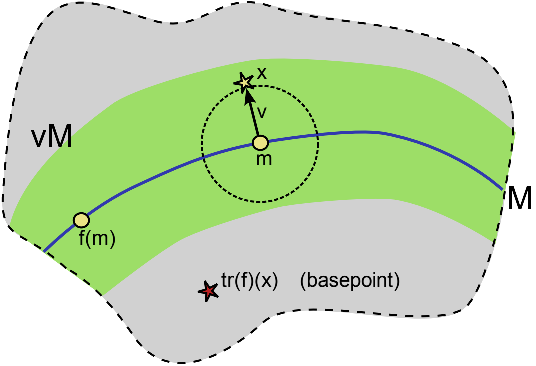

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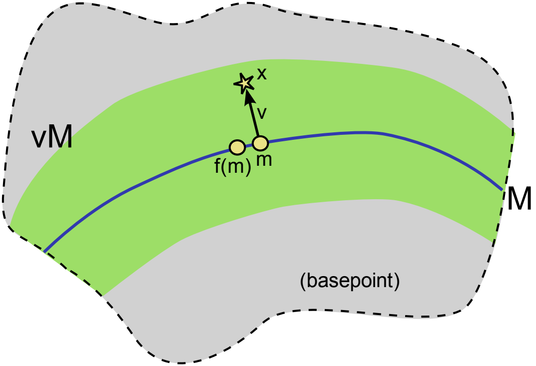

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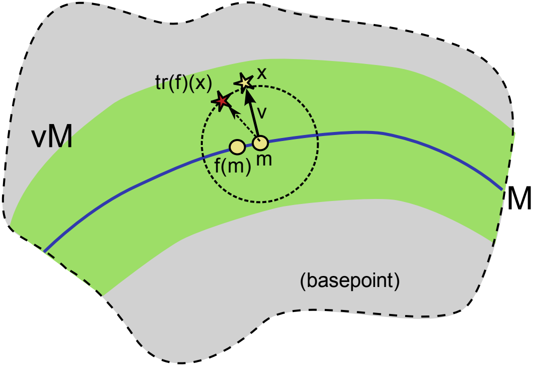

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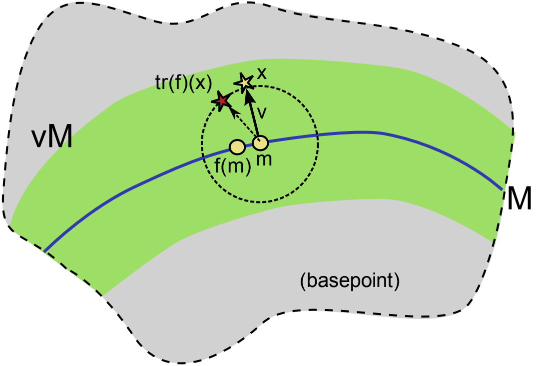

Two cases:

1. $f(m)$ is far from m.
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As m varies near a fixed point, $\mathrm{tr}(f)(m)$ covers the sphere with some degree. Everywhere else, the degree is zero.

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Symmetric monoidal categories

Definition

A symmetric monoidal category is a category $\mathscr{C}$ equipped with a *product* $\otimes\colon \mathscr{C} \times \mathscr{C} \to \mathscr{C}$ and a *unit* $U \in \mathscr{C}$ and isomorphisms

$$A \otimes (B \otimes C) \cong (A \otimes B) \otimes C$$
$$A \otimes U \cong A \cong U \otimes A \qquad A \otimes B \cong B \otimes A$$

satisfying certain natural axioms.

Examples

1. $\mathbf{Vec}_k$ with product $\otimes$ and unit k.
2. $\mathbf{Mod}_R$ (R a commutative ring), with product $\otimes_R$ and unit R.
3. The *stable homotopy category* with product $\wedge$ and unit S.
4. Many others...

Duality in symmetric monoidal categories

Definition

In a symmetric monoidal category $\mathscr{C}$, an object M is dualizable if we have an object M^* and maps

$$\eta\colon U \to M \otimes M^* \qquad \varepsilon\colon M^* \otimes M \to U$$

satisfying certain natural axioms.

