Traces in monoidal categories and bicategories

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What is “applied category theory”?

- **Mathematics** is the study of patterns, usually arising from observations about the real world.
- **Applied mathematics** is the study of those patterns that are typically useful in domains outside of mathematics.
- **Category theory** is one way to study patterns arising from observations about mathematics itself.
- **Applied category theory** is the study of those categorical patterns that are typically useful in fields of mathematics outside of category theory.

This talk is about one particular such categorical pattern: **trace**.
Outline

1. Traces in symmetric monoidal categories
   - Two examples
   - Symmetric monoidal traces: the general case
   - Application: the Lefschetz fixed-point theorem
   - String diagrams for monoidal categories

2. Traces in bicategories
   - Noncommutative traces
   - Bicategories, shadows, and traces
   - String diagrams for bicategories with shadows
1 Traces in symmetric monoidal categories
   - Two examples
     - Symmetric monoidal traces: the general case
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2 Traces in bicategories
   - Noncommutative traces
   - Bicategories, shadows, and traces
   - String diagrams for bicategories with shadows
Let $V$ be a finite-dimensional vector space over a field $k$, and $f: V \to V$ a linear map.

**Definition**

The **trace** of $f$ is the sum

$$\text{tr}(f) = a_{11} + a_{22} + \cdots + a_{nn}$$

where

$$
\begin{pmatrix}
  a_{11} & a_{12} & \cdots & a_{1n} \\
  a_{21} & a_{22} & \cdots & a_{2n} \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{n1} & a_{n2} & \cdots & a_{nn}
\end{pmatrix}
$$

is the matrix of $f$ with respect to some basis of $V$. 
Let $V^*$ be the dual vector space of $V$, let $\varepsilon: V^* \otimes V \to k$ be the evaluation map, and let $\eta: k \to V \otimes V^*$ be defined by

$$\eta(1) = \sum_i v_i \otimes v_i^*$$

for some basis $\{v_i\}$ of $V$, with dual basis $\{v_i^*\}$ for $V^*$.

**Definition**

The trace of $f$ is the composite

$$k \xrightarrow{\eta} V \otimes V^* \xrightarrow{f \otimes \text{id}} V \otimes V^* \xrightarrow{\cong} V^* \otimes V \xrightarrow{\varepsilon} k.$$
Let $V^*$ be the dual vector space of $V$, let $\varepsilon: V^* \otimes V \to k$ be the evaluation map, and let $\eta: k \to V \otimes V^*$ be defined by

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Why is this the same as the classical version?
The trace of a linear map—categorical version

**Definition**

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$$k \xrightarrow{\eta} V \otimes V^* \xrightarrow{f \otimes \text{id}} V \otimes V^* \xrightarrow{\cong} V^* \otimes V \xrightarrow{\varepsilon} k.$$
Let $M$ be a closed smooth $n$-manifold and $f: M \to M$ a map with a discrete (hence finite) set of fixed points.

- If $m$ is a fixed point, then there is a small $(n - 1)$-sphere $S_m$ around $m$ which is approximately mapped to itself by $f$.
- Recall that any self-map $g: S^{n-1} \to S^{n-1}$ of a sphere has a degree $\deg(g) \in \mathbb{Z}$.

**Definition**

The **fixed-point index** of $f: M \to M$ is the sum

$$\sum_{f(m)=m} \deg(f|_{S_m})$$

over all fixed points of $f$. 

Michael Shulman (University of Chicago)

Categorical Traces

CCR La Jolla, 8/25/08
The fixed-point index of a map—categorical version

Let $M$ be embedded in $\mathbb{R}^p$. Let $T\nu$ be the Thom space of the normal bundle of the embedding. Let $\eta: S^p \to M_+ \land T\nu$ be the composite of the Pontryagin-Thom map for $\nu$ with the Thom diagonal. Let $\varepsilon: T\nu \land M_+ \to S^p$ be the Pontryagin-Thom map for the diagonal followed by projection to $S^p$.

