Categories, functors, and profunctors are a free cocompletion

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Outline

Monads

Definition (Bénabou, 1967)

A monad in a bicategory consists of

- An object A,
- A morphism $t : A \to A$,
- 2-cells $m : tt \to t$ and $i : 1_A \to t$, and
- Associativity and unitality axioms.

- Equivalently, a lax functor out of 1.
- Includes ordinary monads, enriched ones, internal ones, indexed ones, monoidal ones, ...
- The "Eilenberg-Moore object" A^t is a lax limit, and the "Kleisli object" A_t is a lax colimit (Street).

Example

A monad in the bicategory of spans (of sets) consists of

- A set A_0,
- A span $A_0 \leftarrow A_1 \rightarrow A_0$,
- Functions $m : A_1 \times_{A_0} A_1 \rightarrow A_1$ and $i : A_0 \rightarrow A_1$, and
- Associativity and unitality axioms.

In other words, it's nothing but a (small) category!

- In similar bicategories, we obtain internal categories, enriched categories, ...

Morphisms of monads

Definition (Street, 1972)

A lax monad morphism $(A, t) \to (B, s)$ consists of

- A morphism $f : A \to B$,
- A 2-cell $\overline{f} : sf \to ft$, and
- Some axioms.

Morphisms of monads

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- Some axioms.

- $=$ a lax natural transformation between lax functors.
- Induces a morphism $A^t \to B^s$ between E-M objects.
- Colax monad morphisms $=$ colax natural transformations, and induce morphisms $A_t \to B_s$ between Kleisli objects.

Morphisms of categories?

Example

A lax monad morphism in the bicategory of spans consists of

- A span $A_0 \leftarrow F_0 \rightarrow B_0$,
- A morphism

$$
\begin{tikzcd}
A_0 & & A_1 \arrow[ll] \arrow[rr] & & A_0 \\
& & & \bullet \arrow[ul] \arrow[dr] \arrow[ur, phantom, "\urcorner", very near start] & \\
F_0 \arrow[uu] \arrow[dd] & & & & F_0 \arrow[uu] \arrow[dd] \\
& \bullet \arrow[ul] \arrow[dr] \arrow[uurr, red] \arrow[dl, phantom, "\llcorner", very near start] & & & \\
B_0 & & B_1 \arrow[ll] \arrow[rr] & & B_0
\end{tikzcd}
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F_0 \arrow[uu] \arrow[dd] & & & & F_0 \arrow[uu] \arrow[dd] \\
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B_0 & B_1 \arrow[l] \arrow[r] & B_0
\end{tikzcd}
$$

Morphisms of categories?

Example

A functor consists of

- A function $f_0 : A_0 \to B_0$
- A morphism

$$\begin{tikzcd}
A_0 \arrow[dd, equal] & A_1 \arrow[l] \arrow[r] \arrow[d, equal] & A_0 \arrow[dd, equal] \\
& \bullet \arrow[dl, red] \arrow[dr] \arrow[dd, red] \arrow[ur, phantom, "\urcorner", very near start] & \\
F_0 \arrow[d] & & F_0 \arrow[d] \\
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Morphisms of categories?

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functor = colax monad morphism such that the span $A_0 \leftarrow F_0 \to B_0$ is a mere function $A_0 \to B_0$.

2-cells of monads

Definition (Street,1972)

For lax f, g, a monad 2-cell
$$\begin{tikzcd}(A,t) \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow"] & (B,s)\end{tikzcd}$$
consists of

- A 2-cell $f \to g$
- Satisfying some axioms.

- $=$ a modification between lax natural transformations.
- Induces a 2-cell $\begin{tikzcd}A^t \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & B^s\end{tikzcd}$ between E-M objects.
- If f, g are colax, monad 2-cells $=$ modifications between colax transformations, and induce 2-cells between Kleisli objects.

2-cells of categories?