The maps η and ε play the role of the maps we saw before in $\mathbf{Vec}_k$:

$$\begin{aligned} k &\xrightarrow{\eta} V \otimes V^* & \qquad V^* \otimes V &\xrightarrow{\varepsilon} k \\ 1 &\mapsto \sum_i v_i \otimes v_i^* & \qquad v_i^* \otimes v_j &\mapsto v_i^*(v_j) = \delta_{ij}. \end{aligned}$$

Thus, in $\mathbf{Vec}_k$ the dualizable objects are the finite-dimensional vector spaces over k.

Trace in symmetric monoidal categories

Definition

If M is dualizable and $f\colon M \to M$, then the trace of f is the composite

$$U \xrightarrow{\eta} M \otimes M^* \xrightarrow{f \otimes \mathrm{id}} M \otimes M^* \xrightarrow{\cong} M^* \otimes M \xrightarrow{\varepsilon} U.$$

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Examples

- In $\mathbf{Vec}_k$, this gives the linear trace.
- In $\mathbf{Mod}_R$, the dualizable objects are the finitely-generated projective modules, and the trace is analogous to the trace for vector spaces.
- In the stable homotopy category, any smooth manifold is dualizable, and the trace is the fixed-point index.
- Many others…

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Graded modules

One more example: let $\mathbf{GrVec}_k$ be the category of graded vector spaces over a field k. All the structure is obvious, except that we take

$$A \otimes B \xrightarrow{\cong} B \otimes A$$

to be defined by

$$a \otimes b \mapsto (-1)^{|b||a|} b \otimes a.$$

Then...

- The dualizable objects are those of finite total dimension, and
- The trace of a map is the alternating sum of its degreewise traces:

$$\operatorname{tr}(f) = \sum_n (-1)^n \operatorname{tr}(f_n).$$

Preservation of traces

Theorem (Preservation of traces)

Let $\mathscr{C}$, $\mathscr{D}$ be symmetric monoidal and M dualizable in $\mathscr{C}$. If $H\colon \mathscr{C} \to \mathscr{D}$ preserves $\otimes$ and U up to isomorphism, then $H(M)$ is dualizable in $\mathscr{D}$, and for any $f\colon M \to M$ we have $\mathrm{tr}(H(f)) = H(\mathrm{tr}(f))$.

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Proof

Here is $\mathrm{tr}(f)$:

$$U \xrightarrow{\eta} M \otimes M^* \xrightarrow{f\otimes \mathrm{id}} M \otimes M^* \xrightarrow{\cong} M^* \otimes M \xrightarrow{\varepsilon} U.$$

(cont. . .)

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Proof

Apply H to get $H(\mathrm{tr}(f))$:

$$H(U) \xrightarrow{H(\eta)} H(M \otimes M^*) \xrightarrow{H(f \otimes \mathrm{id})} H(M \otimes M^*) \xrightarrow{\cong} H(M^* \otimes M) \xrightarrow{H(\varepsilon)} H(U).$$

(cont. . .)

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Proof

Since H preserves the unit and product:

$$\begin{tikzcd}
H(U) \arrow[r, "H(\eta)"] \arrow[d, "\cong"'] & H(M \otimes M^*) \arrow[r, "H(f\otimes\mathrm{id})"] \arrow[d, "\cong"'] & H(M \otimes M^*) \arrow[r, "\cong"] \arrow[d, "\cong"'] & H(M^* \otimes M) \arrow[r, "H(\varepsilon)"] \arrow[d, "\cong"'] & H(U) \arrow[d, "\cong"'] \\
U \arrow[r, "\eta"'] & H(M) \otimes H(M)^* \arrow[r, "H(f)\otimes\mathrm{id}"'] & H(M) \otimes H(M)^* \arrow[r, "\cong"'] & H(M)^* \otimes H(M) \arrow[r, "\varepsilon"'] & U.
\end{tikzcd}$$

which is $\mathrm{tr}(H(f))$. □

The Lefschetz fixed-point theorem

Corollary (The Lefschetz fixed-point theorem)

Let M be a closed smooth manifold and $f\colon M \to M$, and let $H\colon \mathbf{Mfd} \to \mathbf{GrVec}_k$ denote rational homology, $H(M) = H_*(M, \mathbb{Q})$. If

$$\mathrm{tr}(H(f)) = \sum_n (-1)^n \, \mathrm{tr}(H_n(f))$$

is nonzero, then f has a fixed point.