Definition
The fixed-point index of $f$ is the composite

$$ S^p \xrightarrow{\eta} M_+ \land T\nu \xrightarrow{f \otimes \text{id}} M_+ \land T\nu \xrightarrow{\varepsilon} T\nu \land M_+ \xrightarrow{\varepsilon} S^p. $$
The fixed-point index of a map—categorical version

Let $M$ be embedded in $\mathbb{R}^p$. Let $T_\nu$ be the Thom space of the normal bundle of the embedding. Let $\eta: S^p \to M_+ \wedge T_\nu$ be the composite of the Pontryagin-Thom map for $\nu$ with the Thom diagonal. Let $\varepsilon: T_\nu \wedge M_+ \to S^p$ be the Pontryagin-Thom map for the diagonal followed by projection to $S^p$.

**Definition**

The **fixed-point index** of $f$ is the composite

$$
S^p \xrightarrow{\eta} M_+ \wedge T_\nu \xrightarrow{f \otimes \text{id}} M_+ \wedge T_\nu \xrightarrow{\iota} T_\nu \wedge M_+ \xrightarrow{\varepsilon} S^p.
$$

Compare:

$$
k \xrightarrow{\eta} V \otimes V^* \xrightarrow{f \otimes \text{id}} V \otimes V^* \xrightarrow{\iota} V^* \otimes V \xrightarrow{\varepsilon} k.
$$
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$$k \xrightarrow{\eta} V \otimes V^* \xrightarrow{f \otimes \text{id}} V \otimes V^* \xrightarrow{\varepsilon} V^* \otimes V \xrightarrow{\varepsilon} k.$$ 

Why is this the same as the classical version?
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Compare:

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Why is this the same as the classical version?

More fundamentally, what do all these words mean?
To define $T_\nu$:

1. Choose a tubular neighborhood of the embedding $M \hookrightarrow \mathbb{R}^p$.
2. Collapse everything outside this neighborhood to a basepoint.
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The Thom space of the normal bundle

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1. Choose a tubular neighborhood of the embedding $M \hookrightarrow \mathbb{R}^p$.
2. Collapse everything outside this neighborhood to a basepoint.

\begin{itemize}
  \item $vM$
  \item $M$
  \item (basepoint)
\end{itemize}
The topological unit

To define $\eta : S^p \to M_+ \wedge T_v$:

1. Identify $S^p$ with the one-point compactification of $\mathbb{R}^p$.
2. Points outside the tubular neighborhood go to the basepoint.
3. Points inside go to themselves, paired with their projection to $M$.
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\[ \eta(x) = (\text{basepoint}) \]
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\[ \eta(x) = (m, v) \]
The topological counit

To define $\varepsilon : T\nu \land M_+ \to S^p$:

1. If $m$ and $v$ are far apart, then $(m, v)$ goes to the basepoint.
2. If they are close together, then they are added ($T_p M \oplus \nu_p M \cong \mathbb{R}^p$).
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The categorical fixed-point index

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S^p \xrightarrow{\eta} M_+ \land T^\nu \xrightarrow{f \otimes \text{id}} M_+ \land T^\nu \xrightarrow{\mathbb{R}} T^\nu \land M_+ \xrightarrow{\varepsilon} S^p.
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Two cases:

1. \( f(m) \) is far from \( m \).
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Two cases:

1. $f(m)$ is far from $m$.
2. $f(m)$ is close to $m$. 

As $m$ varies near a fixed point, $\text{tr}(f(m))$ covers the sphere with some degree. Everywhere else, the degree is zero.
The categorical fixed-point index

\[ \eta : S^p \rightarrow M_+ \wedge T \nu \xrightarrow{f \otimes \text{id}} M_+ \wedge T \nu \xrightarrow{\mathbb{R}} T \nu \wedge M_+ \xrightarrow{\varepsilon} S^p. \]

Two cases:

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As \( m \) varies near a fixed point, \( \text{tr}(f)(m) \) covers the sphere with some degree. Everywhere else, the degree is zero.
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A symmetric monoidal category is a category $\mathcal{C}$ equipped with a product $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ and a unit $U \in \mathcal{C}$ and isomorphisms

$$A \otimes (B \otimes C) \cong (A \otimes B) \otimes C$$

$$A \otimes U \cong A \cong U \otimes A$$

$$A \otimes B \cong B \otimes A$$

satisfying certain natural axioms.

**Examples**

1. $\text{Vec}_k$ with product $\otimes$ and unit $k$.
2. $\text{Mod}_R$ ($R$ a commutative ring), with product $\otimes_R$ and unit $R$.
3. The *stable homotopy category* with product $\wedge$ and unit $S$.
4. Many others...
Definition

In a symmetric monoidal category $\mathcal{C}$, an object $M$ is dualizable if we have an object $M^*$ and maps

$$\eta: U \to M \otimes M^* \quad \varepsilon: M^* \otimes M \to U$$

satisfying certain natural axioms.

