Homotopy type theory

Synthetic ∞ -groupoids

Homotopy type theory: towards Grothendieck's dream

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Category theory

Topos theory



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Homotopy type theory?



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Category theory

Outline

Grothendieck's problem

2 Homotopy type theory

3 The world of synthetic ∞ -groupoids

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The homotopy hypothesis

... the study of n-truncated homotopy types (of semisimplicial sets, or of topological spaces) [should be] essentially equivalent to the study of so-called n-groupoids.... This is expected to be achieved by associating to any space (say) X its "fundamental n-groupoid" $\Pi_n(X)$ The obvious idea is that 0-objects of $\Pi_n(X)$ should be the points of X, 1-objects should be "homotopies" or paths between points, 2-objects should be homotopies between 1-objects, etc.

- Grothendieck, 1983

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Category theory

Some definitions of higher categories

- 1957: Kan (∞ -groupoids)
- 1983: Grothendieck (∞ -groupoids)
- 1987: Street (ω-categories)
- 1995: Tamsamani-Simpson (*n*-categories)
- 1997: Joyal (ω-categories)
- 1997: Baez-Dolan, Hermida-Makkai-Power (ω -categories)
- 1998: Batanin, Leinster (ω-categories)
- 1999: Penon (ω-categories)
- 1999: Trimble (*n*-categories)
- 2007: Moerdijk-Weiss (n-categories)
- 2009: Barwick $((\infty, n)$ -categories)
- 2009: Rezk $((\infty, n)$ -categories)
- 2010: Maltsiniotis (ω-categories)
- 2012: Ara $((\infty, n)$ -categories)

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Category theory

Some definitions of higher categories

- 1957: Kan (∞ -groupoids)
- 1983: Grothendieck (∞ -groupoids) \leftarrow algebraic, globular
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Styles of ∞ -groupoid

Following Kan, the nonalgebraic, simplicial definitions...

- satisfy the homotopy hypothesis.
- support a rich theory.
- have lots of applications.

So why should we care about algebraic, globular ones?

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The homotopy hypothesis



(image by Leinster, 2010)

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Discs versus morphisms

There is a fundamental mismatch:

- In homotopy theory, spaces are glued together from discs.
- A classical ∞ -groupoid is a structured collection of morphisms.

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Discs versus morphisms

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- In homotopy theory, spaces are glued together from discs.
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What's the difference?

- The 2-sphere S^2 can be built from a point and a 2-disc, but has infinitely many higher morphisms (e.g. $\pi_3(S^2) = \mathbb{Z}$).
- The Eilenberg–Mac Lane space K(ℤ₂, 2) has only one nontrivial 2-morphism, but requires infinitely many discs.

Small examples

I like to write down and work with small examples, but...it seems nearly impossible to write down explicitly any quasicategory that isn't actually the nerve of a category.

– me, 2010

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[This] is a tall order: a formalism which would allow you to compute easily with small examples should give you a recipe for calculating unstable homotopy groups of spheres. (Of course, this depends on exactly what you mean by "small.")

- Lurie, 2010

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Homotopy type theory

Homotopy type theory is a synthetic theory of ∞ -groupoids.

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Synthetic versus analytic

synthetic

analytic

Euclid's geometry

homotopy type theory

geometry in \mathbb{R}^2

 ∞ -groupoids à la Kan, Grothendieck, Batanin, ...





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Why synthetic? I

"An ∞ -groupoid A has a collection of objects, with for each pair of objects a and b an ∞ -groupoid $\underline{A}(a, b)$, plus operations..."

- 1 Analytically, difficult to make sense of as a definition.
- 2 Synthetically, makes perfect sense as an axiom.

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Category theory

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 Analytically, difficult to make sense of as a definition.
 Synthetically, makes perfect sense as an axiom. ...if our ∞-groupoids are globular.

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Why synthetic? II

Everything is an ∞ -functor \Longrightarrow

we cannot distinguish between equivalent objects!

If we stay within the rules of the synthetic theory, everything automatically satisfies the "principle of equivalence".

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Why synthetic? III

What is a natural equivalence between ∞ -functors $f, g : A \rightarrow B$?

- 1 Analytically:
 - for each object $a \in A$, a morphism $\gamma_a : fa \rightarrow ga$,
 - for each morphism $\phi: a \to a'$, a 2-morphism $\gamma_{a'}.f \phi \to g \phi. \gamma_a$,
 - for each 2-morphism $\mu:\phi \to \phi'$, a 3-morphism . . .
 - . . .

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2 Synthetically:

• for each $a \in A$, a morphism $\gamma_a : fa \rightarrow ga$.

"for each" always means "an ∞ -functor assigning to each".



