

Categories, functors, and profunctors are a free cocompletion

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- 1 The problem
- 2 Progress and hints
- 3 The solution
- 4 Concluding remarks

Definition (Bénabou, 1967)

A **monad** in a bicategory consists of

- An object A ,
 - A morphism $t : A \rightarrow A$,
 - 2-cells $m : tt \rightarrow t$ and $i : 1_A \rightarrow t$, and
 - Associativity and unitality axioms.
-
- Equivalently, a lax functor out of 1.
 - Includes ordinary monads, enriched ones, internal ones, indexed ones, monoidal ones, ...
 - The “Eilenberg-Moore object” A^t is a lax limit, and the “Kleisli object” A_t is a lax colimit (Street).

Example

A monad in the bicategory of spans (of sets) consists of

- A set A_0 ,
- A span $A_0 \leftarrow A_1 \rightarrow A_0$,
- Functions $m : A_1 \times_{A_0} A_1 \rightarrow A_1$ and $i : A_0 \rightarrow A_1$, and
- Associativity and unitality axioms.

In other words, it's nothing but a (small) **category**!

- In similar bicategories, we obtain internal categories, enriched categories, ...

Definition (Street, 1972)

A **lax monad morphism** $(A, t) \rightarrow (B, s)$ consists of

- A morphism $f : A \rightarrow B$,
- A 2-cell $\bar{f} : sf \rightarrow ft$, and
- Some axioms.

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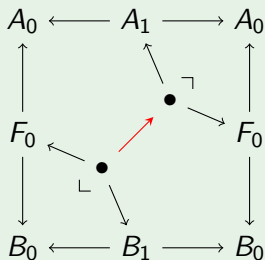
- A morphism $f : A \rightarrow B$,
 - A 2-cell $\bar{f} : sf \rightarrow ft$, and
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-
- = a lax natural transformation between lax functors.
 - Induces a morphism $A^t \rightarrow B^s$ between E-M objects.
 - Colax monad morphisms = colax natural transformations, and induce morphisms $A_t \rightarrow B_s$ between Kleisli objects.

Morphisms of categories?

Example

A lax monad morphism in the bicategory of spans consists of

- A span $A_0 \leftarrow F_0 \rightarrow B_0$,
- A morphism

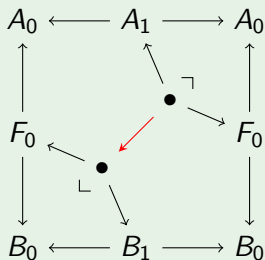


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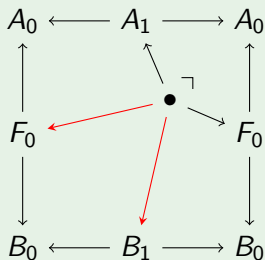


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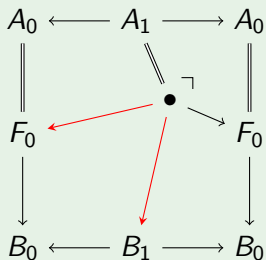


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A **functor** consists of

- A function $f_0 : A_0 \rightarrow B_0$
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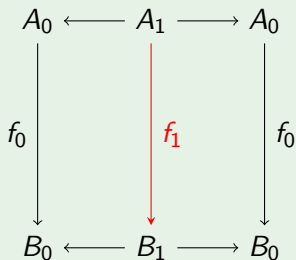


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Morphisms of categories?

Example

A functor consists of

- A function $f_0 : A_0 \rightarrow B_0$
- A morphism

$$\begin{array}{ccccc} A_0 & \longleftarrow & A_1 & \longrightarrow & A_0 \\ \downarrow f_0 & & \downarrow f_1 & & \downarrow f_0 \\ B_0 & \longleftarrow & B_1 & \longrightarrow & B_0 \end{array}$$

functor = colax monad morphism such that the span $A_0 \leftarrow F_0 \rightarrow B_0$ is a mere function $A_0 \rightarrow B_0$.

