Lifting Grothendieck universes to Grothendieck toposes

OR: why Grothendieck should have believed in universes!

Mike Shulman
University of San Diego
jwv Daniel Gratzer and Jon Sterling

Grothendieck Conference, Chapman University
May 27, 2022
Part 1:
Universes in Toposes
A Grothendieck universe is

- A nonempty set $\mathcal{U}$
- Transitive: $A \in B \in \mathcal{U} \Rightarrow A \in \mathcal{U}$
- Unions: $A \in \mathcal{U}$, $B: A \rightarrow \mathcal{U} \Rightarrow \bigcup_{a \in A} B_a \in \mathcal{U}$
- Powersets: $A \in \mathcal{U} \Rightarrow \mathcal{P}(A) \in \mathcal{U}$
A Grothendieck universe is

- A nonempty set $\mathcal{U}$
- Transitive: $A \in B \in \mathcal{U} \Rightarrow A \in \mathcal{U}$
- Unions: $A \in \mathcal{U}, \; B: A \rightarrow \mathcal{U} \Rightarrow \bigcup_{a \in A} B_a \in \mathcal{U}$
- Powersets: $A \in \mathcal{U} \Rightarrow \mathcal{P} A \in \mathcal{U}$

A structural (G.) universe is

- A class $S$ of sets, closed under $\subseteq$
- $0, 1, 2 \in S$
- Disjoint unions: $A \in S$, each $B_a \in S$
  $\Rightarrow \bigsqcup_{a \in A} B_a \in S$
- Products: $A \in S$, each $B_a \in S$
  $\Rightarrow \prod_{a \in A} B_a \in S$
- Generic set: There is a set $\mathcal{U}$ and a family of sets $(E_x)_{x \in \mathcal{U}}$ such that $A \in S \Rightarrow \exists x. A \in E_x$
A structural (6.) universe is

- A class $S$ of sets, closed under $\cong$
- $0, 1, 2 \in S$
- Unions & Products: $A \in S$, each $B_a \in S$
  \[ \bigcup_{a \in A} B_a \in S \text{ and } \prod_{a \in A} B_a \in S \]
- Generic set: There is a set $U$
  and a family of sets $(E_x)_{x \in U}$
  such that $A \in S \Rightarrow \exists x. A \cong E_x$. 


A structural \((G, \cdot)\) universe is

- A class \(S\) of sets, closed under \(\simeq\)
- \(0, 1, 2 \in S\)
- Unions & Products: \(A \in S\), each \(B_a \in S\)
  \[\Rightarrow \biguplus_{a \in A} B_a \in S \text{ and } \prod_{a \in A} B_a \in S\]
- Generic set: There is a set \(U\)
  and a family of sets \(\{E_x\}_{x \in U}\)
  such that \(A \in S \Rightarrow \exists x. A \simeq E_x\).

A Streicher universe in a topos \(E\) is

- A pullback-stable class \(S\) of morphisms
  \((g \in S \Rightarrow f^*g \in S)\)
- All monomorphisms and \(\prod\), are in \(S\)
- Composition: \(f, g \in S \Rightarrow fg \in S\)
- Push forward: \(f, g \in S \Rightarrow \prod f_g \in S\)
  \((fg = \Sigma_fg \text{ where } \Sigma_f = f^* \prod g)\)
- Generic morphism: There is a \((\hat{u} \downarrow x) \in S\)
  such that for any \((A \downarrow x) \in S\) there is
  a pullback
  \[
  \begin{array}{ccc}
  A & \longrightarrow & \hat{u} \\
  \downarrow & & \downarrow \\
  X & \longrightarrow & U
  \end{array}
  \]
A Streicher universe in a topos $\mathcal{E}$ is

- A pullback-stable class $S$ of morphisms $(g \in S \implies f^*g \in S)$
- All monomorphisms $\downarrow \leftarrow$, are in $S$
- Composition : $f, g \in S \implies fg \in S$
- Push forward : $f, g \in S \implies \Pi_f g \in S$
  \[(fg = \Sigma_f (\Sigma g) \text{ where } \Sigma f \leftarrow f^* \leftarrow \Pi g)\]
- Generic morphism : There is a $(\hat{u} \downarrow) \in S$
  such that for any $(\hat{a} \downarrow \hat{x}) \in S$ there is
  a pullback $\xymatrix{A & \hat{u} \ar[l] \ar[d] \ar@{..>}[dl] \ar[d] \ar[r] & \hat{x} \ar[d] \ar@{..>}[dl] & X \ar[l] \ar[r] & U}$
Part 2: Moduli spaces
A topos is ...

