$\boxed{\text{Conservativity of Duals}}$

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Contents

- Conservativity
- Duals
- Conservativity via gluing
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Inclusions of algebraic doctrines

Let D_1, D_2 be algebraic doctrines = algebraic structures on categories
= algebraic 2-theories.

such that $D_1 \subseteq D_2$, i.e. D_2-structure includes D_1-structure.

<u>Examples</u>: categories w/ finite products
categories w/ finite limits
categories w/ finite limits & colimits
monoidal categories
symmetric monoidal categories
⋮

closed symmetric monoidal categories
cartesian closed categories
compact closed categories
traced monoidal categories
$*$-autonomous categories

2-Conservativity of Doctrine extensions

Let $D_1 \subseteq D_2$ be algebraic doctrines.

We say D_2 is 2-conservative over D_1 if the unit components $\mathcal{C} \longrightarrow UF\mathcal{C}$ of the adjunction $F : D_1\text{-Alg} \rightleftarrows D_2\text{-Alg} : U$ are fully faithful.

Remarks: 1) Occasionally called "fully complete."

2) Implies every D_1-algebra embeds fully-faithfully in a D_2-algebra (namely, $F\mathcal{C}$), but the converse is less clear — full-faithfulness doesn't "cancel".

2-Conservativity of Doctrine extensions

D_2 is 2-conservative over D_1 if each $\mathcal{C} \to UF\mathcal{C}$ is fully faithful.

One motivation: This implies we can use formal calculi for D_2-categories (type theories, string diagrams) to reason about D_1-categories.

- $\mathcal{C} \to U\mathcal{E}$ faithful for some $\mathcal{E}$ $\Rightarrow$ If the D_2-calculus proves two morphisms are equal, they are equal in any D_1-category.
- $\mathcal{C} \to U\mathcal{E}$ fully faithful for some $\mathcal{E}$ $\Rightarrow$ If the D_2-calculus defines a morphism, it defines an actual morphism in a D_1-category.
- $\mathcal{C} \to UF\mathcal{C}$ fully faithful $\Rightarrow$ That morphism is uniquely determined (doesn't depend on the choice of $\mathcal{E}$).

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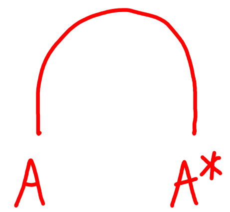
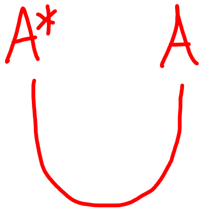
What are duals for?

One answer: parallelism.

A^* is the type of "channels" or "continuations" accepting an A.



creates a channel accepting A and starts a thread listening to use whatever is passed to that channel.



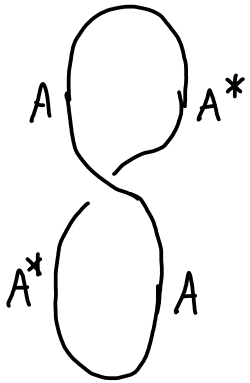
passes a value of type A to a channel.

Non-conservativity of duals

The simplest notion of dual involves $I \longrightarrow A \otimes A^*$ and $A^* \otimes A \longrightarrow I$.

A symmetric monoidal category with all such duals is compact closed.

These are NOT 2-conservative over (closed) symmetric monoidal categories, because they have traces (a.k.a. deadlocks):



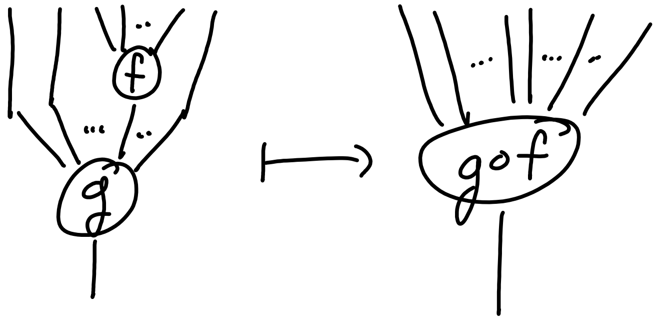
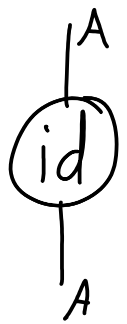
$$\begin{tikzcd} I \arrow[d] \\ A \otimes A^* \arrow[d, "\cong"] \\ A^* \otimes A \arrow[d] \\ I \end{tikzcd}$$

(But compact closed categories are 2-conservative over traced monoidal categories.
–Joyal-Street-Verity)

Is there a weaker sort of "dual" that is 2-conservative over CSMCs?

