

Lifting Grothendieck universes
to Grothendieck toposes

OR: why Grothendieck *should* have
believed in universes!

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Part 1:

Universes in Toposes

A Grothendieck universe is

- A nonempty set \mathcal{U}
- Transitive: $A \in B \in \mathcal{U} \Rightarrow A \in \mathcal{U}$
- Unions: $A \in \mathcal{U}, B: A \rightarrow \mathcal{U} \Rightarrow \bigcup_{a \in A} B_a \in \mathcal{U}$
- Powersets: $A \in \mathcal{U} \Rightarrow \mathcal{P}A \in \mathcal{U}$

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A structural (G.) universe is

- A class S of sets, closed under \cong
- $0, 1, 2 \in S$
- Disjoint unions: $A \in S$, each $B_a \in S$
 $\Rightarrow \bigsqcup_{a \in A} B_a \in S$
- Products: $A \in S$, each $B_a \in S$
 $\Rightarrow \prod_{a \in A} B_a \in S$
- Generic set: There is a set \mathcal{U} and a family of sets $(E_x)_{x \in \mathcal{U}}$ such that $A \in S \Rightarrow \exists x. A \cong E_x$.

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A **Streicher universe** in a topos \mathcal{E} is

- A pullback-stable class S of morphisms
($g \in S \Rightarrow f^*g \in S$)
- All monomorphisms, and $\begin{array}{c} \Omega \\ \downarrow \\ 1 \end{array}$, are in S
- Composition: $f, g \in S \Rightarrow fg \in S$
- Push forward: $f, g \in S \Rightarrow \Pi_f g \in S$
($fg = \Sigma_f g$ where $\Sigma_f \dashv f^* \dashv \Pi_f$)
- Generic morphism: There is a $\begin{pmatrix} \tilde{u} \\ \downarrow \\ u \end{pmatrix} \in S$
such that for any $\begin{pmatrix} A \\ \downarrow \\ X \end{pmatrix} \in S$ there is

a pullback
$$\begin{array}{ccc} A & \longrightarrow & \tilde{u} \\ \downarrow & \lrcorner & \downarrow \\ X & \longrightarrow & u \end{array} .$$

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Part 2:
Moduli spaces

A topos is ...

1) A category that behaves like Set

2) A kind of generalized space (e.g. $\text{Sh}(X)$) ("little")

3) A category whose **objects** are spaces ("big")

- simplicial sets
- condensed sets (Scholze) / pyknotic sets (Barwick-Haine)
- con-sequential spaces (Johnstone)
- simplicial sheaves on any (little) topos

A **moduli space** for $F: \mathcal{E}^{\text{op}} \rightarrow \text{Set}$ is $BF \in \mathcal{E}$ with

1) A natural isomorphism $FX \cong \text{Hom}_{\mathcal{E}}(X, BF)$

or equivalently (by Yoneda)

2) An element $b \in F(BF)$ such that for any $x \in FX$
there exists a unique $g: X \rightarrow BF$ such that $Fg(b) = x$.

A Streicher universe is* the primordial moduli space.

We don't have to do this by hand separately — there is a general machine called **homotopy type theory**.

Natural mathematical language

interpretation in \mathcal{E}

moduli space

$\{ (A, m) \mid A \in \mathcal{U}, m: A \times A \rightarrow A \}$

\mapsto

$B\mathcal{F}_{\text{Mag}}$

$\{ (G, m) \mid m \text{ a group structure on } G \in \mathcal{U} \}$

\mapsto

$B\mathcal{F}_{\text{Grp}}$

$\{ (V, \dots) \mid V \in \mathcal{U} \text{ a vector space} \}$

\mapsto

$B\mathcal{F}_{\text{Vect}}$

etc.

Problem: Usually FX is not a set but a groupoid,
so $FX \cong \text{Hom}_\varepsilon(X, BF)$ is unrealistic.

Solutions: ① Coarse moduli space: $FX \longrightarrow \text{Hom}_\varepsilon(X, BF)$

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② Moduli stack: $FX \cong \text{Map}_{\hat{\mathcal{E}}}(X, BF)$

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so $FX \cong \text{Hom}_{\mathcal{E}}(X, BF)$ is unrealistic.

Solutions: ① Coarse moduli space: $FX \longrightarrow \text{Hom}_{\mathcal{E}}(X, BF)$

② Moduli stack: $FX \simeq \text{Map}_{\hat{\mathcal{E}}}(X, BF)$

③ Generic space: $\text{Hom}_{\mathcal{E}}(X, BF) \xrightarrow{\text{surj.}} FX$

④ A strong homotopy moduli space for F is $BF \in \mathcal{E}$
with a trivial fibration $\text{Hom}_{\mathcal{E}}(-, BF) \xrightarrow{\sim} F$.

For a Streicher universe, this is called *realignment*.

A strong homotopy moduli space IS a moduli stack!

