Lifting Grothendieck universes to Grothendieck toposes

OR: why Grothendieck *should* have believed in universes!

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Part 1:

Universes in Toposes

A **Grothendieck universe** is

- A nonempty set $\mathcal{U}$
- Transitive: $A \in B \in \mathcal{U} \Rightarrow A \in \mathcal{U}$
- Unions: $A \in \mathcal{U},\ B : A \to \mathcal{U} \Rightarrow \bigcup_{a \in A} B_a \in \mathcal{U}$
- Powersets: $A \in \mathcal{U} \Rightarrow \mathcal{P}A \in \mathcal{U}$

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A structural (G.) universe is

- A class S of sets, closed under $\cong$
- $0, 1, 2 \in S$
- Disjoint unions: $A \in S$, each $B_a \in S$ $\Rightarrow \coprod_{a \in A} B_a \in S$
- Products: $A \in S$, each $B_a \in S$ $\Rightarrow \prod_{a \in A} B_a \in S$
- Generic set: There is a set $\mathcal{U}$ and a family of sets $(E_x)_{x \in \mathcal{U}}$ such that $A \in S \Rightarrow \exists x.\, A \cong E_x$.

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A Streicher universe in a topos $\mathcal{E}$ is

- A pullback-stable class S of morphisms $(g \in S \Rightarrow f^* g \in S)$
- All monomorphisms, and $\begin{pmatrix}\Omega \\ \downarrow \\ 1\end{pmatrix}$, are in S
- Composition: $f, g \in S \Rightarrow fg \in S$
- Pushforward: $f, g \in S \Rightarrow \prod_f g \in S$ $(fg = \Sigma_f g$ where $\Sigma_f \dashv f^* \dashv \prod_g)$
- Generic morphism: There is a $\begin{pmatrix}\tilde{\mathcal{U}} \\ \downarrow \\ \mathcal{U}\end{pmatrix} \in S$ such that for any $\begin{pmatrix}A \\ \downarrow \\ X\end{pmatrix} \in S$ there is a pullback

$$\begin{tikzcd} A \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \tilde{\mathcal{U}} \arrow[d] \\ X \arrow[r] & \mathcal{U} \end{tikzcd}.$$

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Part 2: Moduli spaces

A topos is ...

1) A category that behaves like Set

2) A kind of generalized space (e.g. $Sh(X)$) ("little")

3) A category whose objects are spaces ("big")

- simplicial sets
- condensed sets (Scholtz) / pyknotic sets (Barwick-Haine)
- con-sequential spaces (Johnstone)
- simplicial sheaves on any (little) topos

A moduli space for $F: \mathcal{E}^{op} \longrightarrow \mathrm{Set}$ is $BF \in \mathcal{E}$ with

1) A natural isomorphism $FX \cong \mathrm{Hom}_{\mathcal{E}}(X, BF)$

or equivalently (by Yoneda)

2) An element $b \in F(BF)$ such that for any $x \in FX$ there exists a unique $g: X \longrightarrow BF$ such that $Fg(b) = x$.

A Streicher universe is* the primordial moduli space.

We don't have to do this by hand separately — there is a general machine called homotopy type theory.

Natural mathematical language	interpretation in $\mathcal{E}$	moduli space
$\{(A,m) \mid A \in \mathcal{U},\ m : A \times A \to A\}$	$\longmapsto$	BF_{Mag}
$\{(G,m) \mid m \text{ a group structure on } G \in \mathcal{U}\}$	$\longmapsto$	BF_{Grp}
$\{(V,\ldots) \mid V \in \mathcal{U} \text{ a vector space}\}$	$\longmapsto$	BF_{Vect}

etc.

Problem: Usually FX is not a set but a groupoid,

so $FX \cong \mathrm{Hom}_{\mathcal{E}}(X, BF)$ is unrealistic.

