Towards third generation HOTT

Part 3: Univalent universes

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The story so far

Plan for the three talks:

1. Basic syntax of H.O.T.T.
2. Symmetries and semicartesian cubes
3. From semicartesian cubes to univalent universes

Outline

1. The magic of semicartesian cubes
2. Paths in exponentials
3. The parametricity universe
4. The universe of fibrant types
5. Cubical spaces
6. Explaining the universe

The semicartesian cube category

- The semicartesian cube category $\mathbb{\square}$ has, as objects, finite sets.
- A morphism $\phi \in \mathbb{\square}(m, n)$ is a function $\phi : n \to m \sqcup \{-, +\}$ that is injective on the preimage of m.
- The symmetric monoidal structure $m \oplus n$ is disjoint union.
- The automorphisms of n are the symmetric group S_n.
- The presheaf category $\widehat{\mathbb{\square}} = \mathrm{Set}^{\mathbb{\square}^{\mathrm{op}}}$ has a Day convolution monoidal structure. Write $\square^n$ for the representable $\mathbb{\square}(-, n)$.
- A $\widehat{\mathbb{\square}}$-enriched category with $\square^n$-powers has ID-structure:

$$x : A, y : A \vdash \mathrm{Id}_A(x, y) : \mathrm{U} \qquad \longleftrightarrow \qquad \begin{array}{c} \square^1 \pitchfork A \\ \downarrow \\ A \times A \end{array}$$

Cubical paths and cylinders

Let $\mathcal{E}$ be a presheaf category (such as Set). The category $\mathcal{E}^{\square^{\mathrm{op}}}$ of cubical objects is $\widehat{\square}$-enriched with copowers and powers:

$$\widehat{\square}(K, \mathsf{Map}_{\mathcal{E}^{\square^{\mathrm{op}}}}(X, Y)) \cong \mathcal{E}^{\square^{\mathrm{op}}}(K \odot X, Y) \cong \mathcal{E}^{\square^{\mathrm{op}}}(X, K \pitchfork Y).$$

In particular, it has path spaces $\square^1 \pitchfork X$ and cylinders $\square^1 \odot X$. Path spaces are defined by shifting, while cylinders are magic:

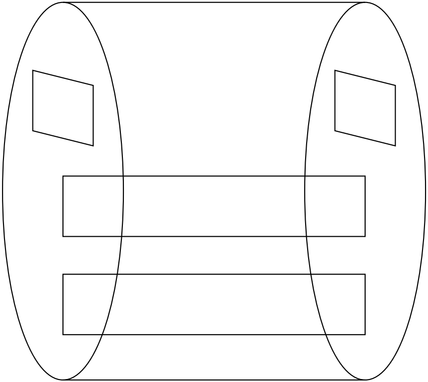
$$\begin{aligned}
(\square^1 \pitchfork X)_n &= X_{n \oplus 1} \\
(\square^1 \odot X)_n &= X_n + X_n + \sum_{k \in n} X_{n \smallsetminus \{k\}} \\
&\cong 2 \cdot X_n + n \cdot X_{n-1}.
\end{aligned}$$

Almost no other cube category satisfies the magic cylinder formula; we need symmetries but no diagonals or connections.

The magic of semicartesian cylinders

For example:

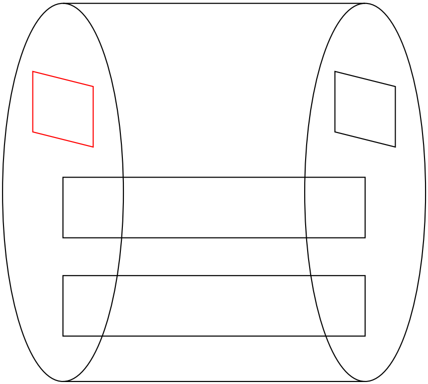
$$(\Box^1 \odot X)_2 = X_2 + X_2 + X_1 + X_1$$



The magic of semicartesian cylinders

For example:

$$(\square^1 \odot X)_2 = {\color{red}X_2} + X_2 + X_1 + X_1$$



The magic of semicartesian cylinders

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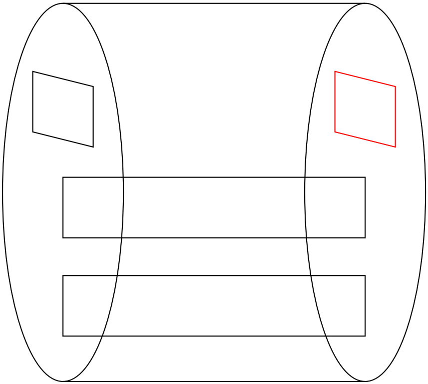
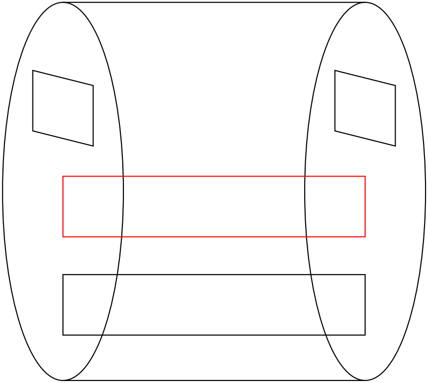
$$(\square^1 \odot X)_2 = X_2 + {\color{red}X_2} + X_1 + X_1$$



The magic of semicartesian cylinders

For example:

$$(\square^1 \odot X)_2 = X_2 + X_2 + {\color{red}X_1} + X_1$$



The magic of semicartesian cylinders

For example:

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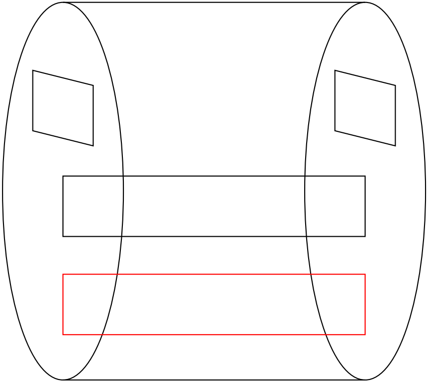

