Abstracting away from cell complexes

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Replacing big messy cell complexes with smaller and simpler but more abstract ones

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Outline

Small simple cell complexes

- Spheres
- Tori
- Projective space
- Manifolds
- ...

Big messy cell complexes, I

Postnikov towers

The n^{th} Postnikov section of X is obtained from X by

- gluing on enough $(n+2)$-cells to kill $\pi_{n+1}(X)$, then
- gluing on enough $(n+3)$-cells to kill π_{n+2} of the result, then
- gluing on enough $(n+4)$-cells to kill π_{n+3} of the result,
- and so on.

Note: Gluing on a k-cell is the same as taking a pushout

$$\begin{array}{ccc} S^{k-1} & \longrightarrow & X \\ \downarrow & & \\ D^k & & \end{array}$$

Big messy cell complexes, II

Localization

The localization of X at a map $f : S \to T$ is obtained by:

- Replacing f by a cofibration,
- Taking its pushout product with all the boundary inclusions $S^n \hookrightarrow D^{n+1}$,
- For each resulting map $\hat{f}_n : A_n \to B_n$, taking one pushout

$$\begin{array}{ccc} A_n & \longrightarrow & X \\ \downarrow & & \\ B_n & & \end{array}$$

  for each map $A_n \to X$,
- Repeating the previous step, perhaps *transfinitely often*,
- . . . until we're done.

Big messy cell complexes, III

. . . and it doesn't get any easier from there.

Big messy cell complexes, III

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Can we package up this machinery better so we don't have to think about it?

Outline

The 1-categorical case

- G. M. Kelly, "A unified treatment of transfinite constructions for free algebras, free monoids, colimits, associated sheaves, and so on", Bull. Austral. Math. Soc. 22 (1980), 1–83

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Theorem (Kelly)

Let $\mathcal{A}$ be a cocomplete category with two cocomplete factorization systems $(\mathcal{E}, \mathcal{M})$ and $(\mathcal{E}', \mathcal{M}')$, let $\mathcal{A}$ be $\mathcal{E}$- and $\mathcal{E}'$-cowellpowered, let S be a well-pointed endofunctor, and for some regular cardinal α let S preserve the $\mathcal{E}$-tightness of $(\mathcal{M}, \alpha)$-cones. Then S-Alg is constructively reflective in $\mathcal{A}$.

The 1-categorical case, really now

Theorem (Kelly?)

Let $\mathcal{C}$ be a locally presentable category. Then:

- *Every accessible endofunctor of $\mathcal{C}$ generates an algebraically-free monad.*
- *Every small diagram of accessible monads on $\mathcal{C}$ has an algebraic colimit.*

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- What does this mean?
- What is it good for?

Review about monads

- Every monad T has a category of algebras T-Alg.
- The forgetful functor $U_T : T\text{-Alg} \to \mathcal{C}$ has a left adjoint F_T.
- $T = U_T F_T$

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- $T = U_T F_T$
- The assignation $T \mapsto T$-Alg is a fully faithful embedding

$$\mathrm{Monads}^{\mathrm{op}} \hookrightarrow \mathrm{Cat}_{/\mathcal{C}}.$$

i.e. we have

$$\begin{array}{ccc} \mathrm{Monads}(T_1, T_2) & \cong & \mathrm{Cat}_{/\mathcal{C}}(T_2\text{-Alg}, T_1\text{-Alg}) \\ \left[\phi : T_1 \to T_2\right] & \mapsto & \left[(T_2X \to X) \mapsto (T_1X \xrightarrow{\phi} T_2X \to X)\right] \end{array}$$

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$$\left[T_1 \xrightarrow{T_1 \eta_2} T_1 T_2 \xrightarrow{\mathrm{act}_{G T_2}} T_2 \right] \quad \leftarrow\!\shortmid \quad \left[\begin{array}{ccc} T_2\text{-Alg} & \xrightarrow{G} & T_1\text{-Alg} \\ & \searrow^{U_{T_2}} \quad \swarrow_{U_{T_1}} & \\ & \mathcal{C} & \end{array} \right]$$

Free monads

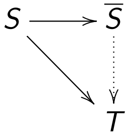
Definition

Every monad has an underlying endofunctor; this defines a functor

$$\text{monads on } \mathcal{C} \longrightarrow \text{endofunctors on } \mathcal{C}.$$

A free monad on an endofunctor S is the value at S of a (partially defined) left adjoint to this:

$$\mathrm{Monads}(\overline{S}, T) \cong \mathrm{Endofrs}(S, T)$$



Algebraically-free monads

Definition

For an endofunctor S, an S-algebra is an object X equipped with a map $SX \to X$.

