GENERATORS AND COLIMIT CLOSURES

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1. Epimorphisms

Let $\mathcal{A}$ be cocomplete and finitely complete. First we remark on some different types of epimorphisms.

(i) e is a **regular epimorphism** if it is a coequalizer.
(ii) e is a **strong epimorphism** if it is left orthogonal to monics.
(iii) e is an **extremal epimorphism** if $e = mf$ with m monic implies m iso.
(iv) e is an **epimorphism** if $fe = ge$ implies $f = g$.

Exercise 1.1. Any map which is both monic and extremal epic is an isomorphism.

Exercise 1.2. Regular $\Rightarrow$ strong $\Leftrightarrow$ extremal $\Rightarrow$ epic (always assuming $\mathcal{A}$ has finite limits).

Without pullbacks, not every extremal epic may be strong. Without equalizers, our definitions of strong and extremal epic need not imply e is epic, so that has to be included explicitly. Note that epis, strong epis, and extremal epis are closed under composition and cointersections (pushouts), but regular ones aren't.

Fact 1.3. *If extremal epis are stable under pullback and every morphism factors as an extremal epic followed by a monic, then all extremal epics are regular.*

The only proof of this I know is kind of fiddly and not enlightening. But the point is that in most categories that appear 'in nature,' regular, strong, and extremal epics are about the same, and all are the correct notion of *quotient*, while ordinary epimorphisms may not be. For example, $\mathbb{N} \to \mathbb{Q}$ is epic in **Rings**, but not extremal epic. In **Top** the epics are the surjective maps, while the regular = strong = extremal epics are the quotient maps.

If you were at Emily's talk on factorization systems, it is natural to wonder: since strong epics are left orthogonal to monics, i.e. $\mathit{StrEpi} = {}^{\perp}\mathit{Mono}$, when is $(\mathit{StrEpi}, \mathit{Mono})$ an orthogonal factorization system? It would suffice to show that any map factors as an extremal epic followed by a monic. Here's a natural way to do that: given $f\colon X \to Y$, take its kernel pair, which is the pullback

$$\begin{tikzcd} K \arrow[r, "a"] \arrow[d, "b"'] & X \arrow[d, "f"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

and take the coequalizer e_1 of a and b. Since $fa = fb$ by definition, f factors through e_1, so we have written $f = f_1 g_1$ where g_1 is even a *regular* epic. So if f_1 were monic, we'd be done. Often this is the case, but if it isn't, then we can form

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the kernel pair of f_1 and thereby factor $f_1 = f_2 g_2$ where g_2 is regular epic, and so on. We can even repeat this transfinitely; thus for any ordinal α we have $f = f_\alpha e_\alpha$, where e_α is the transfinite composition of all the g_i.

$$\begin{tikzcd}
X \arrow[dddd, "f"'] \arrow[ddr, "g_1"] \arrow[ddrr, "e_2"] \arrow[ddrrr, bend left, "e_3"] & & & \\
& & & \\
& Z_1 \arrow[r, "g_2"'] \arrow[ddl, "f_1"] & Z_2 \arrow[r, "g_3"'] \arrow[ddll, "f_2"] & Z_3 \dots \arrow[ddlll, bend left, "f_3"] \\
& & & \\
Y & & &
\end{tikzcd}$$

In particular, note that each Z_α is a quotient of X, in the sense that we have an extremal epic $e_\alpha \colon X \to Z_\alpha$. Now, in most reasonable categories, each object only has a set worth of different quotients. Precisely, we say $\mathcal{A}$ is **extremally well-copowered**[1] if there are only a set of extremal epis, up to isomorphism, with a given domain. In this case, there must be some α such that $g_{\alpha+1}$ is an isomorphism, which means that the two maps in the kernel pair of f_α were equal, which means that f_α was monic; so $f = f_\alpha e_\alpha$ is our desired factorization.

Proposition 1.4. *If $\mathcal{A}$ is cocomplete, finitely complete, and extremally well-copowered, then $(StrEpi, Mono)$ is an orthogonal factorization system.*

2. Adjunctions

Exercise 2.1. In an adjunction $F \dashv G$,

(i) G is full and faithful iff all the counits are isomorphisms.
(ii) G is faithful iff all the counits are epic.
(iii) G is (faithful and) conservative iff the counits are extremal epic.