The magic of semicartesian cylinders

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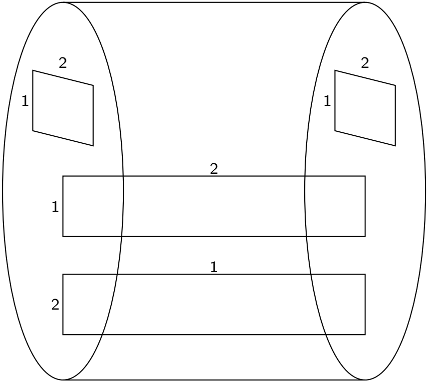
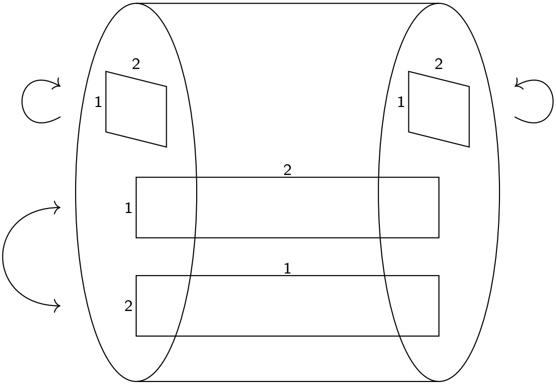


The magic of semicartesian cylinders

For example:

$$(\Box^1 \odot X)_2 = X_2 + X_2 + X_1 + X_1$$



Amazing right adjoints

Since the path-space is shifting, $(\square^1 \pitchfork X)_n = X_{n\oplus 1}$, it preserves all colimits, hence has an ("amazing") right adjoint

$$\mathcal{E}^{\square^{\mathrm{op}}}(\square^1 \pitchfork X, Y) \cong \mathcal{E}^{\square^{\mathrm{op}}}(X, \sqrt{Y})$$

This also has a fiberwise version:

$$\begin{array}{c} Y \\ \downarrow \\ \square^1 \pitchfork W \end{array} \quad \mapsto \quad \begin{array}{ccc} \sqrt{Y}_W & \longrightarrow & \sqrt{Y} \\ \downarrow & \lrcorner & \downarrow \\ W & \longrightarrow & \sqrt{\square^1 \pitchfork W} \end{array}$$

The fiberwise version maps $\mathcal{E}^{\square^{\mathrm{op}}}/(\square^1 \pitchfork W)$ to $\mathcal{E}^{\square^{\mathrm{op}}}/W$.

Outline

1. The magic of semicartesian cubes
2. Paths in exponentials
3. The parametricity universe
4. The universe of fibrant types
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Identity types of exponentials

For cartesian cubes, powers coincide with cartesian exponentials. So $\square^1 \pitchfork (A \to B) \cong A \to (\square^1 \pitchfork B)$, and $\mathsf{Id}_{A\to B}(f,g) \cong \prod_{(x:A)} \mathsf{Id}_B(fx, gx)$.

In the semicartesian case, we need to relate the cartesian exponential $A \to B$ with the monoidal path-space $(\square^1 \pitchfork -)$. To get our desired rule

$$\mathsf{Id}_{A\to B}(f,g) \cong \prod_{(u:A)} \prod_{(v:A)} \prod_{(q:\mathsf{Id}_A(u,v))} \mathsf{Id}_B(f(u), g(v)).$$

we want a pullback in $\mathcal{E}^{\square^{\mathrm{op}}}$:

$$\begin{array}{ccc}
\square^1 \pitchfork (A \to B) & \longrightarrow & \big((\square^1 \pitchfork A) \to (\square^1 \pitchfork B)\big) \\
\downarrow & \lrcorner & \downarrow \\
(A \to B) \times (A \to B) & \longrightarrow & ((\square^1 \pitchfork A) \to B) \times ((\square^1 \pitchfork A) \to B)
\end{array}$$

Identity types of exponentials

We want a pullback in $\mathcal{E}^{\square^{\mathrm{op}}}$:

$$\begin{array}{ccc}
\square^1 \pitchfork (A \to B) & \longrightarrow & \big((\square^1 \pitchfork A) \to (\square^1 \pitchfork B)\big) \\
\downarrow & \lrcorner & \downarrow \\
(A \to B)^2 & \longrightarrow & ((\square^1 \pitchfork A) \to B)^2
\end{array}$$

Identity types of exponentials

By Yoneda, we want a pullback in Set for all $X \in \mathcal{E}^{\square^{\mathrm{op}}}$:

$$
\begin{array}{ccc}
\mathcal{E}^{\square^{\mathrm{op}}}(X, \square^1 \pitchfork (A \to B)) & \longrightarrow & \mathcal{E}^{\square^{\mathrm{op}}}\big(X, (\square^1 \pitchfork A) \to (\square^1 \pitchfork B)\big) \\
\downarrow & \lrcorner & \downarrow \\
\mathcal{E}^{\square^{\mathrm{op}}}(X, (A \to B)^2) & \longrightarrow & \mathcal{E}^{\square^{\mathrm{op}}}(X, ((\square^1 \pitchfork A) \to B)^2)
\end{array}
$$

Identity types of exponentials

Now we apply universal properties...

$$\begin{array}{ccc}
\mathcal{E}^{\square^{\mathrm{op}}}(X, \square^1 \pitchfork (A \to B)) & \longrightarrow & \mathcal{E}^{\square^{\mathrm{op}}}\big(X, (\square^1 \pitchfork A) \to (\square^1 \pitchfork B)\big) \\
\downarrow & \lrcorner & \downarrow \\
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\end{array}$$

Identity types of exponentials

Now we apply universal properties...

$$\begin{array}{ccc}
\mathcal{E}^{\square^{\mathrm{op}}}(\square^1 \odot X, A \to B) & \longrightarrow & \mathcal{E}^{\square^{\mathrm{op}}}\big(X, (\square^1 \pitchfork A) \to (\square^1 \pitchfork B)\big) \\
\downarrow & \lrcorner & \downarrow \\
\mathcal{E}^{\square^{\mathrm{op}}}(X, (A \to B)^2) & \longrightarrow & \mathcal{E}^{\square^{\mathrm{op}}}(X, ((\square^1 \pitchfork A) \to B)^2)
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\mathcal{E}^{\square^{\mathrm{op}}}(X, (A \to B)^2) & \longrightarrow & \mathcal{E}^{\square^{\mathrm{op}}}(X, ((\square^1 \pitchfork A) \to B)^2)
\end{array}$$