Definition (Street,1972)

For lax f, g , a **monad 2-cell** $(A, t) \begin{array}{c} \xrightarrow{f} \\ \Downarrow \\ \xrightarrow{g} \end{array} (B, s)$ consists of

- A 2-cell $f \rightarrow g$
 - Satisfying some axioms.
-
- = a modification between lax natural transformations.
 - Induces a 2-cell $A^t \begin{array}{c} \xrightarrow{\quad} \\ \Downarrow \\ \xrightarrow{\quad} \end{array} B^s$ between E-M objects.
 - If f, g are colax, monad 2-cells = modifications between colax transformations, and induce 2-cells between Kleisli objects.

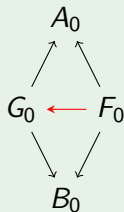
2-cells of categories?

Example

For categories A, B and colax monad morphisms $f, g : A \rightarrow B$,

a monad 2-cell $A \begin{array}{c} \xrightarrow{f} \\ \Downarrow \\ \xrightarrow{g} \end{array} B$ consists of

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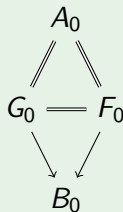
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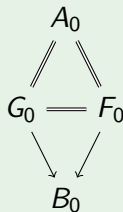
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- Satisfying some axioms.

i.e. just an equality $f = g$!

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Profunctors

Definition (Bénabou, 1967?)

For categories A, B , a **profunctor** $A \nrightarrow B$ consists of

- A functor $H : B^{\text{op}} \times A \rightarrow \text{Set}$

Definition (Bénabou, 1967?)

For categories A, B , a **profunctor** $A \leftrightarrow B$ consists of

- Sets $H(b, a)$ for $a \in A_0, b \in B_0$
- Actions

$$H(b, a) \times A(a, a') \rightarrow H(b, a') \quad \text{and}$$
$$B(b', b) \times H(b, a) \rightarrow H(b', a)$$

- Associativity and unitality axioms.

Definition (Bénabou, 1967?)

For categories A, B , a **profunctor** $A \leftrightarrow B$ consists of

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- Actions

$$H \times_{A_0} A_1 \rightarrow H \quad \text{and}$$

$$B_1 \times_{B_0} H \rightarrow H$$

- Associativity and unitality axioms.

Definition (Bénabou, 1973)

For monads (A, t) , (B, s) in a bicategory, a **module** $A \rightarrow B$ is

- A morphism $h : A \rightarrow B$
- Actions

$$ht \rightarrow h \quad \text{and}$$

$$sh \rightarrow h$$

- Associativity and unitality axioms.
- Modules in spans = profunctors
- Also get enriched profunctors, internal profunctors, ...

Definition

The **composite** of profunctors $H : A \rightarrow B$ and $K : B \rightarrow C$ is

$$(KH)(c, a) = \int^{b \in B} K(c, b) \times H(b, a)$$

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$$= \text{coeq} \left(\sum_{b, b' \in B_0} K(c, b') \times B(b', b) \times H(b, a) \rightrightarrows \sum_{b \in B_0} K(c, b) \times H(b, a) \right)$$

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$$= \text{coeq} (K \times_{B_0} B_1 \times_{B_0} H \rightrightarrows K \times_{B_0} H)$$

Composing modules

Definition

A bicategory \mathcal{B} has **local coequalizers** if all hom-categories have coequalizers and composition preserves them in each variable.

Definition

In this case, the **composite** of modules $h : (A, t) \rightarrow (B, s)$ and $k : (B, s) \rightarrow (C, r)$ is

$$\text{coeq}_1(ksh \rightrightarrows kh)$$

We have a bicategory $\text{Mod}_1(\mathcal{B})$ whose

- objects are monads in \mathcal{B} , and
- morphisms are modules

What are these modules?

Theorem (Street 1981, Carboni-Kasangian-Walters 1987)

- 1 $\text{Mod}_1(\text{Mod}_1(\mathcal{B})) \simeq \text{Mod}_1(\mathcal{B})$.
- 2 \mathcal{C} is of the form $\text{Mod}_1(\mathcal{B})$ iff it has
 - Local coequalizers, and
 - Kleisli objects for monads.

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Example

In $\mathcal{P}rof = Mod_1(Span)$, a monad on a category A is

- A category A' with the same objects as A , and
- An identity-on-objects functor $A \rightarrow A'$.

Its Kleisli object is just A' .

Functors vs Profunctors

Definition (Wood 1982)

A **proarrow equipment** consists of

- Two bicategories \mathcal{K} and \mathcal{M} with the same objects,
- An identity-on-objects and locally fully faithful pseudofunctor $(-)_\bullet : \mathcal{K} \rightarrow \mathcal{M}$,
- Such that every morphism f_\bullet has a right adjoint in \mathcal{M} .