1) A category that behaves like Set
2) A kind of generalized space (e.g. $\mathbb{Sh}(X)$) (“little”)
3) A category whose objects are spaces (“big”)
   - simplicial sets
   - condensed sets (Scholz) / pyknotic sets (Barwick-Haine)
   - con-sequential spaces (Johnstone)
   - simplicial sheaves on any (little) topos
A moduli space for $F : \mathcal{E}^{op} \to \text{Set}$ is $BF \in \mathcal{E}$ with

1) A natural isomorphism $FX \cong \text{Hom}_\mathcal{E}(X, BF)$

or equivalently (by Yoneda)

2) An element $b \in F(BF)$ such that for any $x \in FX$

there exists a unique $g : X \to BF$ such that $Fg(b) = x$. 
A Streicher universe is the primordial moduli space.
We don't have to do this by hand separately — there is a general machine called homotopy type theory.

\[
\begin{align*}
\{ (A, m) \mid A \in U, m : A \times A \to A \} & \quad \mapsto \quad B \mathcal{F}_{\text{Mag}} \\
\{ (G, m) \mid m \text{ a group structure on } G \in U \} & \quad \mapsto \quad B \mathcal{F}_{\text{Gp}} \\
\{ (V, \ldots) \mid V \in U \text{ a vector space} \} & \quad \mapsto \quad B \mathcal{F}_{\text{Vect}} \\
\text{etc.}
\end{align*}
\]
**Problem:** Usually $FX$ is not a set but a groupoid, so $FX \cong \text{Hom}_e(X, BF)$ is unrealistic.

**Solutions:**
1. Coarse moduli space: $FX \rightarrow \text{Hom}_e(X, BF)$
Problem: Usually FX is not a set but a groupoid, so \( FX \cong \text{Hom}_{\mathcal{E}}(X, BF) \) is unrealistic.

Solutions: ① Coarse moduli space: \( FX \to \text{Hom}_{\mathcal{E}}(X, BF) \)

② Moduli stack: \( FX \cong \text{Map}_{\hat{\mathcal{E}}}(X, BF) \)
**Problem:** Usually $FX$ is not a set but a groupoid, so $FX \cong \text{Hom}_\varepsilon(X, BF)$ is unrealistic.

**Solutions:**

1. **Coarse moduli space:** $FX \rightarrow \text{Hom}_\varepsilon(X, BF)$
2. **Moduli stack:** $FX \cong \text{Map}_\varepsilon(X, BF)$
3. **Generic space:** $\text{Hom}_\varepsilon(X, BF) \xrightarrow{\text{surj.}} FX$
A strong homotopy moduli space for $F$ is $\text{BF}\in \mathcal{E}$ with a trivial fibration $\text{Hom}_E(-,\text{BF}) \simeq F$.

For a Streicher universe, this is called realignment.
A strong homotopy moduli space is a moduli stack!

Let $\mathcal{E}$ present an $\infty$-category with (e.g.) a Quillen model structure, hence cylinder objects $X \sqcup X \xrightarrow{\text{mono}} X \times I \xrightarrow{\sim} X$. 
Realignment has many other uses:

- Nested universes $U_1 \subseteq U_2 \subseteq \ldots$ with cumulative operations
- Independence results (e.g. Markov’s principle)
- Canonicity and normalization via synthetic Tait computability (e.g. cubical type theory, multimodal type theory, guarded type theory)
Part 3: Constructing universes
Any Grothendieck universe in $\mathbf{Set}$ satisfies realignment.

\[
\begin{align*}
\begin{array}{ccc}
X & \rightarrow & \tilde{U} \\
\downarrow & & \downarrow \\
Y & \rightarrow & U \\
\downarrow & & \downarrow \\
A & \rightarrow & B
\end{array}
\end{align*}
\]

\[h(b) = \begin{cases} 
g(a) & \text{if } b = f(a) \\
\text{some } Z \in U \text{ s.t. } Z \subseteq Y_b & \text{otherwise}
\end{cases}\]

Hofmann–Streicher (1997) lifted universes to presheaf toposes. These also satisfy realignment.

Streicher (2005) lifted universes further to all (sheaf) toposes. These do not (apparently) satisfy realignment!
Step 1 Suppose \( U \) fails a realignment problem:

\[
\begin{align*}
X \rightarrow \hat{U} \\
\uparrow & \quad \uparrow \\
Y \rightarrow \hat{U} \\
\downarrow & \quad \downarrow \\
A \rightarrow \hat{U} \\
\downarrow & \\
B
\end{align*}
\]
Step 2: If $U$ fails lots of realignment problems, fix them all:

$$\prod X_i \rightarrow \tilde{U}$$

$$\prod Y_i \rightarrow \tilde{U}$$

$$\prod A_i \rightarrow U$$

$$\prod B_i \rightarrow U$$

Step 3: $U'$ may introduce new unsolved problems. Iterate:

$$U \rightarrow U' \rightarrow U'' \rightarrow U''' \rightarrow \ldots$$

even transfinitely

$$\ldots \rightarrow U^{(\alpha)} \rightarrow U^{(\alpha+1)} \rightarrow \ldots$$

But: there could be a proper class of such problems.

[But: When can we stop?]
Finally, the monomorphisms in a topos are cofibrantly generated: There is a set of monos \( \{ f_i : A_i \hookrightarrow B_i \} \) such that

1) Every mono is a (transfinite) composite of pushouts of the generators \( f_i \).

2) There is an ordinal \( \lambda \) such that each \( A_i \) is \( \lambda \)-compact: any map from \( A_i \) into a \( \lambda \)-colimit factors through a stage.

So we can glue in solutions only to realignment problems involving the generators, and stop after \( \lambda \) steps.
Thanks for listening!

arXiv: 2202.12012