The Lefschetz fixed-point theorem

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Proof.

The functor H preserves the product and unit, by the Kunneth theorem:

$$H(M \times N, \mathbb{Q}) \cong H(M, \mathbb{Q}) \otimes H(N, \mathbb{Q}).$$

Therefore, $\mathrm{tr}(H(f)) = H(\mathrm{tr}(f))$. But $\mathrm{tr}(f)$ is the fixed-point index of f, which is zero if f has no fixed points. $\square$

Outline

1. Traces in symmetric monoidal categories
   - Two examples
   - Symmetric monoidal traces: the general case
   - Application: the Lefschetz fixed-point theorem
   - String diagrams for monoidal categories

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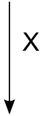
   - Noncommutative traces
   - Bicategories, shadows, and traces
   - String diagrams for bicategories with shadows

String diagrams

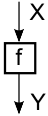
Here is a different way to draw pictures of objects and morphisms in a monoidal category.

Object X



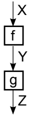
Morphism

$X \xrightarrow{f} Y$



Composition

$X \xrightarrow{f} Y \xrightarrow{g} Z$



String diagrams

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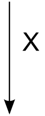
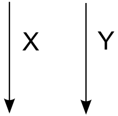
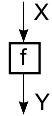


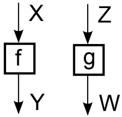
Product
$X \otimes Y$



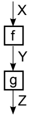
Morphism
$X \xrightarrow{f} Y$



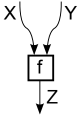
Product
$X \otimes Z \xrightarrow{f \otimes g} Y \otimes W$



Composition
$X \xrightarrow{f} Y \xrightarrow{g} Z$



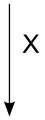
Morphism
$X \otimes Y \xrightarrow{f} Z$



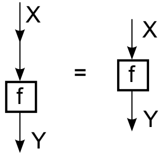
Special string diagrams (plumbing)

Some objects and morphisms are special; we draw them more simply.

Identity morphism
id_X

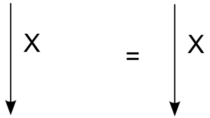


Composite with identity
$X \xrightarrow{\mathrm{id}_X} X \xrightarrow{f} Y$

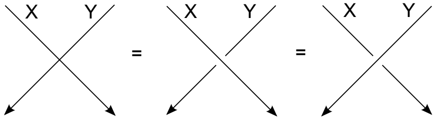


Unit object
U

Product with unit
$X \otimes U \cong X$

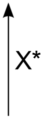


Symmetry
$X \otimes Y \cong Y \otimes X$

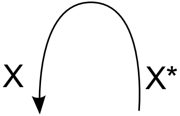


More special string diagrams

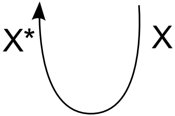
Dual object X^*



$\eta\colon U \to X \otimes X^*$

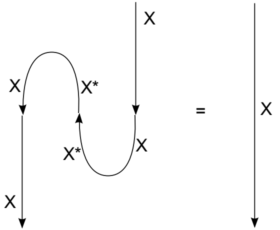


$\varepsilon\colon X^* \otimes X \to U$

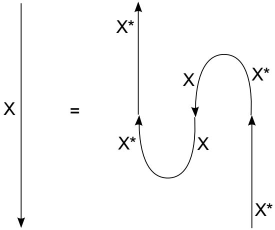


Example of string diagrams: the duality axioms

For example, the axioms for a dualizable object X are the following:



and



Or “bent strings can be straightened.”