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Rethinking the homotopy hypothesis

Old

The (analytic) homotopy theory of ∞ -groupoids is equivalent to that of spaces.

- A theorem for some analytic definitions of ∞ -groupoid.
- A desideratum for all of them.

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Rethinking the homotopy hypothesis

Old

The (analytic) homotopy theory of ∞ -groupoids is equivalent to that of spaces.

- A theorem for some analytic definitions of ∞ -groupoid.
- A desideratum for all of them.

New

The synthetic theory of ∞ -groupoids is modeled by spaces (but also by lots of other things).

• A theorem (Voevodsky, Awodey–Warren, Lumsdaine–S.).

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Category theory

Discs versus morphisms, revisited

- An analytic ∞ -groupoid is defined by
 - for objects a, b, a set of morphisms $a \rightarrow b$;
 - for each $f, g: a \rightarrow b$, a set of 2-morphisms $f \rightarrow g$;
 - for each $u, v : f \to g$, a set of 3-morphisms $u \to v$;
 - . . .
- A synthetic ∞ -groupoid A has
 - for objects a, b, an ∞ -groupoid $\underline{A}(a, b)$ of morphisms $a \rightarrow b$. The 2-morphisms are the morphisms of $\underline{A}(a, b)$, etc.

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Category theory

Discs versus morphisms, revisited

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The synthetic ones...

1 ... are not "put together" out of sets!

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The synthetic ones...

- 1 ... are not "put together" out of sets!
- 2 ... can't be put together any old way have to follow rules.

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Example: Univalence

The synthetic $\infty\text{-}\mathsf{groupoid}\ \infty\text{-}\mathsf{Gpd}$ has

- Objects: (small) (synthetic) ∞ -groupoids A, B, \ldots
- Morphisms: <u>∞-Gpd</u>(A, B) = the ∞-groupoid of equivalences of ∞-groupoids A ≃ B.

The rule which allows us to form ∞ -Gpd is the univalence axiom (Voevodsky).

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The rule which allows us to form ∞ -Gpd is the univalence axiom (Voevodsky).

Recall: everything is an ∞ -functor \implies we cannot distinguish between equivalent ∞ -groupoids!

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Discs are free generators

- The classical space S^2 is built from
 - a 0-disc and
 - a 2-disc.
- The classical ∞ -groupoid $\Pi_\infty(S^2)$ has
 - one 0-morphism,
 - one 1-morphism,
 - a \mathbb{Z} worth of 2-morphisms,
 - a \mathbb{Z} worth of 3-morphisms,
 - a \mathbb{Z}_2 worth of 4-morphisms,
 - ...

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 - a \mathbb{Z}_2 worth of 4-morphisms,
 - ...
- The synthetic ∞ -groupoid S^2 is freely generated by
 - an object $b \in S^2$, and
 - a 2-morphism $s \in \underline{\underline{S}^2(b,b)}(1_b,1_b)$.

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More generators

The rule of higher inductive types allows us to form the synthetic ∞ -groupoid freely generated by any reasonable "generators".

- The torus T^2 is freely generated by $b \in T^2$ and $p, q \in \underline{T^2}(b, b)$ and $s \in \underline{\underline{T^2}(b, b)}(pq, qp)$.
- The coproduct A + B is freely generated by functors $i : A \rightarrow A + B$ and $j : B \rightarrow A + B$.
- 1 is freely generated by an object $\star \in 1$.
- \emptyset is freely generated by no generators.
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- \emptyset is freely generated by no generators.

Relations are just higher generators: all presentations are free.

The pushout A +_C B is freely generated by functors
 i : A → A +_C B and j : B → A +_C B and a natural equivalence between the two functors C → A +_C B.

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Recursive generators

- \mathbb{N} is freely generated by $0 \in \mathbb{N}$ and a functor $s : \mathbb{N} \to \mathbb{N}$.
- The free monoid on A is freely generated by e ∈ LA and a functor A × LA → LA.

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Category theory

Truncation by generators

- The classical space $K(\mathbb{Z},2)$ is built from
 - one 0-disc and one 2-disc, as for S^2 ;
 - enough 4-discs to kill off the undesired 3-morphisms;
 - enough 5-discs to kill off the undesired 4-morphisms;
 - enough 6-discs to kill off the undesired 5-morphisms;
 - . . .
- The analytic ∞ -groupoid $\Pi_{\infty}(K(\mathbb{Z},2))$ has
 - one 0-morphism and one 1-morphism;
 - a \mathbb{Z} worth of 2-morphisms;
 - only identity k-morphisms for $k \ge 3$.