Let \mathcal{E} present an ∞ -category with (e.g.) a Quillen model structure, hence cylinder objects

$$X \sqcup X \xrightarrow{\text{mono}} X \times I \xrightarrow{\sim} X.$$

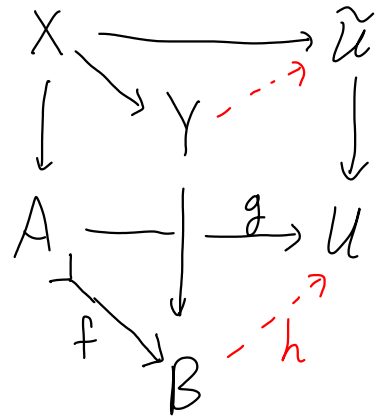
Realignment has many other uses:

- Nested universes $U_1 \in U_2 \in \dots$ with cumulative operations
- Independence results (e.g. Markov's principle)
- Canonicity and normalization via synthetic Tait computability
(e.g. cubical type theory, multimodal type theory, guarded type theory)

Part 3:

Constructing universes

Any Grothendieck universe in Set satisfies realignment.



$$h(b) = \begin{cases} g(a) & \text{if } b = f(a) \\ \text{some } Z \in \mathcal{U} \text{ s.t. } Z \cong Y_b & \text{otherwise.} \end{cases}$$

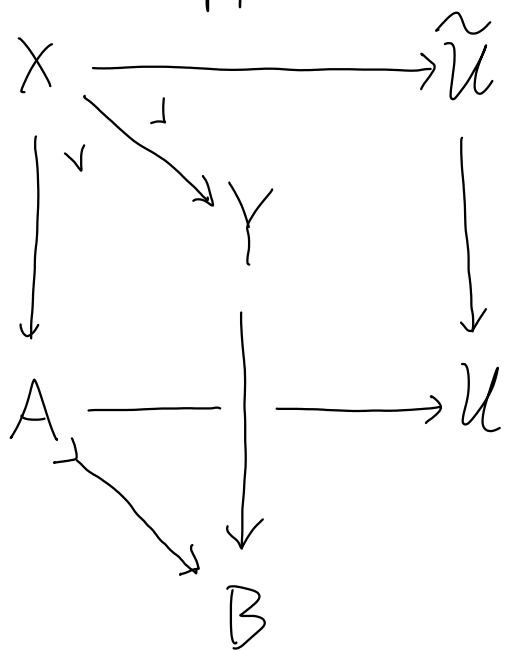
Hofmann-Streicher (1997) lifted universes to **presheaf** toposes.

These also satisfy realignment.

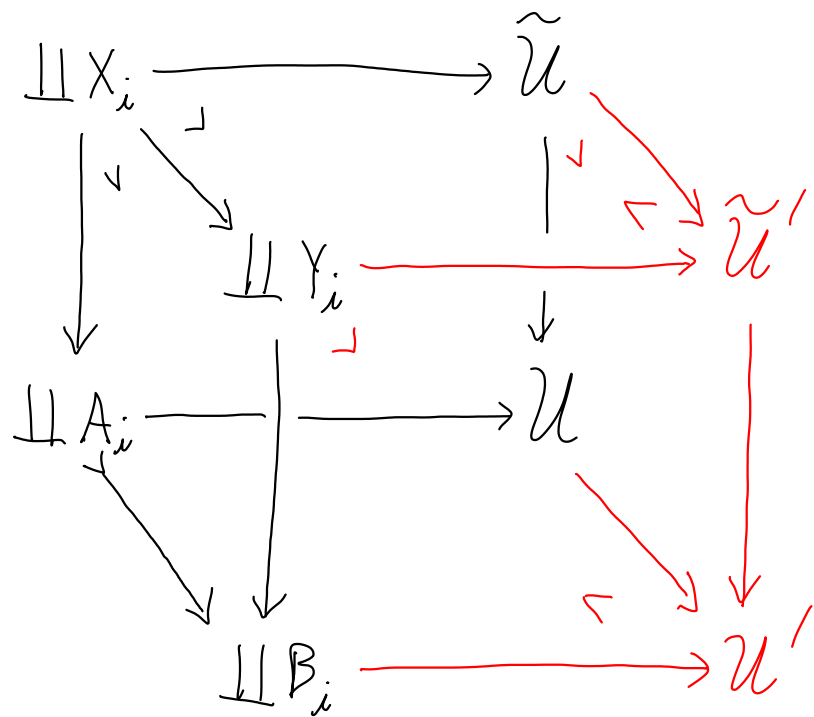
Streicher (2005) lifted universes further to **all** (sheaf) toposes.

These do **not** (apparently) satisfy realignment!

Step 1 Suppose U fails a realignment problem:



Step 2 If \mathcal{U} fails lots of
realignment problems, fix them all:



[But: there could be a proper
class of such problems.]

Step 3 \mathcal{U}' may introduce new
unsolved problems. Iterate:

$$\mathcal{U} \rightarrow \mathcal{U}' \rightarrow \mathcal{U}'' \rightarrow \mathcal{U}''' \rightarrow \dots$$

even transfinitely

$$\dots \rightarrow \mathcal{U}^{(\alpha)} \rightarrow \mathcal{U}^{(\alpha+1)} \rightarrow \dots$$

[But: When can we stop?]

Finally

The monomorphisms in a topos are cofibrantly generated:

There is a set of monos $\{f_i: A_i \rightarrow B_i\}$ such that

1) Every mono is a (transfinite) composite of pushouts of the generators f_i .

2) There is an ordinal λ such that each A_i is λ -compact: any map from A_i into a λ -colimit factors through a stage.

So we can glue in solutions only to realignment problems involving the generators, and stop after λ steps.

Thanks for listening!

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