Identity types of exponentials

Now we apply universal properties...

$$\begin{array}{ccc}
\mathcal{E}^{\square^{\mathrm{op}}}((\square^1 \odot X) \times A, B) & \longrightarrow & \mathcal{E}^{\square^{\mathrm{op}}}\big(X, (\square^1 \pitchfork A) \to (\square^1 \pitchfork B)\big) \\
\downarrow & \lrcorner & \downarrow \\
\mathcal{E}^{\square^{\mathrm{op}}}(X, (A \to B)^2) & \longrightarrow & \mathcal{E}^{\square^{\mathrm{op}}}(X, ((\square^1 \pitchfork A) \to B)^2)
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Identity types of exponentials

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\big\downarrow & \lrcorner & \big\downarrow \\
\mathcal{E}^{\square^{\mathrm{op}}}(2 \cdot X, A \to B) & \longrightarrow & \mathcal{E}^{\square^{\mathrm{op}}}(X, ((\square^1 \pitchfork A) \to B)^2)
\end{array}
$$

Identity types of exponentials

Now we apply universal properties. . .

$$\begin{array}{ccc}
\mathcal{E}^{\square^{\mathrm{op}}}((\square^1 \odot X) \times A, B) & \longrightarrow & \mathcal{E}^{\square^{\mathrm{op}}}\big(X, (\square^1 \pitchfork A) \to (\square^1 \pitchfork B)\big) \\
\downarrow & \lrcorner & \downarrow \\
\mathcal{E}^{\square^{\mathrm{op}}}(2 \cdot X, {\color{red}A \to} B) & \longrightarrow & \mathcal{E}^{\square^{\mathrm{op}}}(X, ((\square^1 \pitchfork A) \to B)^2)
\end{array}$$

Identity types of exponentials

Now we apply universal properties...

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\begin{array}{ccc}
\mathcal{E}^{\square^{\mathrm{op}}}((\square^1 \odot X) \times A, B) & \longrightarrow & \mathcal{E}^{\square^{\mathrm{op}}}\big(X, (\square^1 \pitchfork A) \to (\square^1 \pitchfork B)\big) \\
\big\downarrow & \lrcorner & \big\downarrow \\
\mathcal{E}^{\square^{\mathrm{op}}}((2 \cdot X) \times A, B) & \longrightarrow & \mathcal{E}^{\square^{\mathrm{op}}}(X, ((\square^1 \pitchfork A) \to B)^2)
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$$

Identity types of exponentials

Now we apply universal properties...

$$\begin{array}{ccc}
\mathcal{E}^{\square^{\mathrm{op}}}((\square^1 \odot X) \times A, B) & \longrightarrow & \mathcal{E}^{\square^{\mathrm{op}}}\big(X \times (\square^1 \pitchfork A), \square^1 \pitchfork B\big) \\
\downarrow & \lrcorner & \downarrow \\
\mathcal{E}^{\square^{\mathrm{op}}}((2 \cdot X) \times A, B) & \longrightarrow & \mathcal{E}^{\square^{\mathrm{op}}}(X, ((\square^1 \pitchfork A) \to B)^2)
\end{array}$$

Identity types of exponentials

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\downarrow & \lrcorner & \downarrow \\
\mathcal{E}^{\square^{\mathrm{op}}}((2 \cdot X) \times A, B) & \longrightarrow & \mathcal{E}^{\square^{\mathrm{op}}}(X, ((\square^1 \pitchfork A) \to B)^2)
\end{array}
$$

Identity types of exponentials

Now we apply universal properties. . .

$$\begin{array}{ccc}
\mathcal{E}^{\square^{\mathrm{op}}}((\square^1 \odot X) \times A, B) & \longrightarrow & \mathcal{E}^{\square^{\mathrm{op}}}\left(\color{red}{\square^1 \odot}\, (X \times (\square^1 \pitchfork A)), B\right) \\
\big\downarrow & \lrcorner & \big\downarrow \\
\mathcal{E}^{\square^{\mathrm{op}}}((2 \cdot X) \times A, B) & \longrightarrow & \mathcal{E}^{\square^{\mathrm{op}}}(X, ((\square^1 \pitchfork A) \to B)^2)
\end{array}$$

Identity types of exponentials

Now we apply universal properties...

$$\begin{array}{ccc}
\mathcal{E}^{\square^{\mathrm{op}}}((\square^1 \odot X) \times A, B) & \longrightarrow & \mathcal{E}^{\square^{\mathrm{op}}}\big(\square^1 \odot (X \times (\square^1 \pitchfork A)), B\big) \\
\downarrow & \lrcorner & \downarrow \\
\mathcal{E}^{\square^{\mathrm{op}}}((2 \cdot X) \times A, B) & \longrightarrow & \mathcal{E}^{\square^{\mathrm{op}}}\big(X, ((\square^1 \pitchfork A) \to B)^{\color{red}{2}}\big)
\end{array}$$

Identity types of exponentials

Now we apply universal properties. . .

$$\begin{array}{ccc}
\mathcal{E}^{\square^{op}}((\square^1 \odot X) \times A, B) & \longrightarrow & \mathcal{E}^{\square^{op}}(\square^1 \odot (X \times (\square^1 \pitchfork A)), B) \\
\downarrow & \lrcorner & \downarrow \\
\mathcal{E}^{\square^{op}}((2 \cdot X) \times A, B) & \longrightarrow & \mathcal{E}^{\square^{op}}(2 \cdot X, (\square^1 \pitchfork A) \to B)
\end{array}$$

Identity types of exponentials

Now we apply universal properties. . .