Definition

A monad $\overline{S}$ is algebraically-free on S if we have an equivalence of categories over $\mathcal{C}$:

$$\overline{S} \text{ monad-algebras} \xrightarrow{\simeq} S \text{ endofunctor-algebras}$$

Theorem (Kelly?)

Every algebraically-free monad is free, and the converse holds if $\mathcal{C}$ is locally small and complete.

Examples

$S(X) = A$

- S-algebras are objects X with a map $A \to X$.
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- $\overline{S}(X)$ is the free magma on X; in **Set** its elements are bracketed words $((xy)z)(zy)$.

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Note: can be built as an infinite "cell complex"

$$X \longrightarrow X \sqcup (X \times X) \longrightarrow \cdots$$

Examples, II

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- S-algebras are “pointed magmas”.

$S(X) = A \times X + 1$

- $\overline{S}(\emptyset)$ is the “list object” on A, generated by $\mathrm{nil} : 1 \to X$ and $\mathrm{cons} : A \times X \to X$.
- If $A = 1$, then $\overline{S}(\emptyset) = \mathbb{N}$.

Algebraically-free monads

Theorem (Kelly?)

Every algebraically-free monad is free, and the converse holds if $\mathcal{C}$ is locally small and complete.

Proof.

If $\overline{S}$ is algebraically-free on S, then

$$\begin{aligned}\mathrm{Endofrs}(S, T) &\cong \mathsf{Cat}_{/\mathcal{C}}(T\text{-}\mathsf{Alg}_{\mathsf{monad}}, S\text{-}\mathsf{Alg}_{\mathsf{endofr}}) \\ &\cong \mathsf{Cat}_{/\mathcal{C}}(T\text{-}\mathsf{Alg}_{\mathsf{monad}}, \overline{S}\text{-}\mathsf{Alg}_{\mathsf{monad}}) \\ &\cong \mathrm{Monads}(\overline{S}, T)\end{aligned}$$

□

Algebraically-free monads

Theorem (Kelly?)

Every algebraically-free monad is free, and the converse holds if $\mathcal{C}$ is locally small and complete.

Proof.

If $\mathcal{C}$ is loc sm and complete, $X \in \mathcal{C}$ has an endomorphism monad $\langle X, X\rangle = \mathrm{Ran}_{(X:1\to\mathcal{C})}(X : 1 \to \mathcal{C})$ such that

- For a monad T,

$$\Big(T\text{-algebra structures on } X\Big) \longleftrightarrow \Big(\text{monad maps } T \to \langle X, X\rangle\Big).$$

- For an endofunctor S,

$$\Big(S\text{-algebra structures on } X\Big) \longleftrightarrow \Big(\text{endofr maps } S \to \langle X, X\rangle\Big).$$

Algebraic colimits of monads

Definition

An algebraic colimit of a diagram $D : J \to \text{Monads}$ is a monad T with an equivalence of categories over $\mathcal{C}$:

$$T\text{-Alg} \xrightarrow{\simeq} \lim_{j \in J} \ D_j\text{-Alg}$$

This is a limit in $\text{Cat}_{/\mathcal{C}}$, so it means that

T-algebra structures on X $\longleftrightarrow$ compatible families of D_j-algebra structures on X.

Theorem

Every algebraic colimit is a colimit in the category of monads, and the converse holds if $\mathcal{C}$ is locally small and complete.

Example 0: coproducts

Example

The algebraically-initial monad has, as category of algebras, the terminal object of $\mathsf{Cat}_{/\mathcal{C}}$, namely $\mathcal{C}$ itself. Thus, it is the identity monad Id.

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Example

An algebra structure for the algebraic coproduct $T_1 \sqcup T_2$ consists of unrelated T_1-algebra and T_2-algebra structures.

Example 1: algebra

Recall a semigroup is a magma whose operation is associative.

1. Let $S_2(X) = X \times X$; so S_2-algebras are magmas and $\overline{S_2}X$ is the free magma on X.
2. Let $S_3(X) = X \times X \times X$, so S_3-algebras are sets equipped with a ternary operation.
3. A magma X has two induced ternary operations $x(yz)$ and $(xy)z$. This yields two functors
$$S_2\text{-Alg} \rightrightarrows S_3\text{-Alg}$$
whose equalizer is the category of semigroups.
4. Thus, the algebraic coequalizer of the corresponding two monad morphisms $\overline{S_3} \rightrightarrows \overline{S_2}$ is the monad for semigroups.