Example

For categories A, B and colax monad morphisms $f, g : A \to B$,

a monad 2-cell $\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow"] & B \end{tikzcd}$ consists of

- A morphism

$$\begin{tikzcd} & A_0 & \\ G_0 \arrow[ur] \arrow[dr] & & F_0 \arrow[ul] \arrow[dl] \arrow[ll, red] \\ & B_0 & \end{tikzcd}$$

- Satisfying some axioms.

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- Satisfying some axioms.

i.e. just an equality $f = g$!

Outline

Profunctors

Definition (Bénabou, 1967?)

For categories A, B, a profunctor $A \nrightarrow B$ consists of

- A functor $H : B^{\mathrm{op}} \times A \to \mathrm{Set}$

Profunctors

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For categories A, B, a profunctor $A \nrightarrow B$ consists of

- Sets $H(b, a)$ for $a \in A_0,\ b \in B_0$
- Actions

$$H(b,a) \times A(a,a') \to H(b,a') \qquad \text{and}$$
$$B(b',b) \times H(b,a) \to H(b',a)$$

- Associativity and unitality axioms.

Profunctors

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For categories A, B, a profunctor $A \nrightarrow B$ consists of

- Spans $A_0 \leftarrow H \rightarrow B_0$
- Actions

$$H \times_{A_0} A_1 \rightarrow H \qquad \text{and}$$
$$B_1 \times_{B_0} H \rightarrow H$$

- Associativity and unitality axioms.

Modules

Definition (Bénabou, 1973)

For monads (A, t), (B, s) in a bicategory, a module $A \nrightarrow B$ is

- A morphism $h : A \to B$
- Actions

$$ht \to h \qquad \text{and}$$
$$sh \to h$$

- Associativity and unitality axioms.

- Modules in spans = profunctors
- Also get enriched profunctors, internal profunctors, . . .

Composing profunctors

Definition

The composite of profunctors $H : A \nrightarrow B$ and $K : B \nrightarrow C$ is

$$(KH)(c, a) = \int^{b \in B} K(c, b) \times H(b, a)$$

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$$(KH)(c, a) = \int^{b \in B} K(c, b) \times H(b, a)$$

$$= \operatorname{coeq}\Big(\textstyle\sum_{b,b' \in B_0} K(c,b') \times B(b',b) \times H(b,a) \rightrightarrows \sum_{b \in B_0} K(c,b) \times H(b,a)\Big)$$

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$$= \mathrm{coeq}\left(K \times_{B_0} B_1 \times_{B_0} H \;\rightrightarrows\; K \times_{B_0} H\right)$$

Composing modules

Definition

A bicategory $\mathcal{B}$ has local coequalizers if all hom-categories have coequalizers and composition preserves them in each variable.

Definition

In this case, the composite of modules $h : (A, t) \to (B, s)$ and $k : (B, s) \to (C, r)$ is

$$\mathrm{coeq}(ksh \rightrightarrows kh)$$

We have a bicategory $\mathcal{M}od_1(\mathcal{B})$ whose

- objects are monads in $\mathcal{B}$, and
- morphisms are modules

What are these modules?

Theorem (Street 1981, Carboni-Kasangian-Walters 1987)

1. $\mathcal{M}od_1(\mathcal{M}od_1(\mathcal{B})) \simeq \mathcal{M}od_1(\mathcal{B})$.
2. $\mathcal{C}$ *is of the form* $\mathcal{M}od_1(\mathcal{B})$ *iff it has*
    - *Local coequalizers, and*
    - *Kleisli objects for monads.*

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Example

In $\mathcal{P}rof = \mathcal{M}od_1(\mathcal{S}pan)$, a monad on a category A is

- A category A' with the same objects as A, and
- An identity-on-objects functor $A \to A'$.

Its Kleisli object is just A'.

Functors vs Profunctors

Definition (Wood 1982)

A proarrow equipment consists of

- Two bicategories $\mathcal{K}$ and $\mathcal{M}$ with the same objects,
- An identity-on-objects and locally fully faithful pseudofunctor $(-)_\bullet : \mathcal{K} \to \mathcal{M}$,
- Such that every morphism $f_\bullet$ has a right adjoint in $\mathcal{M}$.