$$\begin{array}{ccc}
\mathcal{E}^{\square^{\mathrm{op}}}((\square^1 \odot X) \times A, B) & \longrightarrow & \mathcal{E}^{\square^{\mathrm{op}}}(\square^1 \odot (X \times (\square^1 \pitchfork A)), B) \\
\downarrow & \lrcorner & \downarrow \\
\mathcal{E}^{\square^{\mathrm{op}}}((2 \cdot X) \times A, B) & \longrightarrow & \mathcal{E}^{\square^{\mathrm{op}}}(2 \cdot X, (\square^1 \pitchfork A) \to B)
\end{array}$$

Identity types of exponentials

Now we apply universal properties...

$$\begin{array}{ccc} \mathcal{E}^{\square^{\mathrm{op}}}((\square^1 \odot X) \times A, B) & \longrightarrow & \mathcal{E}^{\square^{\mathrm{op}}}\big(\square^1 \odot (X \times (\square^1 \pitchfork A)), B\big) \\ \downarrow & \lrcorner & \downarrow \\ \mathcal{E}^{\square^{\mathrm{op}}}((2 \cdot X) \times A, B) & \longrightarrow & \mathcal{E}^{\square^{\mathrm{op}}}((2 \cdot X) \times (\square^1 \pitchfork A), B) \end{array}$$

Identity types of exponentials

Now we apply universal properties...

$$\begin{array}{ccc}
\mathcal{E}^{\square^{\mathrm{op}}}((\square^1 \odot X) \times A, {\color{red}B}) & \longrightarrow & \mathcal{E}^{\square^{\mathrm{op}}}(\square^1 \odot (X \times (\square^1 \pitchfork A)), {\color{red}B}) \\
\downarrow & \lrcorner & \downarrow \\
\mathcal{E}^{\square^{\mathrm{op}}}((2 \cdot X) \times A, {\color{red}B}) & \longrightarrow & \mathcal{E}^{\square^{\mathrm{op}}}((2 \cdot X) \times (\square^1 \pitchfork A), {\color{red}B})
\end{array}$$

And now, by Yoneda again...

Identity types of exponentials

We equivalently want a pushout in $\mathcal{E}^{\square^{\mathrm{op}}}$:

$$\begin{array}{ccc}
(2 \cdot X) \times (\square^1 \pitchfork A) & \longrightarrow & \square^1 \odot (X \times (\square^1 \pitchfork A)) \\
\downarrow & \ulcorner & \downarrow \\
(2 \cdot X) \times A & \longrightarrow & (\square^1 \odot X) \times A
\end{array}$$

Identity types of exponentials

Which means a pushout in $\mathcal{E}$ for all n:

$$\begin{array}{ccc}
((2 \cdot X) \times (\square^1 \pitchfork A))_n & \longrightarrow & (\square^1 \odot (X \times (\square^1 \pitchfork A)))_n \\
\big\downarrow & \ulcorner & \big\downarrow \\
((2 \cdot X) \times A)_n & \longrightarrow & ((\square^1 \odot X) \times A)_n
\end{array}$$

Identity types of exponentials

Which means a pushout in $\mathcal{E}$ for all n:

$$\begin{array}{ccc}
((2 \cdot X) \times (\square^1 \pitchfork A))_n & \longrightarrow & (\square^1 \odot (X \times (\square^1 \pitchfork A)))_n \\
\downarrow & \ulcorner & \downarrow \\
(2 \cdot X)_n \times A_n & \longrightarrow & ((\square^1 \odot X) \times A)_n
\end{array}$$

Identity types of exponentials

Which means a pushout in $\mathcal{E}$ for all n:

$$\begin{array}{ccc}
((2 \cdot X) \times (\square^1 \pitchfork A))_n & \longrightarrow & (\square^1 \odot (X \times (\square^1 \pitchfork A)))_n \\
\downarrow & \ulcorner & \downarrow \\
2 \cdot (X_n \times A_n) & \longrightarrow & ((\square^1 \odot X) \times A)_n
\end{array}$$

Identity types of exponentials

Which means a pushout in $\mathcal{E}$ for all n:

$$\begin{array}{ccc}
(2 \cdot X)_n \times (\square^1 \pitchfork A)_n & \longrightarrow & (\square^1 \odot (X \times (\square^1 \pitchfork A)))_n \\
\downarrow & \ulcorner & \downarrow \\
2 \cdot (X_n \times A_n) & \longrightarrow & ((\square^1 \odot X) \times A)_n
\end{array}$$

Identity types of exponentials

Which means a pushout in $\mathcal{E}$ for all n:

$$\begin{array}{ccc}
2 \cdot (X_n \times A_{n+1}) & \longrightarrow & (\square^1 \odot (X \times (\square^1 \pitchfork A)))_n \\
\downarrow & \ulcorner & \downarrow \\
2 \cdot (X_n \times A_n) & \longrightarrow & ((\square^1 \odot X) \times A)_n
\end{array}$$

Identity types of exponentials

Which means a pushout in $\mathcal{E}$ for all n:

$$\begin{array}{ccc} 2 \cdot (X_n \times A_{n+1}) & \longrightarrow & (\square^1 \odot (X \times (\square^1 \pitchfork A)))_n \\ \downarrow & \ulcorner & \downarrow \\ 2 \cdot (X_n \times A_n) & \longrightarrow & (\square^1 \odot X)_n \times A_n \end{array}$$

Identity types of exponentials

Which means a pushout in $\mathcal{E}$ for all n:

$$\begin{array}{ccc} 2 \cdot (X_n \times A_{n+1}) & \longrightarrow & (\square^1 \odot (X \times (\square^1 \pitchfork A)))_n \\ \downarrow & \ulcorner & \downarrow \\ 2 \cdot (X_n \times A_n) & \longrightarrow & (2 \cdot X_n + n \cdot X_{n-1}) \times A_n \end{array}$$

Identity types of exponentials

Which means a pushout in $\mathcal{E}$ for all n:

$$\begin{array}{ccc} 2 \cdot (X_n \times A_{n+1}) & \longrightarrow & (\square^1 \odot (X \times (\square^1 \pitchfork A)))_n \\ \downarrow & \ulcorner & \downarrow \\ 2 \cdot (X_n \times A_n) & \longrightarrow & 2 \cdot (X_n \times A_n) + n \cdot (X_{n-1} \times A_n) \end{array}$$