The maps $\eta$ and $\varepsilon$ play the role of the maps we saw before in $\text{Vec}_k$:

$$k \xrightarrow{\eta} V \otimes V^* \quad V^* \otimes V \xleftarrow{\varepsilon} k$$

$$1 \mapsto \sum_i v_i \otimes v_i^* \quad v_i^* \otimes v_j \mapsto v_i^*(v_j) = \delta_{ij}.$$ 

Thus, in $\text{Vec}_k$ the dualizable objects are the finite-dimensional vector spaces over $k$. 
Trace in symmetric monoidal categories

Definition

If $M$ is dualizable and $f : M \to M$, then the trace of $f$ is the composite

$$
U \xrightarrow{\eta} M \otimes M^* \xrightarrow{f \otimes \text{id}} M \otimes M^* \xrightarrow{\cong} M^* \otimes M \xrightarrow{\varepsilon} U.
$$

Examples

In $\text{Vec}_k$, this gives the linear trace.

In $\text{Mod}_R$, the dualizable objects are the finitely-generated projective modules, and the trace is analogous to the trace for vector spaces.

In the stable homotopy category, any smooth manifold is dualizable, and the trace is the fixed-point index.

Many others...
Trace in symmetric monoidal categories

Definition
If $M$ is dualizable and $f: M \to M$, then the trace of $f$ is the composite

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\xrightarrow{\cong} M^* \otimes M \\
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\end{array}
\]

Examples
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- In $\textbf{Mod}_R$, the dualizable objects are the finitely-generated projective modules, and the trace is analogous to the trace for vector spaces.
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Graded modules

One more example: let $\text{GrVec}_k$ be the category of graded vector spaces over a field $k$. All the structure is obvious, except that we take $A \otimes B \overset{\cong}{\longrightarrow} B \otimes A$ to be defined by

$$a \otimes b \mapsto (-1)^{|b||a|} b \otimes a.$$ 

Then...

- The dualizable objects are those of finite total dimension, and
- The trace of a map is the alternating sum of its degreewise traces:

$$\text{tr}(f) = \sum_n (-1)^n \text{tr}(f_n).$$
Theorem (Preservation of traces)

Let $\mathcal{C}$, $\mathcal{D}$ be symmetric monoidal and $M$ dualizable in $\mathcal{C}$. If $H: \mathcal{C} \to \mathcal{D}$ preserves $\otimes$ and $U$ up to isomorphism, then $H(M)$ is dualizable in $\mathcal{D}$, and for any $f: M \to M$ we have $\text{tr}(H(f)) = H(\text{tr}(f))$. 
Theorem (Preservation of traces)

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Proof

Here is $\text{tr}(f)$:

$$U \overset{\eta}{\longrightarrow} M \otimes M^* \overset{f \otimes \text{id}}{\longrightarrow} M \otimes M^* \overset{\cong}{\longrightarrow} M^* \otimes M \overset{\varepsilon}{\longrightarrow} U.$$
Preservation of traces

Theorem (Preservation of traces)

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Proof

Apply $H$ to get $H(\text{tr}(f))$:

\[ H(U) \xrightarrow{H(\eta)} H(M \otimes M^*) \xrightarrow{H(f \otimes \text{id})} H(M \otimes M^*) \xrightarrow{\cong} H(M^* \otimes M) \xrightarrow{H(\varepsilon)} H(U). \]

(cont. . . )
Theorem (Preservation of traces)

Let $\mathcal{C}$, $\mathcal{D}$ be symmetric monoidal and $M$ dualizable in $\mathcal{C}$. If $H: \mathcal{C} \to \mathcal{D}$ preserves $\otimes$ and $U$ up to isomorphism, then $H(M)$ is dualizable in $\mathcal{D}$, and for any $f: M \to M$ we have $\text{tr}(H(f)) = H(\text{tr}(f))$.