Review of multicategories

A (symmetric) multicategory consists of

- A set of objects
- Hom-sets $\mathcal{C}(A_1,\dots,A_n;B)$ for all $n \geq 0$ and objects $A_1,\dots,A_n,B$
- Permutation actions $\mathcal{C}(A_1,\dots,A_n;B) \xrightarrow{\cong} \mathcal{C}(A_{\sigma 1},\dots,A_{\sigma n};B)$
- Composition  and 
- Associativity + unit + equivariance axioms.

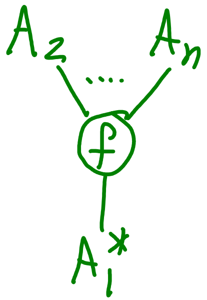
$*$-polycategories

[a.k.a. an augmented cyclic symmetric multicategory]

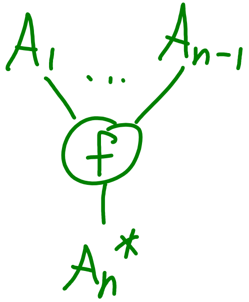
A (symmetric, entries-only) $*$-polycategory consists of

- A set of objects equipped with an involution $A \mapsto A^*$, $A^{**} = A$.
- Hom-sets $\mathcal{E}(A_1, \ldots, A_n)$ for all $n \geq 0$

Draw $f \in \mathcal{E}(A_1, \ldots, A_n)$ as



or ... or



- Permutation actions $\mathcal{E}(A_1, \ldots, A_n) \xrightarrow{\cong} \mathcal{E}(A_{\sigma 1}, \ldots, A_{\sigma n})$
- Composition $\mathcal{E}(A_1, \ldots, A_m, C) \times \mathcal{E}(B_1, \ldots, B_n, C^*) \to \mathcal{E}(A_1, \ldots, A_m, B_1, \ldots, B_n)$ and $id_A \in \mathcal{E}(A, A^*)$
- Associativity + unit + equivariance axioms.

Representability of multicategories and polycategories

A tensor product in a multicategory is $\chi \in \mathcal{C}(A,B\,;\,A\otimes B)$ inducing isomorphisms $\mathcal{C}(C_1,\dots,C_n,A\otimes B\,;\,D) \xrightarrow{\cong} \mathcal{C}(C_1,\dots,C_n,A,B\,;\,D)$.

If all tensor products (& a unit) exist, $\mathcal{C}$ is a symmetric monoidal cat.

A tensor product in a $*$-polycategory is $\chi \in \mathcal{E}(A,B,(A\otimes B)^*)$ inducing isomorphisms $\mathcal{E}(C_1,\dots,C_n,A\otimes B) \xrightarrow{\cong} \mathcal{E}(C_1,\dots,C_n,A,B)$.

If all tensor products (& a unit) exist, $\mathcal{E}$ is a $*$-autonomous category.
(with strictly involutive duals)

Internal-homs in $*$-autonomous categories

A $*$-autonomous category is closed, with $[B,C] = (B \otimes C^*)^*$:

$$\mathcal{E}(A,[B,C]^*) = \mathcal{E}(A, B \otimes C^*) \cong \mathcal{E}(A,B,C^*) \cong \mathcal{E}(A \otimes B, C^*)$$

The duality morphisms are $A^* \otimes A \to I^*$ and $I \to (A^* \otimes A)^*$.
The extra dualization means no trace is induced!

Are $*$-autonomous categories 2-conservative over closed sym. mon. cats?

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Doctrinal Yoneda embeddings

Often, Yoneda $よ : \mathcal{C} \to P\mathcal{C} = [\mathcal{C}^{op}, Set]$ embeds any D_1-category in some D_2-category.

- $P\mathcal{C}$ is complete & cocomplete, cartesian closed, even a topos
- $よ$ preserves all limits and exponentials
- If $\mathcal{C}$ is monoidal, so are $P\mathcal{C}$ and $よ$ by Day convolution.