Identity types of exponentials

Which means a pushout in $\mathcal{E}$ for all n:

$$
\begin{array}{ccc}
2 \cdot (X_n \times A_{n+1}) & \longrightarrow & (\square^1 \odot (X \times (\square^1 \pitchfork A)))_n \\
\downarrow & \ulcorner & \downarrow \\
2 \cdot (X_n \times A_n) & \longrightarrow & 2 \cdot (X_n \times A_n) + n \cdot (X_{n-1} \times A_n)
\end{array}
$$

$$
\begin{aligned}
&(\square^1 \odot (X \times (\square^1 \pitchfork A)))_n \\
&\qquad = 2 \cdot (X \times (\square^1 \pitchfork A))_n + n \cdot (X \times (\square^1 \pitchfork A))_{n-1} \\
&\qquad = 2 \cdot (X_n \times (\square^1 \pitchfork A)_n) + n \cdot (X_{n-1} \times (\square^1 \pitchfork A)_{n-1}) \\
&\qquad = 2 \cdot (X_n \times A_{n+1}) + n \cdot (X_{n-1} \times A_n)
\end{aligned}
$$

Identity types of exponentials

Which means a pushout in $\mathcal{E}$ for all n:

$$\begin{array}{ccc} 2\cdot(X_n \times A_{n+1}) & \longrightarrow & 2\cdot(X_n \times A_{n+1}) + n\cdot(X_{n-1}\times A_n) \\ \downarrow & \ulcorner & \downarrow \\ 2\cdot(X_n \times A_n) & \longrightarrow & 2\cdot(X_n \times A_n) + n\cdot(X_{n-1}\times A_n) \end{array}$$

$$\begin{aligned} &(\square^1 \odot (X \times (\square^1 \pitchfork A)))_n \\ &\qquad = 2\cdot(X\times(\square^1 \pitchfork A))_n + n\cdot(X\times(\square^1 \pitchfork A))_{n-1} \\ &\qquad = 2\cdot(X_n\times(\square^1 \pitchfork A)_n) + n\cdot(X_{n-1}\times(\square^1 \pitchfork A)_{n-1}) \\ &\qquad = 2\cdot(X_n\times A_{n+1}) + n\cdot(X_{n-1}\times A_n) \end{aligned}$$

Identity types of exponentials

$$\begin{array}{ccc} 2\cdot(X_n \times A_{n+1}) & \longrightarrow & 2\cdot(X_n \times A_{n+1}) + n\cdot(X_{n-1}\times A_n) \\ \big\downarrow & \ulcorner & \big\downarrow \\ 2\cdot(X_n \times A_n) & \longrightarrow & 2\cdot(X_n \times A_n) + n\cdot(X_{n-1}\times A_n) \end{array}$$

But this is just a coproduct of two pushout squares:

$$\begin{array}{ccc} 2\cdot(X_n \times A_{n+1}) & = & 2\cdot(X_n \times A_{n+1}) \\ \big\downarrow & \ulcorner & \big\downarrow \\ 2\cdot(X_n \times A_n) & = & 2\cdot(X_n \times A_n) \end{array} \qquad \begin{array}{ccc} \emptyset & \longrightarrow & n\cdot(X_{n-1}\times A_n) \\ \Vert & \ulcorner & \Vert \\ \emptyset & \longrightarrow & n\cdot(X_{n-1}\times A_n) \end{array}$$

Thus, it is a pushout, completing the proof of our desired rule

$$\mathsf{Id}_{A\to B}(f,g) \cong \prod_{(u:A)}\prod_{(v:A)}\prod_{(q:\mathsf{Id}_A(u,v))}\mathsf{Id}_B(f(u),g(v)).$$

The same ideas work for dependent types and for Π-types.

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Paths in the universe

If U "classifies" small maps, then

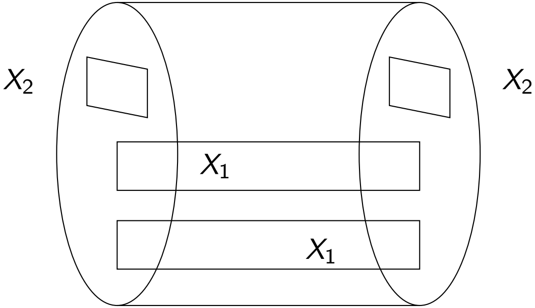
$$\mathcal{E}^{\square^{\mathrm{op}}}(X, \square^1 \pitchfork \mathsf{U}) \cong \mathcal{E}^{\square^{\mathrm{op}}}(\square^1 \odot X, \mathsf{U})$$

so $\square^1 \pitchfork \mathsf{U}$ "classifies" small maps over cylinders.

By "extensivity", a map $Y \to (\square^1 \odot X)$ decomposes Y_n as a coproduct too:

$$\begin{array}{ccc} Y_n & \xrightarrow{\cong} & A_n + B_n + \sum_{k \in n} C_{n,k} \\ \downarrow & & \downarrow \\ (\square^1 \odot X)_n & \xrightarrow[\cong]{} & X_n + X_n + \sum_{k \in n} X_{n \setminus \{k\}}. \end{array}$$

Cubes over cylinders



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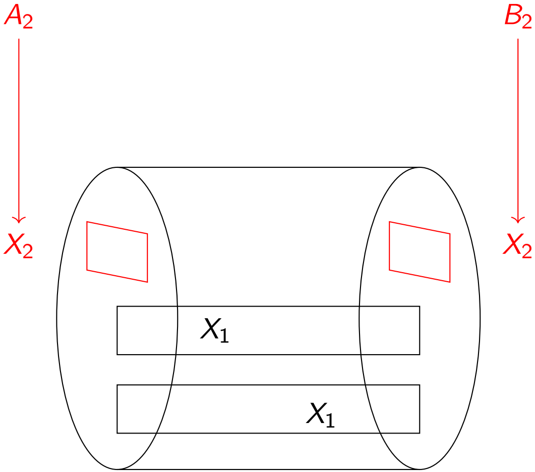