Proof

Since $H$ preserves the unit and product:

\[
\begin{array}{c}
H(U) \xrightarrow{H(\eta)} H(M \otimes M^*) \xrightarrow{H(f \otimes \text{id})} H(M \otimes M^*) \xrightarrow{\eta} H(M^* \otimes M) \xrightarrow{H(\epsilon)} H(U) \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
U \xrightarrow{\eta} H(M) \otimes H(M)^* \xrightarrow{H(f) \otimes \text{id}} H(M) \otimes H(M)^* \xrightarrow{\epsilon} H(M)^* \otimes H(M) \xrightarrow{\epsilon} U.
\end{array}
\]

which is $\text{tr}(H(f))$. 

\[\square\]
Corollary (The Lefschetz fixed-point theorem)

Let $M$ be a closed smooth manifold and $f : M \to M$, and let $H : \text{Mfd} \to \text{GrVec}_k$ denote rational homology, $H(M) = H_*(M, \mathbb{Q})$. If

$$\text{tr}(H(f)) = \sum_n (-1)^n \text{tr}(H_n(f))$$

is nonzero, then $f$ has a fixed point.
Corollary (The Lefschetz fixed-point theorem)

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$$\text{tr}(H(f)) = \sum_{n} (-1)^n \text{tr}(H_n(f))$$

is nonzero, then $f$ has a fixed point.

Proof.

The functor $H$ preserves the product and unit, by the Kunneth theorem:

$$H(M \times N, \mathbb{Q}) \cong H(M, \mathbb{Q}) \otimes H(N, \mathbb{Q}).$$

Therefore, $\text{tr}(H(f)) = H(\text{tr}(f))$. But $\text{tr}(f)$ is the fixed-point index of $f$, which is zero if $f$ has no fixed points.
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Here is a different way to draw pictures of objects and morphisms in a monoidal category.

Object $X$

Morphism $X \xrightarrow{f} Y$

Composition $X \xrightarrow{f} Y \xrightarrow{g} Z$
Here is a different way to draw pictures of objects and morphisms in a monoidal category.

**Object** $X$

**Product** $X \otimes Y$

**Morphism** $f : X \to Y$

**Product** $X \otimes Z \to Y \otimes W$

**Composition** $f : X \to Y, g : Y \to Z$

**Morphism** $f : X \otimes Y \to Z$
Some objects and morphisms are special; we draw them more simply.

Identity morphism $\text{id}_X$

Composite with identity $X \xrightarrow{\text{id}_X} X \xrightarrow{f} Y$

Unit object $U$

Product with unit $X \otimes U \cong X$

Symmetry $X \otimes Y \cong Y \otimes X$
More special string diagrams

Dual object $X^*$

$\eta : U \rightarrow X \otimes X^*$

$\varepsilon : X^* \otimes X \rightarrow U$
Example of string diagrams: the duality axioms

For example, the axioms for a dualizable object $X$ are the following:

\[ X \xrightarrow{X^*} X \xrightarrow{X} X \xrightarrow{X} X \xrightarrow{X^*} X = X \]

and

\[ X \xrightarrow{X^*} X \xrightarrow{X} X \xrightarrow{X} X \xrightarrow{X^*} X = X \]

Or “bent strings can be straightened.”
Example of string diagrams: the duality axioms

For example, the axioms for a dualizable object $X$ are the following:

\[
\begin{align*}
X & \Rightarrow X^* \\
X^* & \Rightarrow X
\end{align*}
\]

and

\[
\begin{align*}
X & \Rightarrow X^* \\
X^* & \Rightarrow X
\end{align*}
\]

Or “bent strings can be straightened.”

What does this mean?
Example of string diagrams: the duality axioms (II)

\[ \eta \otimes \text{id}_X \]

\[ X \otimes X^* \otimes X = U \otimes X \]

\[ \text{id}_X \otimes \epsilon \]

\[ X \otimes U \]

\[ \cong \]

\[ X \]

\[ X \otimes X^* \otimes X \]

\[ = \]

\[ X \]

\[ X \otimes X^* \]

\[ = \]

\[ X \]

\[ X \]

\[ X \]

\[ = \]

\[ \text{id}_X \]

\[ X \]
If $X$ is dualizable, the trace of $f : X \to X$ is given by

$$\begin{align*}
X & \xrightarrow{\eta''} U \\
X & \xrightarrow{\eta'} U \\
X & \xrightarrow{\eta} U
\end{align*}$$

or just

$$\begin{align*}
X & \xrightarrow{f \otimes \text{id}} U \\
X & \xrightarrow{\text{id} \otimes \varepsilon} U
\end{align*}$$
Theorem (Joyal-Street-Verity)

Any two string diagrams which are topologically equivalent represent equal morphisms in a symmetric monoidal category.