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Examples

- $\mathcal{K} = \mathcal{C}at$, $\mathcal{M} = \mathcal{P}rof$, and for a functor $f : A \to B$,

$$f_\bullet(b, a) = B(b, f(a))$$

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Examples

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$$f_\bullet(b, a) = B(b, f(a))$$

- $\mathcal{K} = \mathrm{Set}$, $\mathcal{M} = \mathcal{S}pan$, and for a function $f : A \to B$,

$$f_\bullet \quad = \quad A = A \xrightarrow{f} B$$

Those pesky 2-cells

Theorem (Lack-Street 2002)

In the free cocompletion of $\mathcal{K}$ under Kleisli objects,

- *The objects are monads in $\mathcal{K}$,*
- *The morphisms are colax monad morphisms in $\mathcal{K}$,*
- *The 2-cells* $(A,t) \underset{g}{\overset{f}{\Downarrow}} (B,s)$ *are*
  - *2-cells $f \to sg$ in $\mathcal{K}$,*
  - *Satisfying axioms.*

2-cells of categories

Example

For categories A, B and colax monad morphisms $f, g : A \to B$:

$$
\begin{tikzcd}
 & A_0 & \\
G_0 \arrow[ur] \arrow[d] & \bullet \arrow[l] \arrow[d] \arrow[dl, phantom, "\llcorner", very near start] & F_0 \arrow[ul] \arrow[l, red] \arrow[d] \\
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$$

2-cells of categories

Example

For categories A, B and functors $f, g : A \to B$:

$$\begin{tikzcd}
& A_0 \arrow[dl, equal] \arrow[dr, equal] & \\
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Example

For categories A, B and functors $f, g : A \to B$:

$$\begin{tikzcd} & A_0 \arrow[dl, "g_0"'] \arrow[d, red] \arrow[dr, "f_0"] & \\ B_0 & B_1 \arrow[l] \arrow[r] & B_0 \end{tikzcd}$$

A natural transformation!

$\mathscr{F}$-categories

Observation (Lack-S. 2012)

There is a cartesian closed 2-category $\mathscr{F}$ such that

1. $\mathscr{F}$-enriched categories

are the same as

2. data consisting of
   - 2-categories $\mathcal{K}$ and $\mathcal{M}$ with the same objects, and
   - an identity-on-objects and locally fully faithful 2-functor $\mathcal{K} \to \mathcal{M}$.

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The objects of $\mathscr{F}$ are fully faithful functors $A_\tau \hookrightarrow A_\lambda$.
An $\mathscr{F}$-category $\mathcal{B}$ has homs

$$\mathcal{B}(x, y) = \Big(\mathcal{B}_\tau(x, y) \hookrightarrow \mathcal{B}_\lambda(x, y)\Big).$$

We call the objects of $\mathcal{B}_\tau(x, y)$ tight morphisms, and those of $\mathcal{B}_\lambda(x, y)$ loose.

Outline

The story so far

- Categories are monads in spans.
- Functors are colax monad morphisms whose underlying span is a function.
- Natural transformations are 2-cells between these in the free cocompletion under Kleisli objects.

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- So do functors and profunctors.
- Strict proarrow equipments are special $\mathscr{F}$-enriched categories.

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- So do functors and profunctors.
- Strict proarrow equipments are special $\mathscr{F}$-enriched categories.
- Profunctors are modules in spans.
- Modules exist in any bicategory with local coequalizers.
- $\mathcal{M}od_1$ is idempotent on bicategories with local coequalizers, and its image is those with Kleisli objects.

The solution, part 1

Theorem (Garner-S.)

There is a monoidal bicategory $\mathscr{F}_1$ such that

1. *$\mathscr{F}_1$-enriched bicategories*

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2. *data consisting of*
    - *bicategories $\mathcal{K}$ and $\mathcal{M}$ with the same objects, and*
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Theorem (Garner-S.)