If D_1 contains some colimits $\mathcal{d}$, we can restrict to $P_{\mathcal{d}}\mathcal{C}$ = presheaves preserving $\mathcal{d}$, to make $よ_{\mathcal{d}} : \mathcal{C} \to P_{\mathcal{d}}\mathcal{C}$ also preserve $\mathcal{d}$.

How can we enhance this to 2-conservativity?

Gluing with the Yoneda embedding (Lafont)

Suppose $\mathcal{P}\mathcal{C}$ has D_2-structure, and let $\mathcal{G}$ be the comma category

$$\begin{tikzcd} & \mathcal{G} \arrow[r] \arrow[d] \arrow[dr, phantom, "\Leftarrow"] & \mathcal{P}\mathcal{C} \arrow[d, equal] \\ \mathcal{C} \arrow[r, "\eta"'] & \mathcal{F}\mathcal{C} \arrow[r, "\mathcal{L}_\eta"'] & \mathcal{P}\mathcal{C} \end{tikzcd}$$

Here $\mathcal{L}_\eta(x)(a) = \mathcal{F}\mathcal{C}(\eta a, x)$ is the restricted Yoneda embedding (= "nerve").

Usually, $\mathcal{G}$ also has D_2-structure, preserved by the projections $\mathcal{G} \to \mathcal{F}\mathcal{C}$ and $\mathcal{G} \to \mathcal{P}\mathcal{C}$. (Abstract nonsense sometimes applies.)

The map to the gluing category

We have a functor $\theta : \mathcal{C} \to \mathcal{G}$

$$\begin{tikzcd}
\mathcal{C} \arrow[dr, "\theta"] \arrow[rr, bend left, "\alpha"] \arrow[ddr, bend right, "\eta"'] & & P\mathcal{C} \arrow[dd, equal] \\
& \mathcal{G} \arrow[ur] \arrow[d] \arrow[dr, phantom, "\Leftarrow"] & \\
& F\mathcal{C} \arrow[r, "\alpha_\eta"'] & P\mathcal{C}
\end{tikzcd}$$

induced by the 2-cell

$$\begin{tikzcd}
\mathcal{C} \arrow[rr, bend left, "\alpha"] \arrow[dr, bend right, "\eta"'] \arrow[rr, phantom, bend right=10, "\Swarrow"] & & P\mathcal{C} \arrow[d, equal] \\
& F\mathcal{C} \arrow[r, "\alpha_\eta"'] & P\mathcal{C}
\end{tikzcd}$$

having components

$$\alpha b(a) = \mathcal{C}(a,b) \xrightarrow{\eta} F\mathcal{C}(\eta a, \eta b) = \alpha_\eta(\eta b)(a) = (\alpha_\eta \circ \eta)(b)(a).$$

Usually, θ is a D_1-functor (e.g. since η and α are).

The section of the gluing category

$$\begin{tikzcd}
 & \mathcal{G} \arrow[r] \arrow[d, shift left] \arrow[dr, phantom, "\Leftarrow"] & P\mathcal{C} \arrow[d, equal] \\
\mathcal{C} \arrow[ur, "\theta"] \arrow[r, "\eta"'] & F\mathcal{C} \arrow[r, "\alpha_\eta"'] \arrow[u, bend left, "\omega"] & P\mathcal{C}
\end{tikzcd}$$

Hence, the universal property of $F\mathcal{C}$ yields a D_2-functor $\omega : F\mathcal{C} \to \mathcal{G}$, whose action on homs specializes to

$$h \in F\mathcal{C}(\eta a, \eta b) \to \mathcal{G}(\theta a, \theta b) = \left\{ \begin{tikzcd}[ampersand replacement=\&]
\eta a \arrow[d, "h"] \& \alpha a \arrow[d] \& \alpha a \arrow[r] \arrow[d] \& \alpha_\eta(\eta a) \arrow[d, "h \circ -"] \\
\eta b, \& \alpha b, \& \alpha b \arrow[r] \& \alpha_\eta(\eta b)
\end{tikzcd} \right\}.$$

this must be h, since ω is a section

by the Yoneda lemma, this is determined by $\bar{\omega}(h) \in \mathcal{C}(a,b)$.

then this says $\eta(\bar{\omega}(h)) = h$.