Presentations of monads

Definition

A presentation of a group (or other algebraic structure) is a coequalizer of maps between free groups:

$$F\langle R\rangle \rightrightarrows F\langle X\rangle \to G$$

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Definition

A generalized presentation of an object is an iterated colimit of diagrams of free objects.

Definition

A generalized presentation of a monad is an (algebraic) iterated colimit of diagrams of (algebraically-)free monads.

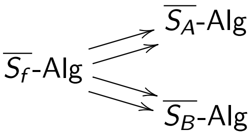
Example 2: localization

Let $f : A \to B$ be a fixed morphism.

1. Let $S_f(X) = \mathcal{C}(A, X) \cdot B$ and $S_A(X) = \mathcal{C}(A, X) \cdot A$ and $S_B(X) = \mathcal{C}(B, X) \cdot B$.
2. An S_f-algebra X is equipped with a map $\mathcal{C}(A, X) \cdot B \to X$, or equivalently $\mathcal{C}(A, X) \to \mathcal{C}(B, X)$.
3. An S_f-algebra has two S_A-algebra structures: "evaluation" and the composite $\mathcal{C}(A, X) \cdot A \xrightarrow{f} \mathcal{C}(A, X) \cdot B \to X$.
4. These coincide iff the S_f-algebra structure $\mathcal{C}(A, X) \to \mathcal{C}(B, X)$ is a right inverse to $(- \circ f) : \mathcal{C}(B, X) \to \mathcal{C}(A, X)$.
5. Similarly, an S_f-algebra has two S_B-algebra structures, which coincide iff the S_f-algebra structure is a left inverse to $(- \circ f)$.

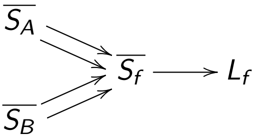
Example 2: localization

6. The joint equalizer of the two parallel pairs



consists of X for which $(- \circ f) : \mathcal{C}(B, X) \to \mathcal{C}(A, X)$ has a (necc. unique) two-sided inverse — i.e. the *f*-local objects.

7. Thus, the joint algebraic coequalizer of the pairs



is the *f*-localization (the "free *f*-local object" monad).

Example 3: combining structures

Suppose instead of one morphism $f : A \to B$ we have a set of them, $\{f_i : A_i \to B_i\}_{i \in I}$.

1. We could generalize the construction of L_f to $L_{\{f_i\}}$.
2. Or we could simply take the algebraic coproduct $\coprod_{i \in I} L_{f_i}$, whose algebras are equipped with (unrelated) L_{f_i}-algebra structures, hence f_i-local for all i.

Outline

Bringing in the homotopy theory

Assumption

$\mathcal{C}$ is a presheaf category with a right proper simplicial model structure in which the cofibrations are the monomorphisms.

(For most of what follows, this is much more than necessary.)

Examples:

- Simplicial sets
- The injective model structure on simplicial presheaves
- Any right proper localization of the latter
- Any locally presentable, locally cartesian closed ∞-category has such a presentation (Cisinski, Gepner–Kock)

Algebraic fibrant replacement

Let $\mathcal{J}$ be the generating acyclic cofibrations. Copying only the first half of the localization construction, we obtain a monad R whose algebras are algebraically fibrant: objects X equipped with chosen lifts against all $\mathcal{J}$-maps.

Theorem (Garner)

Each unit map $\eta_X : X \to RX$ is an acyclic cofibration.

Thus, R is a "fibrant replacement monad".

Algebraic factorization

More generally, we have a monad $\mathbf{R}$ on $\mathcal{C}^{\to}$ whose algebras are algebraic fibrations: maps $g : Y \to X$ equipped with chosen lifts against all $\mathcal{J}$-maps.

Theorem (Garner)

For any $g : Y \to X$, the unit $\eta_g : g \to \mathbf{R}g$ looks like

$$\begin{array}{ccc} Y & \longrightarrow & Eg \\ \downarrow & & \downarrow \\ X & = & X \end{array}$$

and $Y \to Eg$ is an acyclic cofibration.

(In fact, we have a whole "algebraic weak factorization system": cofibrant replacement is also a comonad.)