Recall that *conservative* means isomorphism-reflecting (if Gf is an isomorphism, then so is f).

3. Generators

Let $G \subset \mathcal{A}$ be a set of objects with $\mathcal{G}$ the full subcategory it spans. Here's the notion you've probably seen: G is a **generator** (or **generating set**) if for any $f, g \colon X \to Y$, if $fe = ge$ for all $e \colon P \to X$ with $P \in G$ then $f = g$.

Exercise 3.1. The following are equivalent.

(i) G is a generator.
(ii) The restricted Yoneda embedding $\mathcal{A} \to \mathbf{Set}^G$ is faithful.
(iii) The restricted Yoneda embedding $\mathcal{A} \to [\mathcal{G}^{op}, \mathbf{Set}]$ is faithful.
(iv) For all X, the map $\varepsilon_X \colon \coprod_{Q \to X} Q \to X$ is epic.

Exercise 3.2. The following are equivalent. When they hold, we say G is a **strong generator** (although **extremal generator** would be more appropriate).

(i) if m is a monic such that every $Q \to Y$ with $Q \in G$ factors through m, then m is an isomorphism.

[1] This is a nonstandard term, but I prefer it to "well-copowered with respect to extremal epimorphisms" or "every object of $\mathcal{A}$ has only a small set of extremal epimorphic quotients." Note, though, that "strongly well-copowered" would be misleading, since this is a *weaker* condition than plain well-copoweredness (which refers to all epimorphisms).

(ii) $\mathcal{A} \to \mathbf{Set}^G$ is conservative.
(iii) $\mathcal{A} \to [\mathcal{G}^{op}, \mathbf{Set}]$ is conservative.
(iv) $\varepsilon_X \colon \coprod_{Q \to X} Q \to X$ is extremal epic for all X.

For example, the one-point set 1 is a strong generator in **Set**, and $\mathbb{Z}$ is a strong generator in **Ab**, but 1 is not a strong generator in **Top**, although it is a generator. In fact, **Top** has no strong generator. However, 1 *is* a strong generator in the category **CptHaus** of compact Hausdorff spaces.

Definition 3.3. We say G is a **dense generator** (or, $\mathcal{G}$ is **dense**) if $\mathcal{A} \to [\mathcal{G}^{op}, \mathbf{Set}]$ is full and faithful.

Evidently any dense generator is a strong generator, but the converse is false. For example, 1 is a dense generator in **Set**. But 1 is not a dense generator in **CptHaus**, and in fact **CptHaus** has no dense generator.

Remark 3.4. This sort of stuff tends to lead you fairly quickly into set-theoretic waters. For example $\mathbf{Set}^{op}$ has a dense generator if and only if there does not exist a proper class of measurable cardinals.

Since the left adjoint of $\mathcal{A} \to [\mathcal{G}^{op}, \mathbf{Set}]$ is, essentially, $J \mapsto \operatorname{colim}^J \mathrm{Id}_{\mathcal{G}}$, G being dense is equivalent to saying that each object of $\mathcal{A}$ is a colimit of objects of G in a canonical way. This suggests two other notions of generator.

Definition 3.5. G is a **colimit-dense generator** if every object of $\mathcal{A}$ is a colimit of objects of G in *some* way.

Definition 3.6. G is a **colimit-generator** (this is a nonstandard term) if $\mathcal{A}$ **is the colimit-closure of** G, i.e. the only subcategory of $\mathcal{A}$ containing G and closed under colimits is $\mathcal{A}$ itself.

The idea is that with a colimit-generator, we can get to all of $\mathcal{A}$ by starting with G and taking colimits, but we might have to iterate; i.e. we might have to take a colimit of objects of G, then take a colimit of *those* objects, and so on. In fact, in general the process may not converge at any stage, even transfinitely.

Evidently any dense generator is colimit-dense, and any colimit-dense generator is a colimit-generator. Conversely, $\mathbb{R}$ is colimit-dense in $\mathbf{Vect}_{\mathbb{R}}$, but it is not dense.[2] ($\mathbb{R} \oplus \mathbb{R}$ is, however, dense in $\mathbf{Vect}_{\mathbb{R}}$.) Colimit-generators that are not colimit-dense are somewhat harder to come by, but it turns out that $\mathbb{2}$ is such in **Cat**.