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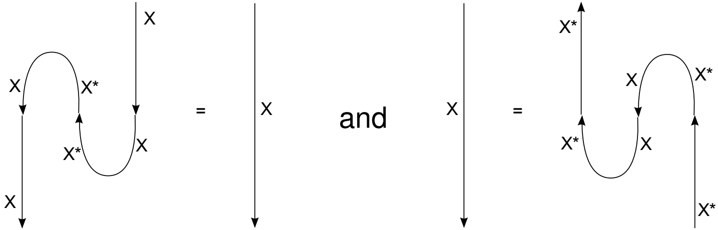
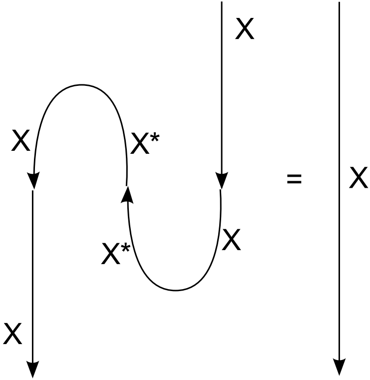


Or “bent strings can be straightened.”

What does this mean?

Example of string diagrams: the duality axioms (II)

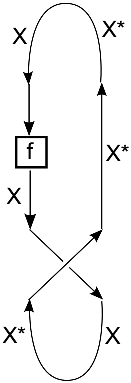
$$
\begin{tikzcd}
X \arrow[d, "\cong"] \\
U \otimes X \arrow[d, "\eta \otimes \mathrm{id}_X"] \\
X \otimes X^* \otimes X \arrow[d, "\mathrm{id}_X \otimes \varepsilon"] \\
X \otimes U \arrow[d, "\cong"] \\
X
\end{tikzcd}
\quad = \quad
$$



$$
= \quad
\begin{tikzcd}
X \arrow[d, "\mathrm{id}_X"] \\
X
\end{tikzcd}
$$

Second example of strings: the trace

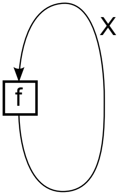
If X is dualizable, the trace of $f\colon X \to X$ is given by



$=$

$$
\begin{tikzcd}
U \arrow[d, "\eta"] \\
X \otimes X^* \arrow[d, "f \otimes \mathrm{id}"] \\
X \otimes X^* \arrow[d, "\cong"] \\
X^* \otimes X \arrow[d, "\varepsilon"] \\
U
\end{tikzcd}
$$

or just



Why string diagrams are so useful

Theorem (Joyal-Street-Verity)

Any two string diagrams which are topologically equivalent represent equal morphisms in a symmetric monoidal category.

So we can prove theorems by drawing pictures!

Why string diagrams are so useful

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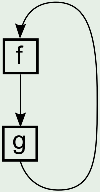
So we can prove theorems by drawing pictures!

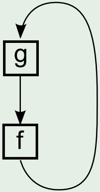
Example

If X and Y are dualizable, $X \xrightarrow{f} Y$, and $Y \xrightarrow{g} X$, then

tr(gf) =



=



= tr(fg).

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Noncommutative traces: the problem

Suppose R is a noncommutative ring, M is a finitely generated projective right R-module, and $f\colon M \to M$ is a map; how can we define the trace of f? We want to write:

$$R \xrightarrow{\eta} M \otimes M^* \xrightarrow{f \otimes \mathrm{id}} M \otimes M^* \xrightarrow{\cong} M^* \otimes M \xrightarrow{\varepsilon} R.$$

where $M^* = \mathrm{Hom}_R(M, R)$, but…

- M is a right R-module while M^* is a left R-module, and thus
- $M^* \otimes M = M^* \otimes_{\mathbb{Z}} M$ is an R-R-bimodule, while
- $M \otimes_R M^*$ is just an abelian group ($= \mathbb{Z}$-$\mathbb{Z}$-bimodule). And they certainly aren't isomorphic!

Noncommutative traces: the solution

Definition

For any R-R-bimodule N, the shadow of N is the abelian group

$$\langle\!\langle N \rangle\!\rangle = N \Big/ \langle r \cdot n = n \cdot r \mid n \in N, r \in R \rangle.$$

This is the "circular tensor product of N with itself", by analogy with

$$P \otimes_R Q = P \otimes_{\mathbb{Z}} Q \Big/ \langle pr \otimes q = p \otimes rq \rangle.$$

Note: if $R = \mathbb{Z}$, then $\langle\!\langle N \rangle\!\rangle \cong N$.