From gluing to 2-conservativity

$$\begin{tikzcd}
 & \mathcal{G} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & P\mathcal{C} \arrow[d, equal] \\
\mathcal{C} \arrow[ur, "\theta"] \arrow[r, "\eta"'] & F\mathcal{C} \arrow[u, bend left, "\omega"] \arrow[r, "\partial_\eta"'] & P\mathcal{C}
\end{tikzcd}$$

So for $h \in F\mathcal{C}(\eta a, \eta b)$ we have $\bar{\omega}(h) \in \mathcal{C}(a,b)$ with $\eta(\bar{\omega}(h)) = h$.

And $\omega \circ \eta = \theta$ implies $\bar{\omega}(\eta(g)) = g$ for $g \in \mathcal{C}(a,b)$.

Thus $\mathcal{C}(a,b) \underset{\bar{\omega}}{\overset{\eta}{\rightleftarrows}} F\mathcal{C}(\eta a, \eta b)$ are inverse isomorphisms,

so $\eta : \mathcal{C} \to F\mathcal{C}$ is fully faithful, i.e. D_2 is 2-conservative over D_1.

Lafont gluing: summary

This technique requires

(1) D_1 & D_2 lift to comma categories, coherently.

(2) For a D_1-category $\mathcal{C}$, $P\mathcal{C}$ has D_2-structure and $\text{よ} : \mathcal{C} \hookrightarrow P\mathcal{C}$ is a D_1-functor.

If D_1 includes colimits $\mathcal{d}$, then in (2) we can use $P_{\mathcal{d}}\mathcal{C}$ instead.

But (2) fails when D_2 contains duals — $P\mathcal{C}$ is not $*$-autonomous.

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From multicategories to polycategories, I

The forgetful functor $V : *Poly \to Multi$, $V\mathcal{E}(A_1,\dots,A_n ; B) = \mathcal{E}(A_1,\dots,A_n, B^*)$, has a left adjoint $L : Multi \to *Poly$.

- $ob(L\mathcal{C}) = ob(\mathcal{C}) \sqcup ob(\mathcal{C})$ — objects $A \in \mathcal{C}$ and their formal duals $\overline{A}$
- The involution is $A^* = \overline{A}$, $\overline{A}^* = A$.
- The only nonempty homsets of $L\mathcal{C}$ have exactly one formal dual:

  $$L\mathcal{C}(A_1,\dots,A_m, \overline{B}, C_1,\dots,C_n) = \mathcal{C}(A_1,\dots,A_m, C_1,\dots,C_n ; B)$$

- Composition, etc. is induced from $\mathcal{C}$.

Tensor products in the left adjoint

Even if $\mathcal{C}$ is monoidal, $L\mathcal{C}$ doesn't have all $\otimes$s, but it has some:

- $A \otimes B$ remains a tensor product in $L\mathcal{C}$.
- If $\mathcal{C}$ is closed, then $\overline{[B,C]}$ is a tensor product $B \otimes \overline{C}$:

$$
\begin{aligned}
L\mathcal{C}(A_1,\dots,A_n,\overline{[B,C]}) &= \mathcal{C}(A_1,\dots,A_n;[B,C]) \\
&\cong \mathcal{C}(A_1,\dots,A_n,B;C) \\
&= L\mathcal{C}(A_1,\dots,A_n,B,\overline{C})
\end{aligned}
$$

Instead of a CSMC $\mathcal{C}$, consider the $*$-polycat. $L\mathcal{C}$ w/ these $\otimes$s.

From multicategories to polycategories, II

The forgetful functor $V : *\mathrm{Poly} \to \mathrm{Multi}$, $V\mathcal{E}(A_1,\dots,A_n;B) = \mathcal{E}(A_1,\dots,A_n,B^*)$, also has a right adjoint $R : \mathrm{Multi} \to *\mathrm{Poly}$.