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Examples

- $\mathcal{K} = \mathit{Cat}$, $\mathcal{M} = \mathit{Prof}$, and for a functor $f : A \rightarrow B$,

$$f_\bullet(b, a) = B(b, f(a))$$

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- $\mathcal{K} = \mathit{Cat}$, $\mathcal{M} = \mathit{Prof}$, and for a functor $f : A \rightarrow B$,

$$f_\bullet(b, a) = B(b, f(a))$$

- $\mathcal{K} = \mathit{Set}$, $\mathcal{M} = \mathit{Span}$, and for a function $f : A \rightarrow B$,

$$f_\bullet = A \multimap A \xrightarrow{f} B$$

Those pesky 2-cells

Theorem (Lack-Street 2002)

In the *free cocompletion* of \mathcal{K} under Kleisli objects,

- The objects are monads in \mathcal{K} ,
- The morphisms are colax monad morphisms in \mathcal{K} ,

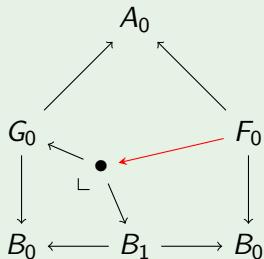
- The 2-cells $(A, t) \begin{array}{c} \xrightarrow{f} \\ \Downarrow \\ \xrightarrow{g} \end{array} (B, s)$ are

- 2-cells $f \rightarrow sg$ in \mathcal{K} ,
- Satisfying axioms.

2-cells of categories

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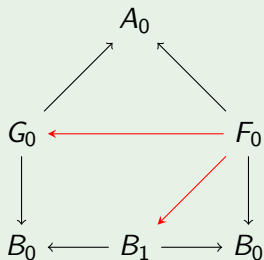
For categories A, B and colax monad morphisms $f, g : A \rightarrow B$:



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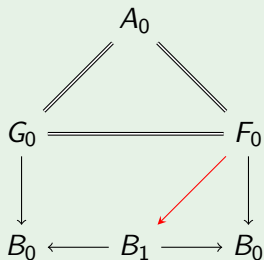
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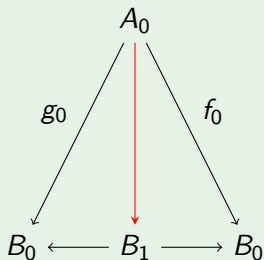
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2-cells of categories

Example

For categories A, B and functors $f, g : A \rightarrow B$:



A natural transformation!

Observation (Lack-S. 2012)

There is a cartesian closed 2-category \mathcal{F} such that

① \mathcal{F} -enriched categories

are the same as

② data consisting of

- 2-categories \mathcal{K} and \mathcal{M} with the same objects, and
- an identity-on-objects and locally fully faithful 2-functor $\mathcal{K} \rightarrow \mathcal{M}$.

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The objects of \mathcal{F} are fully faithful functors $A_\tau \hookrightarrow A_\lambda$.

An \mathcal{F} -category \mathcal{B} has homs

$$\mathcal{B}(x, y) = \left(\mathcal{B}_\tau(x, y) \hookrightarrow \mathcal{B}_\lambda(x, y) \right).$$

We call the objects of $\mathcal{B}_\tau(x, y)$ **tight** morphisms, and those of $\mathcal{B}_\lambda(x, y)$ **loose**.

Outline

- ① The problem
- ② Progress and hints
- ③ The solution**
- ④ Concluding remarks

The story so far

- Categories are monads in spans.
- Functors are colax monad morphisms whose underlying span is a function.
- Natural transformations are 2-cells between these in the free cocompletion under Kleisli objects.

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- So do functors and profunctors.
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- So do functors and profunctors.
- Strict proarrow equipments are special \mathcal{F} -enriched categories.
- Profunctors are modules in spans.
- Modules exist in any bicategory with local coequalizers.
- Mod_1 is idempotent on bicategories with local coequalizers, and its image is those with Kleisli objects.

Theorem (Garner-S.)

There is a monoidal bicategory \mathcal{F}_1 such that

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② *data consisting of*

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Theorem (Garner-S.)