Noncommutative traces: the solution (II)

Lemma

If P is an R-S-bimodule and Q is an S-R-bimodule, then

$$\langle\!\langle P \otimes_S Q \rangle\!\rangle \cong \langle\!\langle Q \otimes_R P \rangle\!\rangle.$$

In particular, $M \otimes_R M^* \cong \langle\!\langle M \otimes_R M^* \rangle\!\rangle \cong \langle\!\langle M^* \otimes_{\mathbb{Z}} M \rangle\!\rangle$.

Noncommutative traces: the solution (II)

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In particular, $M \otimes_R M^* \cong \langle\!\langle M \otimes_R M^* \rangle\!\rangle \cong \langle\!\langle M^* \otimes_{\mathbb{Z}} M \rangle\!\rangle$.

Definition

If M is a finitely generated projective left R-module, the (Hattori-Stallings) trace of $f \colon M \to M$ is the composite

$$\mathbb{Z} \xrightarrow{\eta} M \otimes_R M^* \xrightarrow{f \otimes \mathrm{id}} M \otimes_R M^* \xrightarrow{\cong} \langle\!\langle M^* \otimes_{\mathbb{Z}} M \rangle\!\rangle \xrightarrow{\langle\!\langle \varepsilon \rangle\!\rangle} \langle\!\langle R \rangle\!\rangle.$$

How can we express this idea categorically?

Outline

Bicategories

Definition

A bicategory consists of

- A collection of objects: $R, S, T, \ldots,$
- A collection of categories $\mathscr{B}(R, S)$,
- Product functors $\odot : \mathscr{B}(R, S) \times \mathscr{B}(S, T) \to \mathscr{B}(R, T)$, and
- Units $U_R \in \mathscr{B}(R, R)$,
- such that $M \odot (N \odot P) \cong (M \odot N) \odot P$, $M \odot U_S \cong M$, and $U_R \odot M \cong M$, coherently.

Note: there is no symmetry! $M \odot N$ and $N \odot M$ are objects of different categories, and in general need not both exist.

Example

In $\mathcal{M}od$, the objects are (noncommutative) rings, $\mathcal{M}od(R, S)$ consists of R-S-bimodules, $\odot$ is tensor product, and $U_R = R$.

Pictures in bicategories

In a bicategory $\mathscr{B}$, we sometimes think of an object $M \in \mathscr{B}(R, S)$ as an arrow $R \xrightarrow{M} S$, and call it a 1-cell. Then the product $\odot$ is a sort of 'composition' of 1-cells:

$$R \xrightarrow{M} S \xrightarrow{N} T \qquad = \qquad R \xrightarrow{M \odot N} S.$$

Similarly, we think of a morphism $f\colon M \to N$ in $\mathscr{B}(R, S)$ as a 'higher-dimensional' arrow or 2-cell:

$$\begin{tikzcd} R \arrow[r, bend left, "M"] \arrow[r, bend right, "N"'] \arrow[r, phantom, "\Downarrow f"] & S. \end{tikzcd}$$

Duality in bicategories

Definition

A 1-cell $M \in \mathscr{B}(R, S)$ in a bicategory is right dualizable if there exists a 1-cell $M^* \in \mathscr{B}(S, R)$ and 2-cells $\eta \colon U_R \to M \odot M^*$ and $\varepsilon \colon M^* \odot M \to U_S$, satisfying the same axioms as before.

Example

A $\mathbb{Z}$-R-bimodule is right dualizable in $\mathcal{M}od$ if and only if it is a finitely generated projective right R-module.

Shadows in bicategories

Definition (Ponto)

A shadow on a bicategory $\mathscr{B}$ consists of functors

$$\langle\!\langle - \rangle\!\rangle \colon \mathscr{B}(R,R) \longrightarrow \mathbf{T},$$

for some fixed category $\mathbf{T}$, such that $\langle\!\langle M \odot N \rangle\!\rangle \cong \langle\!\langle N \odot M \rangle\!\rangle$ coherently.