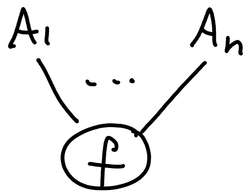
- $ob(R\mathcal{E}) = ob(\mathcal{E}) \times ob(\mathcal{C})$ — pairs (A^+,A^-) of objects $A^+,A^- \in \mathcal{C}$.
- The involution is $(A^+,A^-)^* = (A^-,A^+)$.
- $R\mathcal{C}(A_1,\dots,A_n) = \mathcal{C}(A_2^+,\dots,A_n^+;A_1^-) \times \cdots \times \mathcal{C}(A_1^+,\dots,A_{n-1}^+;A_n^-)$

  In particular, $R\mathcal{C}(A,B^*) = \mathcal{C}(A^+,B^+) \times \mathcal{C}(B^- A^-)$
- Composition, etc. is induced from $\mathcal{C}$.

The Chu construction

An augmented multicategory has additionally

- hom-sets $\mathcal{C}(A_1, \dots, A_n ; \cdot)$ with empty codomain:



- compositions, etc.

Example: For any multicategory $\mathcal{C}$ and $\bot \in \mathcal{C}$, we have an augmented multicategory $(\mathcal{C}, \bot)$ extending $\mathcal{C}$ by $(\mathcal{C}, \bot)(A_1, \dots, A_n ; \cdot) = \mathcal{C}(A_1, \dots, A_n ; \bot)$

The forgetful functor $V_+ : *\mathrm{Poly} \longrightarrow \mathrm{AugMulti}$, $V_+\mathcal{E}(A_1, \dots, A_n ; \cdot) = \mathcal{E}(A_1, \dots, A_n)$, also has a right adjoint, called the Chu construction, e.g. $\mathrm{Chu}(\mathcal{C}, \bot)$.

Modules for polycategories (Hyland)

For $\mathcal{E}$ a $*$-polycategory, an $\mathcal{E}$-module consists of

- "hom" sets $M(A_1, \ldots, A_n)$ with permutation actions
- "composition" actions $M(A_1, \ldots, A_m, C) \times \mathcal{E}(B_1, \ldots, B_n, C^*) \longrightarrow M(A_1, \ldots, A_m, B_1, \ldots, B_n)$ + axioms.

Examples: (1) $\mathcal{L}_B(A_1, \ldots, A_n) = \mathcal{E}(A_1, \ldots, A_n, B^*)$ — representable modules

(2) $\mathcal{E}(A_1, \ldots, A_n) = \mathcal{E}(A_1, \ldots, A_n)$ — the tautological module

Yoneda embeddings for polycategories (Hyland)

Modules form a multicategory $\mathcal{E}$-Mod with

e.g. $f \in \mathcal{E}\text{-Mod}(\mathcal{M}, \mathcal{N}; Q)$ consisting of maps

$$\mathcal{M}(A_1, \dots, A_m) \times \mathcal{N}(B_1, \dots, B_n) \longrightarrow Q(A_1, \dots, A_m, B_1, \dots, B_n).$$

The representables define $\text{ よ} : V_+(\mathcal{E}) \longrightarrow (\mathcal{E}\text{-Mod}, \mathcal{E})$ in AugMulti,

hence $\text{よ} : \mathcal{E} \longrightarrow \mathrm{Chu}(\mathcal{E}\text{-Mod}, \mathcal{E}) = \mathrm{Env}(\mathcal{E})$, the envelope of $\mathcal{E}$.

Theorem (Hyland) $\text{よ} : \mathcal{E} \to \mathrm{Env}(\mathcal{E})$ is fully faithful, and

$\mathrm{Env}(\mathcal{E})$ is $*$-autonomous.

Modified envelopes

If $\mathcal{E}$ has some $\otimes$s, say $\mathcal{U}$, let $\mathcal{E}\text{-Mod}_{\mathcal{U}}$ be those modules that preserve them: $M(A_1,\dots,A_n,B,C) \cong M(A_1,\dots,A_n,B\otimes C)$.

and $\mathrm{Env}_{\mathcal{U}}(\mathcal{E}) = \mathrm{Chu}(\mathcal{E}\text{-Mod}_{\mathcal{U}}, \mathcal{E})$.

Theorem $\mathcal{L}_{\mathcal{U}} : \mathcal{E} \longrightarrow \mathrm{Env}_{\mathcal{U}}(\mathcal{E})$ is fully faithful, preserves the $\otimes$s in $\mathcal{U}$, and $\mathrm{Env}_{\mathcal{U}}(\mathcal{E})$ is $*$-autonomous.