Composing algebraic fibrations

Theorem (Garner)

If $g : Y \to X$ and $f : Z \to Y$ are algebraic fibrations, then gf is naturally an algebraic fibration, and the square

$$\begin{array}{ccc} Z & \xrightarrow{f} & Y \\ {\scriptstyle gf}\downarrow & & \downarrow{\scriptstyle g} \\ X & = & X \end{array}$$

is a morphism of $\mathbf{R}$-algebras.

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is a morphism of $\mathbf{R}$*-algebras.*

Corollary

If $f : Z \to Y$ is an algebraic fibration and Y is algebraically fibrant, then Z is algebraically fibrant and f is a map of R-algebras.

Outline

Homotopical localization

We can repeat the 1-categorical construction, with mapping spaces and homotopies instead of hom-sets and equalities.

1. $S_f(X) = \mathrm{Map}(A, X) \otimes B$, with $\mathrm{Map}(A, X)$ the simplicial mapping space and $\otimes$ the simplicial tensor.
2. Instead of the coequalizer of $\overline{S_A} \rightrightarrows \overline{S_f}$ we take the pushout of

$$\begin{array}{ccc} \overline{\mathbf{2} \otimes S_1} = \overline{S_A} \sqcup \overline{S_A} & \longrightarrow & \overline{S_f} \\ \downarrow & & \downarrow \\ \overline{\Delta^1 \otimes S_A} & \longrightarrow & P_f \end{array}$$

A P_f-algebra is equipped with a right homotopy inverse to $(- \circ f) : \mathrm{Map}(B, X) \to \mathrm{Map}(A, X)$.

Homotopical localization

3. Instead of a joint coequalizer/pushout using a single copy of $\overline{S_f}$, we take the coproduct of P_f and its dual.
   - The algebras are equipped with both a left and a right homotopy inverse to $(- \circ f)$, perhaps different.
   - The existence of such is still equivalent to $(- \circ f)$ being a homotopy equivalence.
   - The space of (left inverse, right inverse) pairs is contractible (if nonempty); the space of two-sided homotopy inverses is not.

Let's call the resulting monad $\tilde{L}_f$.

Ensuring fibrancy

BUT!!!

- $\tilde{L}_f$ does not produce fibrant objects.
- $\mathrm{Map}(A, X)$ has the wrong homotopy type if X is not fibrant! So non-fibrant $\tilde{L}_f$-algebras need not even have f-local homotopy type.
- In particular, $\tilde{L}_f X$ need not have f-local homotopy type.

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- In particular, $\tilde{L}_f X$ need not have f-local homotopy type.

Simple answer

Take the algebraic coproduct $L_f = \tilde{L}_f \sqcup R$ with the fibrant replacement monad.

- L_f-algebras are $\tilde{L}_f$-algebras with an unrelated R-structure.
- In particular, they are fibrant, hence f-local; including $L_f X$.

Homotopy initiality

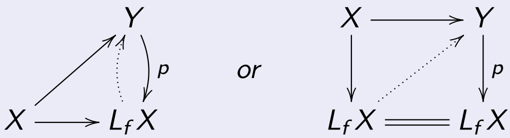
So $L_f X$ is f-local; but is it the f-localization?

- $L_f X$ is strictly initial in the category of "algebraically f-local", algebraically fibrant objects under X (and maps that preserve the algebraic structure).
- The f-localization is supposed to be homotopy initial in the category of f-local, fibrant objects under X (and all maps).

Acyclically cofibrant algebras

Theorem

If $p : Y \to L_f X$ is a (non-algebraic) fibration where Y is f-local, then any section of p over X extends to a section of p:



Acyclically cofibrant algebras

If $p : Y \to L_f X$ is a (non-algebraic) fibration where Y is f-local, then any section of p over X extends to a section of p.

Proof.

1. Since Y is f-local and p is a fibration, we can choose an $\tilde{L}_f$-algebra structure on Y making p an $\tilde{L}_f$-algebra map.
2. Choose an arbitrary $\mathbf{R}$-algebra structure on p.
3. By composition, Y becomes an R-algebra, hence an $L_f = (\tilde{L}_f \sqcup R)$-algebra, and p an L_f-algebra map.
4. Since $L_f X$ is the free L_f-algebra on X, the given section $X \to Y$ induces an L_f-algebra map $L_f X \to Y$.
5. Since the composite $L_f X \to Y \to L_f X$ is again an L_f-algebra map, by uniqueness it is the identity.

□

Homotopy initiality

Theorem

$L_f X$ is the f-localization of X.

Proof.