Example

We have a shadow on the bicatgory $\mathcal{M}od$ defined by

$$\langle\!\langle N \rangle\!\rangle = N \big/ \langle r \cdot n = n \cdot r \rangle$$

as before.

Traces in bicategories with shadows

Definition

If $\mathscr{B}$ is equipped with a shadow, $M \in \mathscr{B}(R, S)$ is right dualizable, and $f\colon M \to M$ is a 2-cell, then the trace of f is the composite

$$\langle\!\langle U_R \rangle\!\rangle \xrightarrow{\langle\!\langle \eta \rangle\!\rangle} \langle\!\langle M \odot M^* \rangle\!\rangle \xrightarrow{\langle\!\langle f \odot \mathrm{id} \rangle\!\rangle} \langle\!\langle M \odot M^* \rangle\!\rangle \xrightarrow{\cong} \langle\!\langle M^* \odot M \rangle\!\rangle \xrightarrow{\langle\!\langle \varepsilon \rangle\!\rangle} \langle\!\langle U_S \rangle\!\rangle.$$

Example

For finitely generated projective right R-modules, regarded as 1-cells from $\mathbb{Z}$ to R in $\mathcal{M}od$, this trace recaptures the Hattori-Stallings trace. (Remember that $\langle\!\langle N \rangle\!\rangle \cong N$ when N is a $\mathbb{Z}$-$\mathbb{Z}$-bimodule.)

Another example: Euler characteristics and characters

Definition

If M is dualizable, its Euler characteristic is $\mathrm{tr}(\mathrm{id}_M)$.

In the symmetric monoidal case:

- In $\mathbf{Vec}_k$ this computes the dimension of a finite dimensional vector space.
- In $\mathbf{GrVec}_k$, it computes the alternating sum of the degreewise dimensions, $\sum_i (-1)^i \dim(V_i)$.
- Thus, by preservation of traces, in the stable homotopy category it computes the usual Euler characteristic of a manifold.

Another example: Euler characteristics and characters

Definition

If M is dualizable, its Euler characteristic is $\mathrm{tr}(\mathrm{id}_M)$.

In the bicategorical case:

Let k be a field, G a group, and V a finite-dimensional left kG-module (i.e. a representation of G over k). Regarding V as a 1-cell in $\mathcal{M}od(kG, k)$, it is right dualizable, with dual

$$V^* = \mathrm{Hom}_k(V, k) \in \mathcal{M}od(k, kG)$$

(The usual dual vector space, with the induced *right* kG-module structure.) Then:

- The shadow $\langle\!\langle kG \rangle\!\rangle$ is a vector space with a basis given by the conjugacy classes of G, and
- The Euler characteristic $\mathrm{tr}(\mathrm{id}_V) \colon \langle\!\langle kG \rangle\!\rangle \to k$ is the character of the representation V.

More examples: generalized fixed-point theory

To prove a converse to the Lefschetz fixed-point theorem, one needs to incorporate information about the fundamental group of M.

Using traces in a ‘stable homotopy bicategory of bimodules’, one can define an invariant called the Reidemeister trace, identify it with an algebraic version, and then prove:

Theorem

Let $f : M \to M$ be continuous, where M is a closed smooth manifold of dimension ≥ 3. Then the Reidemeister trace of f is zero if and only if f is homotopic to a map with no fixed points.

There are also fiberwise and equivariant generalizations. In all cases the notion of bicategorical trace provides a formal framework and a line of inquiry which is often fruitful.

Reference: Kate Ponto, “Fixed point theory and trace for bicategories”

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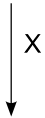
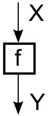
String diagrams for bicategories

The string diagrams for monoidal categories are obtained from the 'ordinary' ones by 'Poincaré duality':

Object
X



Morphism
$X \xrightarrow{f} Y$

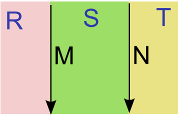


We do the same for bicategories, one dimension up.