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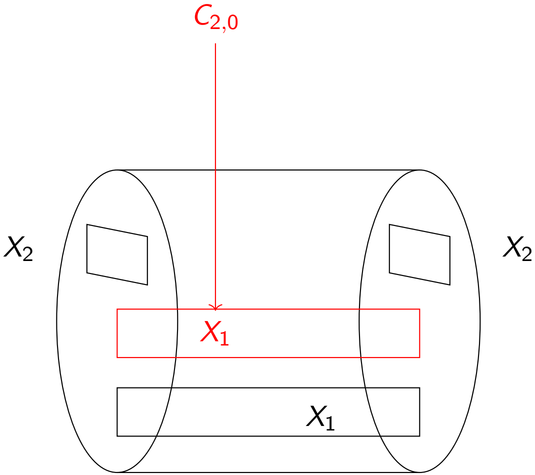

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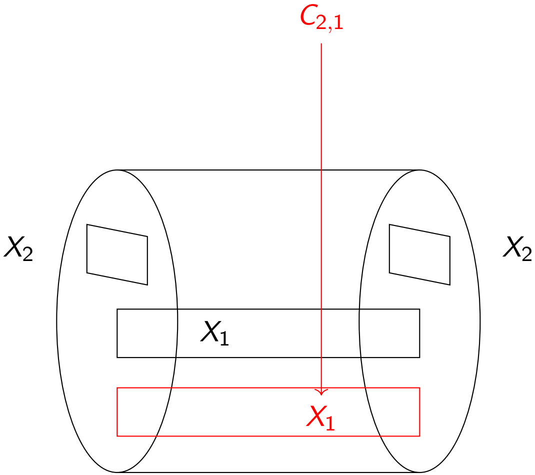

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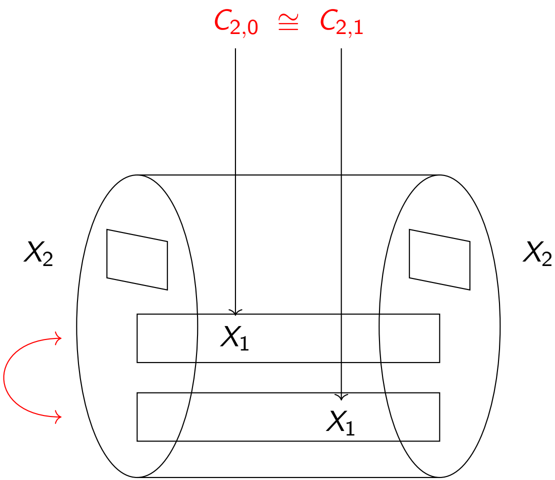

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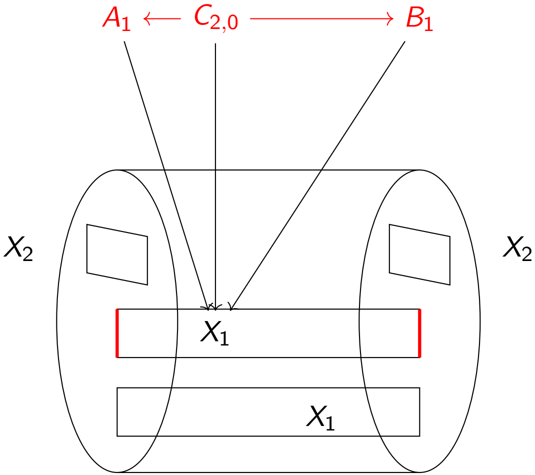

Cubical operators over cylinders

Let $\phi \in \square(m,n)$, so $\phi : n \to m \sqcup \{-,+\}$.

$$\begin{array}{ccc} A_n + B_n + \sum_{k\in n} C_{n,k} & \xrightarrow{\phi^*} & A_m + B_m + \sum_{\ell\in m} C_{m,\ell} \\ \downarrow & & \downarrow \\ X_n + X_n + \sum_{k\in n} X_{n\smallsetminus\{k\}} & \xrightarrow{\phi^*} & X_m + X_m + \sum_{\ell\in m} X_{m\smallsetminus\{\ell\}} \end{array}$$

- Any ϕ preserves the first two summands, so we have $A, B \in \mathcal{E}^{\square^{\mathrm{op}}}$ with maps $A \to X$ and $B \to X$.
- $S_n \subseteq \square(n,n)$ permutes the summands $X_{n\smallsetminus\{k\}}$, and its subgroup $S_{n\smallsetminus\{k\}}$ acts on $X_{n\smallsetminus\{k\}}$. Thus, $C_{n,k} \cong C_{n,k'}$ $\forall k, k'$.
- If $\phi(k) = \ell \in m$, then ϕ maps $X_{n\smallsetminus\{k\}}$ to $X_{m\smallsetminus\{\ell\}}$. These assemble the $C_{n,k}$ into $C \in \mathcal{E}^{\square^{\mathrm{op}}}$ with a map $C \to X$.
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Type families over cylinders

Thus, from $Y \to (\square^1 \odot X)$, we extract a span in $\mathcal{E}^{\boxbar^{\mathrm{op}}}/X$:

$$\begin{array}{ccccc} A & \longleftarrow & C & \longrightarrow & B \\ & \searrow & \downarrow & \swarrow & \\ & & X & & \end{array}$$

Theorem

For $X \in \mathcal{E}^{\boxbar^{\mathrm{op}}}$, we have an equivalence of categories

$$\mathcal{E}^{\boxbar^{\mathrm{op}}}/(\square^1 \odot X) \;\simeq\; (\mathcal{E}^{\boxbar^{\mathrm{op}}}/X)^{(\cdot\leftarrow\cdot\rightarrow\cdot)}$$

(Also generalizes to many other $K \in \widehat{\boxbar}$ replacing $\square^1$.)

Identity types of the parametricity universe

Corollary

There is a $\mathsf{U}^0 \in \mathcal{E}^{\square^{\mathrm{op}}}$ *that classifies small maps, and such that there is a trivial fibration (i.e. a map with RLP against monos)*

$$(\square^1 \pitchfork \mathsf{U}^0) \xrightarrow{\sim} \textstyle\sum_{(A,B:\mathsf{U}^0)}(A \to B \to \mathsf{U}^0)$$

This is not an isomorphism: the isomorphic copies $C_{n,k}$ have to be classified separately.