So we can prove theorems by drawing pictures!
**Theorem (Joyal-Street-Verity)**

*Any two string diagrams which are topologically equivalent represent equal morphisms in a symmetric monoidal category.*

So we can prove theorems by drawing pictures!

**Example**

If $X$ and $Y$ are dualizable, $X \overset{f}{\rightarrow} Y$, and $Y \overset{g}{\rightarrow} X$, then

$$
\text{tr}(gf) = = \text{tr}(fg).
$$
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Suppose $R$ is a noncommutative ring, $M$ is a finitely generated projective right $R$-module, and $f : M \rightarrow M$ is a map; how can we define the trace of $f$? We want to write:

$$
R \xrightarrow{\eta} M \otimes M^* \xrightarrow{f \otimes \text{id}} M \otimes M^* \xrightarrow{\cong} M^* \otimes M \xrightarrow{\varepsilon} R.
$$

where $M^* = \text{Hom}_R(M, R)$, but . . .

- $M$ is a right $R$-module while $M^*$ is a left $R$-module, and thus
- $M^* \otimes M = M^* \otimes_{\mathbb{Z}} M$ is an $R$-$R$-bimodule, while
- $M \otimes_R M^*$ is just an abelian group ($= \mathbb{Z}$-$\mathbb{Z}$-bimodule). And they certainly aren’t isomorphic!
Definition

For any $R$-$R$-bimodule $N$, the **shadow** of $N$ is the abelian group

$$\langle N \rangle = N \left/ \langle r \cdot n = n \cdot r \mid n \in N, r \in R \rangle \right.$$  

This is the “circular tensor product of $N$ with itself”, by analogy with

$$P \otimes_R Q = P \otimes_{\mathbb{Z}} Q \left/ \langle pr \otimes q = p \otimes rq \rangle \right.$$  

**Note:** if $R = \mathbb{Z}$, then $\langle N \rangle \cong N$. 
Lemma

If $P$ is an $R$-$S$-bimodule and $Q$ is an $S$-$R$-bimodule, then

$$\langle \langle P \otimes_S Q \rangle \rangle \simeq \langle \langle Q \otimes_R P \rangle \rangle.$$  

In particular, $M \otimes_R M^* \simeq \langle \langle M \otimes_R M^* \rangle \rangle \simeq \langle \langle M^* \otimes_Z M \rangle \rangle.$
Lemma

If $P$ is an $R$-$S$-bimodule and $Q$ is an $S$-$R$-bimodule, then

$$\langle \langle P \otimes_S Q \rangle \rangle \cong \langle \langle Q \otimes_R P \rangle \rangle.$$

In particular, $M \otimes_R M^* \cong \langle \langle M \otimes_R M^* \rangle \rangle \cong \langle \langle M^* \otimes_{\mathbb{Z}} M \rangle \rangle$.

Definition

If $M$ is a finitely generated projective left $R$-module, the (Hattori-Stallings) trace of $f: M \to M$ is the composite

$$\mathbb{Z} \xrightarrow{\eta} M \otimes_R M^* \xrightarrow{f \otimes \text{id}} M \otimes_R M^* \xrightarrow{\cong} \langle \langle M^* \otimes_{\mathbb{Z}} M \rangle \rangle \xrightarrow{\langle \langle \varepsilon \rangle \rangle} \langle \langle R \rangle \rangle.$$ 

How can we express this idea categorically?
1. Traces in symmetric monoidal categories
   - Two examples
   - Symmetric monoidal traces: the general case
   - Application: the Lefschetz fixed-point theorem
   - String diagrams for monoidal categories

2. Traces in bicategories
   - Noncommutative traces
   - Bicategories, shadows, and traces
   - String diagrams for bicategories with shadows
Bicategories

Definition

A bicategory consists of

- A collection of objects: $R, S, T, \ldots$,
- A collection of categories $\mathcal{B}(R, S)$,
- Product functors $\circ: \mathcal{B}(R, S) \times \mathcal{B}(S, T) \to \mathcal{B}(R, T)$, and
- Units $U_R \in \mathcal{B}(R, R)$,

such that $M \circ (N \circ P) \simeq (M \circ N) \circ P$, $M \circ U_S \simeq M$, and $U_R \circ M \simeq M$, coherently.