Corollary Any CSMC $\mathcal{C}$ embeds fully by a closed monoidal functor in the $*$-autonomous category $\mathrm{Env}_{\mathcal{U}}(L\mathcal{C})$.

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Lafont gluing for $\ast$-autonomous categories

Let $\mathcal{E} \xrightarrow{\eta} F_{cl}\mathcal{E}$ be the free $\ast$-autonomous category generated by the $\ast$-polycategory $\mathcal{E}$ (e.g. $L\mathcal{C}$) preserving the $\otimes$s in cl. To prove η fully faithful, we replace $P\mathcal{C}$ by $Env(\mathcal{E})$, with two other changes:

$$\begin{tikzcd}
 & \mathcal{G} \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & Chu(\mathcal{E}\text{-}Mod_{cl}, \mathcal{E}) = Env_{cl}(\mathcal{E}) \arrow[d] \\
\mathcal{E} \arrow[ur, "\theta"] \arrow[r, "\eta"] & F_{cl}\mathcal{E} \arrow[r, "\alpha_\eta"'] \arrow[u, bend left, "\omega"] & Chu(\mathcal{E}\text{-}Mod_{cl}, (F_{cl}\mathcal{E})_\eta)
\end{tikzcd}$$

this change is fairly trivial, so I won't discuss it further.

This one is more interesting!

The double Chu construction

The double Chu construction, $\mathrm{Chu}(\mathcal{C},\perp)$ is a poly double category,

i.e. an internal category in $*\mathrm{Poly}$, whose:

- horizontal $*$-polycategory is $\mathrm{Chu}(\mathcal{C},\perp)$
- vertical category is $\mathcal{C}\times\mathcal{C}$ (both covariant!)
- "2-cells" are tuples of commuting squares,

e.g.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "h"'] \arrow[dr, phantom, "\Downarrow"] & B \arrow[d, "k"] \\ C \arrow[r, "g"'] & D \end{tikzcd} \quad \text{is} \quad \begin{tikzcd} A^+ \arrow[r, "f^+"] \arrow[d, "h^+"'] & B^+ \arrow[d, "k^+"] \\ C^+ \arrow[r, "g^+"'] & D^+ \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} A^- \arrow[d, "h^-"'] & B^- \arrow[l, "f^-"'] \arrow[d, "k^-"] \\ C^- & D^- \arrow[l, "g^-"] \end{tikzcd} .$$

Double gluing (Hyland-Tan-Schalk)

$$\begin{tikzcd}
\mathcal{G} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Chu}(\mathcal{E}\text{-}\mathrm{Mod}_{\mathcal{C}}, \mathcal{E}) = \mathrm{Env}_{\mathcal{C}}(\mathcal{E}) \arrow[d] \\
F_{\mathcal{C}}\mathcal{E} \arrow[r, "\alpha"'] & \mathrm{Chu}(\mathcal{E}\text{-}\mathrm{Mod}_{\mathcal{C}}, (F_{\mathcal{C}}\mathcal{E})_{\eta})
\end{tikzcd}$$

The comma object w.r.t vertical 2-cells is the double gluing.

Theorem The free extension $\mathcal{E} \xrightarrow{\eta} F_{\mathcal{C}}\mathcal{E}$ of any $*$-polycategory to a $*$-autonomous category preserving $\otimes$s in $\mathcal{C}$ is fully faithful.

Corollary $*$-autonomous categories are 2-conservative over closed symmetric monoidal categories. (use $\mathcal{E} = L\mathcal{C}$.)

Vistas

(1) $*$-autonomous cats are also 2-conservative over linearly distributive cats. (Previously proven by Blute-Cockett-Seely-Trimble w/ cut elimination.)

(2) By adding some limits/colimits to $\mathcal{C}$, we can include these on both sides — except that CSMCs can include only nonempty limits/colimits, since $\mathcal{C} \mapsto L\mathcal{C}$ doesn't preserve $0/\mathbb{1}$.

(3) In fact, $*$-autonomous cats w/ initial object are NOT conservative over CSMCs w/ initial object! (Schellinx)

$*$-autonomous envelopes

arXiv: 2004.08487

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