Lemma 3.7. *Any colimit-generator is a strong generator.*

Proof. Write $Q \perp f$ if the map $0 \to Q$ from the initial object is left orthogonal to $f \colon X \to Y$, or equivalently $\mathcal{A}(Q, X) \to \mathcal{A}(Q, Y)$ is a bijection. Let G be a colimit-generator and let $G^{\perp}$ be the class of all f such that $Q \perp f$ for all $Q \in G$. Then it is easy to see that the class ${}^{\perp}(G^{\perp}) = \{A \mid (\forall f \in G^{\perp})(A \perp f)\}$ is closed under colimits, and it evidently contains G; hence it is all of $\mathcal{A}$. However, any map which is orthogonal to its domain and codomain must be an isomorphism, so $G^{\perp} = \mathit{Iso}$. Thus, if $X \overset{m}{\rightarrowtail} Y$ is a monic such that every map to Y from an object of G factors through m, then evidently $m \in G^{\perp}$, and hence is an isomorphism; so G is a strong generator. ∎

[2] The original version of these notes claimed incorrectly that $\mathbb{Z}$ was colimit-dense but not dense in **Ab**. The error was pointed out on MathOverflow by Qiaochu Yuan, and a corrected example was supplied by Jiří Rosický. It is true that $\mathbb{Z}$ is colimit-dense but not dense in the category of *free* abelian groups, although this is not as interesting since the latter is not cocomplete.

Thus we have a hierarchy of notions

dense $\Rightarrow$ *colimit-dense* $\Rightarrow$ *colimit-generator* $\Rightarrow$ *strong generator* $\Rightarrow$ *generator.*

The only one of these implications that we haven't seen a counterexample to the converse of is "colimit-generator $\Rightarrow$ strong generator." That's because, in all reasonable categories, the two are equivalent.

Lemma 3.8. *If $\mathcal{A}$ is cocomplete, finitely complete, and extremally well-copowered, then any strong generator is a colimit-generator.*

Proof. Suppose G is a strong generator, let $\mathcal{B}$ be its colimit-closure (that is, the smallest subcategory of $\mathcal{A}$ containing G and closed under colimits), and let $RX = \coprod_{P\to X} P$ with $\varepsilon_X \colon RX \to X$ the canonical map (the counit), which is (by assumption) extremal epic. Consider some $f\colon Y \to Z$ with $Y \in \mathcal{B}$, and factor it as $X \xrightarrow{g_1} Z_1 \xrightarrow{f_1} Y$ as in §1, by taking the coequalizer of its kernel pair $K \rightrightarrows X$. Now the object K in the kernel pair may not be in $\mathcal{B}$, but since $\varepsilon_k \colon RK \to K$ is epic, this coequalizer is the same as the coequalizer of $RK \rightrightarrows X$. Hence Z_1 is a colimit of objects in $\mathcal{B}$, so it is also in $\mathcal{B}$, since $\mathcal{B}$ is closed under colimits.

Now, pick any $X \in \mathcal{A}$, and consider the construction of the extepi-mono factorization of ε_X by transfinitely iterating the above construction. The above argument shows that $Z_\alpha \in \mathcal{B}$ for all α. Since $\mathcal{A}$ is extremally well-copowered, the process terminates with some monic $f_\alpha \colon Z_\alpha \to X$. But we assumed that ε_X is already extremal epic, so if it factors through the monic f_α, then f_α must be an isomorphism; thus $X \cong Z_\alpha$ and so $X \in \mathcal{B}$. Therefore, $\mathcal{B} = \mathcal{A}$, so G is a colimit-generator. □

Thus, although examples of strong generators that are not colimit-generators do exist, most of them are rather contrived, since nearly all categories in nature are extremally well-copowered.

The point of this is that it's generally quite easy to check that something is a strong generator, while it would be quite hard to check explicitly that it is a colimit-generator (especially if it's not colimit-dense). But knowing that we have a colimit-generator is sometimes quite useful.