Note: there is no symmetry! $M \circ N$ and $N \circ M$ are objects of different categories, and in general need not both exist.

Example

In $\text{Mod}$, the objects are (noncommutative) rings, $\text{Mod}(R, S)$ consists of $R$-$S$-bimodules, $\circ$ is tensor product, and $U_R = R$. 
In a bicategory $\mathcal{B}$, we sometimes think of an object $M \in \mathcal{B}(R, S)$ as an arrow $R \xrightarrow{M} S$, and call it a 1-cell. Then the product $\odot$ is a sort of ‘composition’ of 1-cells:

$$R \xrightarrow{M} S \xrightarrow{N} T = R \xrightarrow{M \odot N} S.$$ 

Similarly, we think of a morphism $f: M \to N$ in $\mathcal{B}(R, S)$ as a ‘higher-dimensional’ arrow or 2-cell:

$$R \xrightarrow{M} S \xrightarrow{N} T.$$
Definition

A 1-cell $M \in \mathcal{B}(R, S)$ in a bicategory is right dualizable if there exists a 1-cell $M^* \in \mathcal{B}(S, R)$ and 2-cells $\eta: U_R \to M \circ M^*$ and $\varepsilon: M^* \circ M \to U_S$, satisfying the same axioms as before.

Example

A $\mathbb{Z}$-$R$-bimodule is right dualizable in $\mathcal{M}od$ if and only if it is a finitely generated projective right $R$-module.
A shadow on a bicategory $B$ consists of functors $\langle \langle - \rangle \rangle : B(R, R) \to T$, for some fixed category $T$, such that $\langle \langle M \circ N \rangle \rangle \simeq \langle \langle N \circ M \rangle \rangle$ coherently.

We have a shadow on the bicategory $Mod$ defined by

$$\langle \langle N \rangle \rangle = N / \langle r \cdot n = n \cdot r \rangle$$

as before.
Traces in bicategories with shadows

**Definition**

If $B$ is equipped with a shadow, $M \in B(R, S)$ is right dualizable, and $f : M \to M$ is a 2-cell, then the trace of $f$ is the composite

$$
\langle \langle U_R \rangle \rangle \to \langle \langle M \otimes M^* \rangle \rangle \xrightarrow{\langle \langle \eta \rangle \rangle} \langle \langle M \otimes M^* \rangle \rangle \xrightarrow{\langle \langle f \circ \text{id} \rangle \rangle} \langle \langle M^* \otimes M \rangle \rangle \xrightarrow{\approx} \langle \langle M^* \otimes M \rangle \rangle \xrightarrow{\langle \langle \varepsilon \rangle \rangle} \langle \langle U_S \rangle \rangle.
$$

**Example**

For finitely generated projective right $R$-modules, regarded as 1-cells from $\mathbb{Z}$ to $R$ in $\text{Mod}$, this trace recaptures the Hattori-Stallings trace. (Remember that $\langle \langle N \rangle \rangle \cong N$ when $N$ is a $\mathbb{Z}-\mathbb{Z}$-bimodule.)
Another example: Euler characteristics and characters

Definition

If $M$ is dualizable, its **Euler characteristic** is $\text{tr}(\text{id}_M)$.

In the symmetric monoidal case:

- In $\text{Vec}_k$, this computes the dimension of a finite dimensional vector space.
- In $\text{GrVec}_k$, it computes the alternating sum of the degree-wise dimensions, $\sum_i (-1)^i \dim(V_i)$.
- Thus, by preservation of traces, in the stable homotopy category it computes the usual Euler characteristic of a manifold.
Another example: Euler characteristics and characters

**Definition**
If $M$ is dualizable, its **Euler characteristic** is $\text{tr}(\text{id}_M)$. 

In the bicategorical case:

Let $k$ be a field, $G$ a group, and $V$ a finite-dimensional left $kG$-module (i.e. a representation of $G$ over $k$). Regarding $V$ as a 1-cell in $\mathcal{M}od(kG, k)$, it is right dualizable, with dual 

$$V^* = \text{Hom}_k(V, k) \in \mathcal{M}od(k, kG)$$

(The usual dual vector space, with the induced right $kG$-module structure.) Then:

- The shadow $\langle kG \rangle$ is a vector space with a basis given by the conjugacy classes of $G$, and
- The Euler characteristic $\text{tr}(\text{id}_V) : \langle kG \rangle \to k$ is the character of the representation $V$. 