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Trivial fibrations have sections, so we interpret a syntactic retraction

$$(A \to B \to \mathsf{U}^0) \xrightarrow{\uparrow} \mathsf{Id}_{\mathsf{U}^0}(A, B) \xrightarrow{\downarrow} (A \to B \to \mathsf{U}^0) \qquad p{\uparrow}{\downarrow} \equiv p$$

Thus $\widehat{\mathbb{I}}$ with U^0 models a theory of internal parametricity, whose "identity types" consist of arbitrary correspondences.

Outline

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Towards a universe of fibrant types

We'd like to define U to be

"the subtype of U^0 whose identity type correspondences are one-to-one."

For any span $A \leftarrow C \rightarrow B$, i.e. correspondence $C : A \rightarrow B \rightarrow \mathsf{U}^0$, we have the type of assertions that it is one-to-one:

$$\mathsf{is11}(C) :\equiv \Big(\prod_{(a:A)} \mathsf{isContr}(\sum_{(b:B)} C(a,b))\Big) \times \Big(\prod_{(b:B)} \mathsf{isContr}(\sum_{(a:A)} C(a,b))\Big)$$

If A, B, C lie in a slice $\mathcal{E}^{\square^{\mathrm{op}}}/X$, so does $\mathsf{is11}(C) \in \mathcal{E}^{\square^{\mathrm{op}}}/X$.

The universal correspondence

We pull back the universal type family along the adjunction counit:

$$\begin{array}{ccc} \bullet & \longrightarrow & \widetilde{\mathsf{U}^0} \\ \downarrow & \lrcorner & \downarrow \\ \square^1 \odot (\square^1 \pitchfork \mathsf{U}^0) & \longrightarrow & \mathsf{U}^0 \end{array}$$

This yields a type family over the cylinder $\square^1 \odot (\square^1 \pitchfork \mathsf{U}^0)$, hence a universal correspondence over $\square^1 \pitchfork \mathsf{U}^0$:

$$\begin{array}{ccccc} A^0 & \longleftarrow & C^0 & \longrightarrow & B^0 \\ & \searrow & \downarrow & \swarrow & \\ & & \square^1 \pitchfork \mathsf{U}^0 & & \end{array}$$

Thus we have the classifying object $\mathsf{is11}(C^0) \in \mathcal{E}^{\square^{\mathrm{op}}}/(\square^1 \pitchfork \mathsf{U}^0)$.

BUT: this is a predicate on $\square^1 \pitchfork \mathsf{U}^0$, not U^0 itself.

A first-order approximation

We can fix this with the fiberwise amazing right adjoint:

$$\mathsf{U}^1 = \sqrt{\mathsf{is11}(C^0)}_{\mathsf{U}^0}.$$

Theorem

The classifying map $\Delta \to \mathsf{U}^0$ of a type family $\Delta \vdash P : \mathsf{U}^0$ lifts to U^1 if and only if the correspondence $\mathsf{Id}^{\varrho}_{\Delta.P}$ is one-to-one.

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BUT: This correspondence is still U^0-valued: even for $\Delta \vdash P : \mathsf{U}^1$,

$$\mathsf{Id}^{\varrho}_{\Delta.P} : P[\delta] \to P[\delta'] \to \mathsf{U}^0.$$

So we can't consistently use U^1 as "the" universe.

A second-order approximation

For any $A \leftarrow C \rightarrow B$ we have a type classif(C, U^1) of U^1-valued classifying maps for C, i.e. pullback squares

$$\begin{array}{ccc} C & \longrightarrow & \widetilde{\mathsf{U}^1} \\ \downarrow & \lrcorner & \downarrow \\ A \times B & \longrightarrow & \mathsf{U}^1 \end{array}$$

Then we define a further improved universe:

$$\mathsf{U}^2 = \sqrt{\mathsf{is11}(C^1) \times \mathsf{classif}(C^1, \mathsf{U}^1)}_{\mathsf{U}^1}$$

The identity types of $\Delta \vdash P : \mathsf{U}^2$ are one-to-one and U^1-valued.

A limit construction

We continue inductively and take a limit:

$$\mathsf{U}^{n+1} = \sqrt{\mathsf{is11}(C^n) \times \mathsf{classif}(C^n, \mathsf{U}^n)}_{\mathsf{U}^n}$$

$$\mathsf{U} = \lim_n (\cdots \to \mathsf{U}^n \to \cdots \to \mathsf{U}^1 \to \mathsf{U}^0)$$

Theorem

The identity types of $\Delta \vdash P : \mathsf{U}$ are one-to-one and U-valued.

U classifies maps with contractible spaces of uniform Kan fillers.

Higher coinduction for Id_U

U is a higher coinductive type: the terminal coalgebra of a functor involving $\sqrt{\ }$.

- Its higher destructors assemble into

$$\downarrow : \mathsf{Id}_\mathsf{U}(A, B) \to \text{1-1-Corr}(A, B)$$

- The magic cylinder formula implies a formula for paths in $\sqrt{\ }$. Thus, Id_U is also a higher coinductive type.
- By higher coinduction (univ. prop. of lim and $\sqrt{\ }$) we define

$$\uparrow : \text{1-1-Corr}(A, B) \to \mathsf{Id}_\mathsf{U}(A, B)$$

  such that $p{\uparrow}{\downarrow} \equiv p$.

Even if $\uparrow/\downarrow$ for U^0 were an isomorphism, this wouldn't be: Id_U contains more data than 1-1-Corr.

Fibrancy of type-formers

We lift all the type-formers from U^0 to U by higher coinduction. E.g. for Σ-types:

$$\begin{array}{ccc} \sum_{(A:\mathsf{U})}(A \to \mathsf{U}) & \overset{\Sigma}{\dashrightarrow} & \mathsf{U} \\ \downarrow & & \downarrow \\ \sum_{(A:\mathsf{U}^0)}(A \to \mathsf{U}^0) & \xrightarrow{\Sigma^0} & \mathsf{U}^0 \end{array}$$

We must show that:

- Σ takes identifications to one-to-one correspondences.
- These correspondences are isomorphic to some Σ-type.