1. We will show $\mathrm{Map}(L_f X, Z) \to \mathrm{Map}(X, Z)$ is an acyclic fibration for any f-local fibrant Z, by lifting in an arbitrary

$$\begin{array}{ccc} X & \xrightarrow{r} & Z^{\Delta^n} \\ \downarrow & & \downarrow \\ L_f X & \xrightarrow[s]{} & Z^{\partial\Delta^n} \end{array}$$

2. Since $L_f X$, $Z^{\partial\Delta^n}$, and Z^{Δ^n} are f-local, so is $s^* Z^{\Delta^n}$.
3. The map $s^* Z^{\Delta^n} \to L_f X$ is a fibration, so it has a section by the previous theorem.

Outline

Well-behaved monads

What is special about localization, and what isn't? We need:

1. T-algebra structures can be rectified along fibrations.
2. T-algebra structures lift to path objects, etc.

Well-behaved monads

What is special about localization, and what isn't? We need:

1. T-algebra structures can be rectified along fibrations.
2. T-algebra structures lift to path objects, etc.

These work for localization because the pushout

$$\begin{array}{ccccc} \overline{\mathbf{2} \otimes S_1} & = & \overline{S_A} \sqcup \overline{S_A} & \longrightarrow & \overline{S_f} \\ \downarrow & & & & \downarrow \\ \overline{\Delta^1 \otimes S_A} & & \longrightarrow & & P_f \end{array}$$

is homotopically well-behaved.

Cell monads

Definition

- A generating cofibration of monads is a map

$$\overline{S \times A} \to \overline{S \times B}$$

  where S is a well-behaved endofunctor and $A \to B$ is a generating cofibration.
- A cell complex of monads is a composite of pushouts of generating cofibrations.
- A cell monad T is such that $\mathrm{Id} \to T$ is a cell complex. (Recall Id is the initial object in the category of monads.)

Homotopy theory of cell monads, I

"T-algebra structures lift to path objects, etc."

Theorem

If T is a cell monad, then the category $T\text{-Alg}_{\mathbf{f}}$ of fibrant T-algebras is a fibration category á là Brown.

Sketch of proof.

Fibrations, weak equivalences, and limits are inherited from the base category. For factorizations, we argue inductively up the cell complex $\mathrm{Id} \to T$. In the case of a generating cofibration, we factor downstairs and lift the $(S \times B)$-algebra structure, using the pushout product. □

Homotopy theory of cell monads, II

Theorem (in progress)

1. *If $T_1 \to T_2$ is a cell complex, then $T_2\text{-Alg}_{\mathbf{f}} \to T_1\text{-Alg}_{\mathbf{f}}$ is almost a "fibration of fibration categories" á là Szumiło.*
2. *The pushouts in a cell complex yield homotopy pullbacks of fibration categories, hence (Szumiło) pullbacks of ∞-cats.*
3. *For a cell monad T,*
   - *the homotopy ∞-category of $T\text{-Alg}_{\mathbf{f}}$ coincides with the algebras for the analogous ∞-monad;*
   - *any map of ∞-algebras can be presented by a T-algebra fibration; and hence*
   - *free $(T \sqcup R)$-algebras present free objects of this ∞-category.*

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tl;dr: Cell monads are homotopically meaningful.

Replacing big messy cell complexes with smaller and simpler but more abstract ones

Before

The f-localization of a space is a cell complex with transfinitely many cells.

After

The f-localization monad is a cell monad with four cells.

∞-monads

Question

Why not work directly with presentations of ∞-monads?

In theory, we could. But reasons to use cell monads include:

1. We can leverage Kelly's existing "package".
2. Fits in a model-categorical framework, if we prefer that for other reasons.
3. Free $(T \sqcup R)$-algebras have a stronger universal property than just ∞-freeness.

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3. Free $(T \sqcup R)$-algebras have a stronger universal property than just ∞-freeness. . . . and it corresponds exactly to "induction principles" in type theory.

Outline

Higher Inductive Types

Slogan

Higher inductive types are a notation for describing families of cell monads that exist "uniformly" in all nice model categories.

Example

For $f : A \to B$, the f-localization $L_f X$ is generated by four "constructors" (monad cells):

1. $\mathsf{ext} : \forall(g : A \to L_f X), B \to L_f X$
2. $\mathsf{rinv} : \forall(g : A \to L_f X), \forall(a : A), \mathsf{ext}_g(f(a)) = g(a)$
3. $\mathsf{ext}' : \forall(g : A \to L_f X), B \to L_f X$
4. $\mathsf{linv} : \forall(h : B \to L_f X), \forall(b : B), \mathsf{ext}'_{h \circ f}(b) = h(b)$

Homotopy type theory

Slogan

Type theory is a system of notations for describing constructions that exist "uniformly" in all categories of a certain sort.