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③ Generic space: $\mathrm{Hom}_{\mathcal{E}}(X, BF) \xrightarrow{\text{surj.}} FX$

④ A strong homotopy moduli space for F is $BF \in \mathcal{E}$ with a trivial fibration $\mathrm{Hom}_{\mathcal{E}}(-, BF) \xrightarrow{\sim} F$.

For a Streicher universe, this is called realignment.

A strong homotopy moduli space IS a moduli stack!

Let $\mathcal{E}$ present an ∞-category with (e.g.) a Quillen model structure, hence cylinder objects $X \sqcup X \overset{\text{mono}}{\rightarrowtail} X \times I \xrightarrow{\sim} X$.

Realignment has many other uses:

- Nested universes $\mathcal{U}_1 \in \mathcal{U}_2 \in \dots$ with cumulative operations
- Independence results (e.g. Markov's principle)
- Canonicity and normalization via synthetic Tait computability
  (e.g. cubical type theory, multimodal type theory, guarded type theory)

Part 3: Constructing universes

Any Grothendieck universe in **Set** satisfies realignment.

$$\begin{tikzcd}
X \arrow[rr] \arrow[dr] \arrow[dd] & & \tilde{\mathcal{U}} \arrow[dd] \\
& Y \arrow[ur, dashed] \arrow[dd] & \\
A \arrow[rr, "g"] \arrow[dr, "f"'] & & \mathcal{U} \\
& B \arrow[ur, dashed, "h"'] &
\end{tikzcd}$$

$$h(b) = \begin{cases} g(a) & \text{if } b = f(a) \\ \text{some } Z \in \mathcal{U} \text{ s.t. } Z \cong Y_b & \text{otherwise.} \end{cases}$$

Hofmann-Streicher (1997) lifted universes to presheaf toposes.
These also satisfy realignment.

Streicher (2005) lifted universes further to all (sheaf) toposes.
These do not (apparently) satisfy realignment!

$\boxed{\text{Step 1}}$ Suppose $\mathcal{U}$ fails a realignment problem:

$$\begin{tikzcd}
X \arrow[rr] \arrow[dr, tail] \arrow[dd, tail] & & \tilde{\mathcal{U}} \arrow[dd] \\
& Y \arrow[dd] & \\
A \arrow[rr] \arrow[dr, tail] & & \mathcal{U} \\
& B &
\end{tikzcd}$$

$\boxed{\text{Step 2}}$ If $\mathcal{U}$ fails lots of realignment problems, fix them all:

$$\begin{tikzcd}
\coprod X_i \arrow[rr] \arrow[dr] \arrow[dd] & & \tilde{\mathcal{U}} \arrow[dr] \arrow[dd] & \\
& \coprod Y_i \arrow[rr] \arrow[dd] & & \tilde{\mathcal{U}}' \arrow[dd] \\
\coprod A_i \arrow[rr] \arrow[dr] & & \mathcal{U} \arrow[dr] & \\
& \coprod B_i \arrow[rr] & & \mathcal{U}'
\end{tikzcd}$$

[But: there could be a proper class of such problems.]

$\boxed{\text{Step 3}}$ $\mathcal{U}'$ may introduce new unsolved problems. Iterate:

$$\mathcal{U} \to \mathcal{U}' \to \mathcal{U}'' \to \mathcal{U}''' \to \cdots$$

even transfinitely

$$\cdots \to \mathcal{U}^{(\alpha)} \to \mathcal{U}^{(\alpha+1)} \to \cdots$$

[But: When can we stop?]

$\boxed{\text{Finally}}$ The monomorphisms in a topos are cofibrantly generated:

There is a set of monos $\{ f_i : A_i \rightarrowtail B_i \}$ such that

1) Every mono is a (transfinite) composite of pushouts of the generators f_i.

2) There is an ordinal λ such that each A_i is λ-compact: any map from A_i into a λ-colimit factors through a stage.

So we can glue in solutions only to realignment problems involving the generators, and stop after λ steps.

Thanks for listening!

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