More examples: generalized fixed-point theory

To prove a converse to the Lefschetz fixed-point theorem, one needs to incorporate information about the fundamental group of $M$.

Using traces in a ‘stable homotopy bicategory of bimodules’, one can define an invariant called the Reidemeister trace, identify it with an algebraic version, and then prove:

**Theorem**

Let $f : M \to M$ be continuous, where $M$ is a closed smooth manifold of dimension $\geq 3$. Then the Reidemeister trace of $f$ is zero if and only if $f$ is homotopic to a map with no fixed points.

There are also fiberwise and equivariant generalizations. In all cases the notion of bicategorical trace provides a formal framework and a line of inquiry which is often fruitful.

**Reference:** Kate Ponto, “Fixed point theory and trace for bicategories”
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String diagrams for bicategories

The string diagrams for monoidal categories are obtained from the ‘ordinary’ ones by ‘Poincaré duality’:

![Diagram]

We do the same for bicategories, one dimension up.
String diagrams for bicategories, II

Object $R$

Product $M \otimes N$

1-cell

$R \xrightarrow{M} S$

$U_R \xrightarrow{\eta} M \otimes M^*$

2-cell

$R \xrightarrow{M} S$

$M^* \otimes M \xrightarrow{\varepsilon} U_S$
String diagrams for shadows

Shadow
\[ \langle M \rangle \]

Cyclicity
\[ \langle M \odot N \rangle \cong \langle N \odot M \rangle \]
The string diagram for bicategorical trace

\[
\langle \langle U_R \rangle \rangle \langle \langle \eta \rangle \rangle \downarrow \langle \langle M \otimes M^* \rangle \rangle \langle \langle f \otimes \text{id} \rangle \rangle \downarrow \langle \langle M \otimes M^* \rangle \rangle \sim \downarrow \langle \langle \epsilon \rangle \rangle \downarrow \langle \langle U_S \rangle \rangle
\]
The string diagram for bicategorical trace

\[
\begin{array}{c}
\langle \langle UR \rangle \rangle \\
\langle \langle U_R \rangle \rangle \\
\langle \langle \eta \rangle \rangle \\
\langle \langle M \times M^* \rangle \rangle \\
\langle \langle f \otimes \text{id} \rangle \rangle \\
\langle \langle M \times M^* \rangle \rangle \\
\langle \langle M^* \otimes M \rangle \rangle \\
\langle \langle \epsilon \rangle \rangle \\
\langle \langle U_S \rangle \rangle \\
\end{array}
\]

\[\Rightarrow \]

\[
\begin{array}{c}
\langle \langle UR \rangle \rangle \\
\langle \langle U_R \rangle \rangle \\
\langle \langle \eta \rangle \rangle \\
\langle \langle M \times M^* \rangle \rangle \\
\langle \langle f \otimes \text{id} \rangle \rangle \\
\langle \langle M \times M^* \rangle \rangle \\
\langle \langle M^* \otimes M \rangle \rangle \\
\langle \langle \epsilon \rangle \rangle \\
\langle \langle U_S \rangle \rangle \\
\end{array}
\]

or just

\[
\begin{array}{c}
\langle \langle UR \rangle \rangle \\
\langle \langle U_R \rangle \rangle \\
\langle \langle \eta \rangle \rangle \\
\langle \langle f \otimes \text{id} \rangle \rangle \\
\langle \langle M \times M^* \rangle \rangle \\
\langle \langle M^* \otimes M \rangle \rangle \\
\langle \langle \epsilon \rangle \rangle \\
\langle \langle U_S \rangle \rangle \\
\end{array}
\]
Trace in symmetric monoidal categories unifies many different notions, including the linear trace and the fixed-point index. Formal comparison results can then produce contentful theorems (e.g. the Lefschetz fixed-point theorem).

Trace in bicategories generalizes this to include examples such as noncommutative linear traces and basepoint-free, fiberwise, and equivariant fixed-point theory.

In both cases, string diagrams can be used to obtain intuition and simplify proofs.