Strictifying identity types

This amounts to specifying the computation rules for ap_Σ and Id_Σ:

$$\mathsf{ap}_{X.Y.\sum_{(x:X)}Y(x)}(A_2, B_2) \equiv (\mathsf{Id}^{A_2,B_2}_{X.Y.\sum_{(x:X)}Y(x)},$$  $,$  $)$

$$\mathsf{Id}^{\varrho}_{\Delta.\sum_{(x:A)}B}(s,t) \equiv \sum_{(q:\mathsf{Id}^{\varrho}_{\Delta.A}(\pi_1 s, \pi_1 t))} \mathsf{Id}^{\varrho,q}_{(\Delta,x:A).B}(\pi_2 s, \pi_2 t)$$

such that the latter equality holds up to isomorphism for powers $(\square^1 \pitchfork -)$ in $\mathcal{E}^{\square^{\mathrm{op}}}$.

- This works because the identity types of a Σ-type are another Σ-type (and similarly for all other type-formers).
- This is the coherence theorem strictifying $\mathsf{Id}_\Sigma \cong$  to a definitional equality.

Conclusion: cubical universes

Theorem-in-progress

H.O.T.T. has a model in $\mathcal{E}^{\square^{\mathrm{op}}}$, for any presheaf topos $\mathcal{E}$.
In particular, it has a model in $\widehat{\square}$.

Conjecture

By gluing with a global-sections or nerve functor valued in $\widehat{\square}$ or presheaves thereof, we can prove canonicity and normalization.

Note that

1. We must have symmetry in $\square$, to interpret Id_Π and Id_U.
2. We must have symmetry in syntax, for the nerve to lie in $\widehat{\square}$.

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Towards higher topos models

Symmetry solves syntactic problems, but creates semantic ones:
The "syntax-like" model in $\widehat{\square}$ doesn't present classical homotopy.

There could be an equivariant version. But there's another way.

Two approaches to defining higher homotopical structures:

1. As diagrams of sets
   - E.g. quasicategories
   - More parsimonious
2. As diagrams of spaces
   - E.g. complete Segal spaces
   - Often better-behaved

Cubical spaces

Let $\mathcal{E}$ be a type-theoretic model presheaf topos, e.g.:

- $\mathcal{E} =$ sSet, simplicial sets, with the Kan model structure (presents the homotopy theory of spaces).
- $\mathcal{E} =$ simplicial presheaves, with a left exact localization of the injective model structure (presents an $(\infty, 1)$-topos).

Theorem (cf. Rezk–Schwede–Shipley for the simplicial version)

The injective model structure on $\mathcal{E}^{\square^{\mathrm{op}}}$ admits a left Bousfield localization, called the realization model structure, such that:

1. *It is Quillen equivalent to $\mathcal{E}$.*
2. *It is also a type-theoretic model topos.*
   (Though not a left exact localization of the injective one.)

The universe of realization fibrations

Theorem

If $\mathsf{U}^{0,\mathsf{rlz}}$ *classifies realization fibrations, and* $\mathsf{U}^{\mathsf{rlz}} = \lim_n \mathsf{U}^{n,\mathsf{rlz}}$ *as before, there is a trivial fibration*

$$(\square^1 \pitchfork \mathsf{U}^{\mathsf{rlz}}) \xrightarrow{\sim} \textstyle\sum_{(A,B:\mathsf{U}^{\mathsf{rlz}})} \text{1-1-Corr}(A,B).$$

Corollary

The realization model structure interprets all of H.O.T.T.
Thus, H.O.T.T. has models in all Grothendieck $(\infty,1)$*-toposes.*

Why Id_U has no η-rule

1. $\mathsf{Id}_{\mathsf{U}^0}(A, B)$ is not isomorphic to $A \to B \to \mathsf{U}^0$.
2. Id_U contains higher destructors in addition to 1-1-Corr.
3. Injective fibration structures over a cylinder contain more data than those on a span.
4. Homotopical constancy structures over a cylinder contain more data than those on a span.
5. Syntactically, Id_U must contain additional sym data.

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What kind of type is the universe?

Traditionally, the universe is thought of (informally) as inductively defined, with constructors $\Sigma, \Pi, \ldots$, and Tarski eliminator El defined by recursion.

- Not true internally, but informs “meaning explanations” and inductive-recursive universe constructions.
- An observational Id would also be defined by recursion over these constructors, with clauses for Id_Σ, Id_Π, etc.

But:

- It’s hard (not impossible) to make an inductively defined universe univalent.
- Suggests a closed universe, which has to be redefined whenever we add new type formers.

Co-meaning explanations

We instead consider U (still informally) to be coinductively defined.

- Now El and Id are destructors.
- Each type former Σ, Π, . . . is defined by corecursion, specifying its elements, and its identity types "of the same class". E.g. Σ is a corecursive function $\left(\sum_{(A:\mathsf{U})}(A \to \mathsf{U})\right) \to \mathsf{U}$, which makes sense because Id_{Σ} is another Σ-type.
- Explains an open universe: can introduce new type formers without redefining U, just applying its corecursion principle.
- The semantic universe of fibrant types is higher coinductive.

This gives a *philosophical* reason for the "coinductive" behavior of Id_{U}, having β but no η.

Back to Bishop

Recall Bishop's dicta:

> *A set is defined by describing exactly what must be done in order to construct an element of the set and what must be done in order to show that two elements are equal.*
>
> *An operation f from A into B is called a function if whenever $a, a' \in A$ and $a = a'$, we have $f(a) = f(a')$.*

Coinductive synthetic ∞-groupoids

Under propositions-as-types, this naturally becomes coinductive:

> *A type is defined by describing exactly what must be done in order to construct an element of the type and defining a type of ways to identify any two such elements.*
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> *An operation f from A into B is called a function if for $a, a' : A$ we have a function from $a = a'$ to $f(a) = f(a')$.*

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If we augment it with a bit of univalence:

> *We define a type U whose elements are types, where two types are identified by a one-to-one correspondence.*
>
> *Every element of every type is identified with itself. For a type $A : \mathsf{U}$, this yields its own type of identifications.*

We get a philosophical vision that leads ineluctably to H.O.T.T., as a theory of coinductive ∞-groupoids.