The \mathcal{F}_1 -bicategory $(\text{Cat} \rightarrow \text{Prof})$ is the *free cocompletion* of the \mathcal{F}_1 -bicategory $(\text{Set} \rightarrow \text{Span})$ under a type of \mathcal{F}_1 -enriched colimit called *tight Kleisli objects*.

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The \mathcal{C}_1 -bicategory $\mathcal{M}od_1(\mathcal{B})$ is the free cocompletion of a \mathcal{C}_1 -bicategory \mathcal{B} under \mathcal{C}_1 -enriched Kleisli objects.

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Theorem (Garner-S.)

The \mathcal{C}_1 -bicategory $\text{Mod}_1(\mathcal{B})$ is the free cocompletion of a \mathcal{C}_1 -bicategory \mathcal{B} under \mathcal{C}_1 -enriched Kleisli objects.

Moreover, Kleisli objects are **Cauchy** \mathcal{C}_1 -colimits. This explains why $\text{Mod}_1(\text{Mod}_1(\mathcal{B})) \simeq \text{Mod}_1(\mathcal{B})$.

The first ingredient: enriched bicategories

Theorem (Garner-S.)

The classical theory of enriched categories, weighted limits, and free cocompletions can all be categorified into a theory of bicategories enriched over a monoidal bicategory.

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Proof.

40 pages. Thanks Richard!!



The second ingredient: local coequalizers

Definition

\mathcal{C}_1 = the 2-category of categories with coequalizers and coequalizer-preserving functors.

Theorem

\mathcal{C}_1 has a (bicategorical) monoidal structure such that functors $A \otimes B \rightarrow C$ are equivalent to functors $A \times B \rightarrow C$ preserving coequalizers in each variable.

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Theorem

In \mathcal{C}_1 , Eilenberg-Moore objects (which are constructed as in $\mathcal{C}at$) are also Kleisli objects.

Note: \mathcal{C}_1 itself is a bicategory with local coequalizers!

The third ingredient: tight morphisms

Definition

\mathcal{F}_1 = the 2-category of fully faithful functors $A_\tau \hookrightarrow A_\lambda$, where A_λ has coequalizers.

Theorem

\mathcal{F}_1 has a (bicategorical) monoidal structure where $A \otimes B$ is the fully-faithful factorization of

$$A_\tau \times B_\tau \longrightarrow A_\lambda \otimes B_\lambda.$$

The fourth ingredient: tight colimits

Let (A, t) be a monad in an \mathcal{F}_1 -bicategory \mathcal{B} .

Definition

A **tight Kleisli object** of (A, t) consists of

- 1 A Kleisli object A_t of (A, t) in the bicategory \mathcal{B}_λ .
- 2 The left adjoint $f : A \rightarrow A_t$ is tight.
- 3 A morphism $A_t \rightarrow B$ is tight iff its composite with f is tight.

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All \mathcal{F}_1 -weighted colimits look like this: colimits in \mathcal{B}_λ such that a certain group of coprojections are tight and “detect tightness”.

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Bicategory-enriched categories

- A monoidal category $\mathcal{V} \rightsquigarrow$ a one-object bicategory $\mathbf{B}\mathcal{V}$.
- Monads in $\mathbf{B}\mathcal{V} =$ monoids in \mathcal{V} .

Definition (Bénabou)

A **bicategory-enriched category** (or **polyad**) is the thing such that when you do it in $\mathbf{B}\mathcal{V}$, it gives you \mathcal{V} -enriched categories.

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A bicategory-enriched category (or polyad) is the thing such that when you do it in $\mathbf{B}\mathcal{V}$, it gives you \mathcal{V} -enriched categories.

- If \mathcal{B} is locally cocomplete, have a bicategory $\mathcal{M}od_{\infty}(\mathcal{B})$ of these and modules/profunctors.
- It is the free cocompletion of \mathcal{B} under \mathcal{C}_{∞} -enriched **collages**, i.e. **lax colimits**.
- Also have $\mathcal{F}_{\infty} \dots$

A couple of open problems

$\text{Mod}(\mathcal{B})$ is not the Cauchy completion of \mathcal{B} as a \mathcal{C} -bicategory (e.g. its idempotents don't split).

Question

Can we characterize the class of colimits that it does have?

Question

What is the full \mathcal{C} -enriched Cauchy completion?