Each "type constructor" represents a category-theoretic operation, e.g. $x : A \vdash \prod_{y:B(x)} C$ represents the right adjoint to pullback along a fibration $B \to A$.

Slogan

Homotopy type theory is a system of notations for describing constructions that exist in all "categories with homotopy theory".

Fine print: there are unresolved coherence questions in making this completely precise in generality.

Some history

- Traditional type theory has "ordinary" inductive types, which describe free monads and coproducts of monads.
- These include coproducts of spaces, but not other colimits.
- HITs were invented to describe other colimits in type theory.
- Only afterwards did we discover cell monads, when trying to model them categorically.

Recursion and induction for $\mathbb{N}$

Recursion principle

Given any X together with $x_0 : 1 \to X$ and $x_s : \mathbb{N} \times X \to X$, there is an $f : \mathbb{N} \to X$ such that

$$f(0) = x_0$$
$$f(n+1) = x_s(n, f(n))$$

Recursion and induction for $\mathbb{N}$

Recursion principle

Given any X together with $x_0 : 1 \to X$ and $x_s : \mathbb{N} \times X \to X$, there is an $f : \mathbb{N} \to X$ such that

$$f(0) = x_0$$
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Induction principle

Given any property $P(n)$ such that $P(0)$ and $P(n) \to P(n+1)$, we have $\forall n, P(n)$.

Type-theoretic induction for $\mathbb{N}$

Generalized induction principle

Given any map $p : C \to \mathbb{N}$ together with $x_0 : p^{-1}(0)$ and functions $x_{s,n} : p^{-1}(n) \to p^{-1}(n+1)$ for all n, we have a section $f : \mathbb{N} \to C$ of p such that

$$f(0) = x_0$$
$$f(n+1) = x_{s,n}(f(n))$$

Type-theoretic induction for $\mathbb{N}$

Generalized induction principle

Given any map $p : C \to \mathbb{N}$ together with $x_0 : p^{-1}(0)$ and functions $x_{s,n} : p^{-1}(n) \to p^{-1}(n+1)$ for all n, we have a section $f : \mathbb{N} \to C$ of p such that

$$f(0) = x_0$$
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- $C = \mathbb{N} \times X$ gives recursion.
- $C = \{ n \in \mathbb{N} \mid P(n) \}$ gives induction.

Categorical induction for $\mathbb{N}$

Generalized induction principle

Given any map $p : C \to \mathbb{N}$ together with $x_0 : 1 \to C$ and $x_s : C \to C$ such that

$$\begin{array}{ccc} 1 & \xrightarrow{x_0} & C \\ \downarrow & & \downarrow p \\ 1 & \xrightarrow[0]{} & \mathbb{N} \end{array} \qquad \text{and} \qquad \begin{array}{ccc} C & \xrightarrow{x_s} & C \\ \downarrow & & \downarrow \\ \mathbb{N} & \xrightarrow[+1]{} & \mathbb{N} \end{array} \tag{1}$$

we have a section $f : \mathbb{N} \to C$ of p such that

$$f(0) = x_0$$
$$f(n+1) = x_s(f(n))$$

Endofunctor induction

Generalized induction principle

Let $S(X) = 1 + X$. Then any S-algebra map $C \to \mathbb{N}$ has a section.

In type theory, these maps are restricted to be fibrations, corresponding to "dependent types".

Endofunctor induction

Generalized induction principle

Let $S(X) = 1 + X$. Then any S-algebra map $C \to \mathbb{N}$ has a section.

In type theory, these maps are restricted to be fibrations, corresponding to "dependent types".

Ordinary inductive types

For any well-behaved endofunctor S, there is an S-algebra W_S such that any S-algebra fibration $C \to W_S$ has a section.

(It's $(\overline{S} \sqcup R)(\emptyset)$.)

Higher inductive types

Higher inductive types

For any cell monad T, there is a T-algebra W_T such that any T-algebra fibration $C \to W_T$ has a section.

Gives type-theoretic notations for:

- Small concrete cell complexes (spheres, tori, etc.)
- Homotopy colimits
- Localization
- Postnikov towers
- Spectrification
- And much more!