The $\mathscr{F}_1$-bicategory $(\mathcal{C}at \to \mathcal{P}rof)$ is the free cocompletion of the $\mathscr{F}_1$-bicategory $(\mathrm{Set} \to \mathcal{S}pan)$ under a type of $\mathscr{F}_1$-enriched colimit called tight Kleisli objects.

The solution, part 0

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Theorem (Garner-S.)

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Theorem (Garner-S.)

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Theorem (Garner-S.)

The $\mathscr{C}_1$-bicategory $\mathcal{M}od_1(\mathcal{B})$ is the free cocompletion of a $\mathscr{C}_1$-bicategory $\mathcal{B}$ under $\mathscr{C}_1$-enriched Kleisli objects.

Moreover, Kleisli objects are Cauchy $\mathscr{C}_1$-colimits. This explains why $\mathcal{M}od_1(\mathcal{M}od_1(\mathcal{B})) \simeq \mathcal{M}od_1(\mathcal{B})$.

The first ingredient: enriched bicategories

Theorem (Garner-S.)

The classical theory of enriched categories, weighted limits, and free cocompletions can all be categorified into a theory of bicategories enriched over a monoidal bicategory.

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Proof.

40 pages. Thanks Richard!! □

The second ingredient: local coequalizers

Definition

$\mathscr{C}_1$ = the 2-category of categories with coequalizers and coequalizer-preserving functors.

Theorem

$\mathscr{C}_1$ has a (bicategorical) monoidal structure such that functors $A \otimes B \to C$ are equivalent to functors $A \times B \to C$ preserving coequalizers in each variable.

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Theorem

In $\mathscr{C}_1$, Eilenberg-Moore objects (which are constructed as in $\mathcal{C}at$) are also Kleisli objects.

Note: $\mathscr{C}_1$ itself is a bicategory with local coequalizers!

The third ingredient: tight morphisms

Definition

$\mathscr{F}_1$ = the 2-category of fully faithful functors $A_\tau \hookrightarrow A_\lambda$, where A_λ has coequalizers.

Theorem

$\mathscr{F}_1$ has a (bicategorical) monoidal structure where $A \otimes B$ is the fully-faithful factorization of

$$A_\tau \times B_\tau \longrightarrow A_\lambda \otimes B_\lambda.$$

The fourth ingredient: tight colimits

Let (A, t) be a monad in an $\mathscr{F}_1$-bicategory $\mathcal{B}$.

Definition

A tight Kleisli object of (A, t) consists of

1. A Kleisli object A_t of (A, t) in the bicategory $\mathcal{B}_\lambda$.
2. The left adjoint $f : A \to A_t$ is tight.
3. A morphism $A_t \to B$ is tight iff its composite with f is tight.

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3. A morphism $A_t \to B$ is tight iff its composite with f is tight.

All $\mathscr{F}_1$-weighted colimits look like this: colimits in $\mathcal{B}_\lambda$ such that a certain group of coprojections are tight and "detect tightness".

Outline

Bicategory-enriched categories

- A monoidal category $\mathcal{V} \rightsquigarrow$ a one-object bicategory $\mathbf{B}\mathcal{V}$.
- Monads in $\mathbf{B}\mathcal{V}$ = monoids in $\mathcal{V}$.

Definition (Bénabou)

A bicategory-enriched category (or polyad) is the thing such that when you do it in $\mathbf{B}\mathcal{V}$, it gives you $\mathcal{V}$-enriched categories.

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- If $\mathcal{B}$ is locally cocomplete, have a bicategory $\mathcal{M}od_\infty(\mathcal{B})$ of these and modules/profunctors.
- It is the free cocompletion of $\mathcal{B}$ under $\mathscr{C}_\infty$-enriched collages, i.e. lax colimits.
- Also have $\mathscr{F}_\infty \ldots$

A couple of open problems

$\mathcal{M}od(\mathcal{B})$ is not the Cauchy completion of $\mathcal{B}$ as a $\mathscr{C}$-bicategory (e.g. its idempotents don't split).

Question

Can we characterize the class of colimits that it does have?

Question

What is the full $\mathscr{C}$-enriched Cauchy completion?