String diagrams for bicategories, II

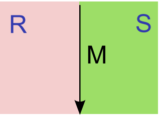
Object R

Product
$M \odot N$

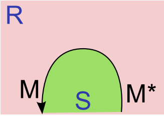


1-cell

$R \xrightarrow{M} S$

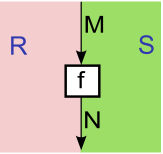


$U_R \xrightarrow{\eta} M \odot M^*$

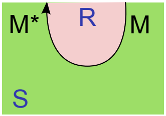


2-cell

$$\begin{tikzcd} R \arrow[r, bend left, "M"] \arrow[r, bend right, "N"'] \arrow[r, phantom, "\Downarrow f"] & S \end{tikzcd}$$

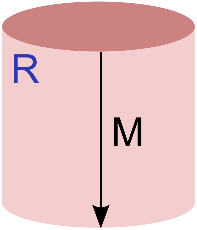


$M^* \odot M \xrightarrow{\varepsilon} U_S$

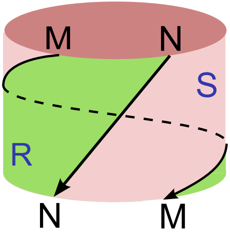


String diagrams for shadows

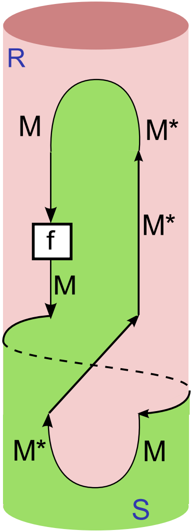
Shadow
$\langle\!\langle M \rangle\!\rangle$



Cyclicity
$\langle\!\langle M \odot N \rangle\!\rangle \cong \langle\!\langle N \odot M \rangle\!\rangle$

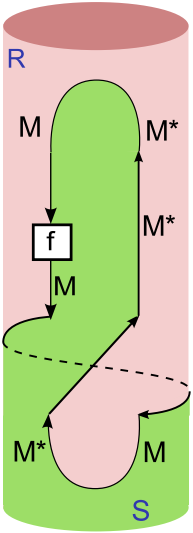


The string diagram for bicategorical trace



$$=\quad \begin{array}{c} \langle\!\langle U_R \rangle\!\rangle \\ \big\downarrow {\scriptstyle \langle\!\langle \eta \rangle\!\rangle} \\ \langle\!\langle M \otimes M^* \rangle\!\rangle \\ \big\downarrow {\scriptstyle \langle\!\langle f \otimes \mathrm{id} \rangle\!\rangle} \\ \langle\!\langle M \otimes M^* \rangle\!\rangle \\ \big\downarrow {\scriptstyle \cong} \\ \langle\!\langle M^* \otimes M \rangle\!\rangle \\ \big\downarrow {\scriptstyle \langle\!\langle \varepsilon \rangle\!\rangle} \\ \langle\!\langle U_S \rangle\!\rangle \end{array}$$

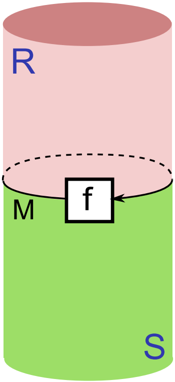
The string diagram for bicategorical trace



$=$

$$\begin{tikzcd}
\langle\!\langle U_R \rangle\!\rangle \arrow[d, "\langle\!\langle \eta \rangle\!\rangle"] \\
\langle\!\langle M \otimes M^* \rangle\!\rangle \arrow[d, "\langle\!\langle f \otimes \mathrm{id} \rangle\!\rangle"] \\
\langle\!\langle M \otimes M^* \rangle\!\rangle \arrow[d, "\cong"] \\
\langle\!\langle M^* \otimes M \rangle\!\rangle \arrow[d, "\langle\!\langle \varepsilon \rangle\!\rangle"] \\
\langle\!\langle U_S \rangle\!\rangle
\end{tikzcd}$$

or just



Conclusion

- Trace in symmetric monoidal categories unifies many different notions, including the linear trace and the fixed-point index. Formal comparison results can then produce contentful theorems (e.g. the Lefschetz fixed-point theorem).
- Trace in bicategories generalizes this to include examples such as noncommutative linear traces and basepoint-free, fiberwise, and equivariant fixed-point theory.
- In both cases, string diagrams can be used to obtain intuition and simplify proofs.