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- The synthetic ∞ -groupoid $K(\mathbb{Z},2)$ is freely generated by
 - an object $b \in K(\mathbb{Z},2)$;
 - a 2-morphism $s \in \underline{K(\mathbb{Z},2)}(b,b)(1_b,1_b);$
 - for every parallel pair of 3-morphisms f, g in $K(\mathbb{Z}, 2)$, a 4-morphism from f to g.

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Aside: the Yoneda lemma

For each *a*, the hom-functor $\underline{A}(a, -) : A \to \infty$ -Gpd is freely generated by the identity $1_a \in \underline{A}(a, a)$.

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Remarks:

1 This is a definition of $\underline{A}(a, b)$.

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- **1** This is a definition of $\underline{A}(a, b)$.
- 2 It implies all the composition and coherence structure of an ∞ -groupoid. (Lumsdaine, Garner–van den Berg)

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- S We get essentially Grothendieck's definition of ∞-groupoid! (Brunerie)

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Progress in synthetic homotopy theory

- $\pi_1(S^1) = \mathbb{Z}$ (S., Licata)
- $\pi_k(S^n) = 0$ for k < n (Brunerie, Licata)
- $\pi_n(S^n) = \mathbb{Z}$ (Licata, Brunerie)
- The long exact sequence of a fibration (Voevodsky)
- The Hopf fibration and $\pi_3(S^2) = \mathbb{Z}$ (Lumsdaine, Brunerie)

•
$$\pi_4(S^3) = \mathbb{Z}_2$$
 (Brunerie – almost)

- The Freudenthal suspension theorem (Lumsdaine)
- The Blakers–Massey theorem (Lumsdaine, Finster, Licata)
- The van Kampen theorem (S.)
- Whitehead's theorem for *n*-types (Licata)
- Covering space theory (Hou)

Some of these are new proofs.

Grothendieck's problem $\underline{\pi_1(S^1)} = \mathbb{Z}$

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 S^1 is freely generated by $b \in S^1$ and $\ell \in \underline{S^1}(b, b)$.

Theorem (S.) $\underline{S^1}(b,b) \simeq \mathbb{Z}.$

Proof (Licata).

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Define $f : \mathbb{Z} \to \underline{S^1}(b, b)$ by $f(n) = \ell^n$.

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$G(b) = \mathbb{Z}$	$\in \infty ext{-}Gpd$
$G(\ell) = \operatorname{succ}$	$\in \infty ext{-}Gpd(\mathbb{Z},\mathbb{Z})$

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$G(b) = \mathbb{Z}$	$\in \infty ext{-}Gpd$	
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Prove $g \circ f = 1_{\mathbb{Z}}$ by induction.

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 S^1 is freely generated by $b \in S^1$ and $\ell \in \underline{S^1}(b, b)$.

Theorem (S.) $\underline{S^1}(b,b) \simeq \mathbb{Z}.$

Proof (Licata). Define $f : \mathbb{Z} \to \underline{S^1}(b, b)$ by $f(n) = \ell^n$. Define $g : \underline{S^1}(b, b) \to \mathbb{Z}$ by $g(\alpha) = G(\alpha)(0)$, for $G : S^1 \to \infty$ -Gpd $G(b) = \mathbb{Z} \qquad \in \infty$ -Gpd $G(\ell) = \text{succ} \qquad \in \infty$ -Gpd(\mathbb{Z}, \mathbb{Z}).

Prove $g \circ f = 1_{\mathbb{Z}}$ by induction.

Prove $f \circ g = 1_{\underline{S^1}(b,b)}$ by ...?

Grothendieck's problem $\pi_1(S^1) = \mathbb{Z}$

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 S^1 is freely generated by $b \in S^1$ and $\ell \in \underline{S^1}(b, b)$.

Theorem (S.) $\underline{S^1}(b,b) \simeq \mathbb{Z}.$

Proof (Licata).

Generalize f and g to

$$egin{aligned} & f_x: G(x) o \underline{S^1}(b,x) & ext{ for all } x \in S^1 \ & g_x: \underline{S^1}(b,x) o G(x) & ext{ for all } x \in S^1 \end{aligned}$$

and prove $f_x(g_x(p)) = p$ for all $x \in S^1$ and $p \in \underline{S^1}(b, x)$.

Grothendieck's problem $\pi_1(S^1) = \mathbb{Z}$

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and prove $f_x(g_x(p)) = p$ for all $x \in S^1$ and $p \in \underline{S^1}(b, x)$.

But $\underline{\infty\text{-}\mathsf{Gpd}}(b, -)$ is freely generated by 1_b , so it suffices to check $f_b(g_b(1_b)) = f(g(1_b)) = f(0) = 1_b$.

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synthetic	analytic
Euclid's geometry	geometry in \mathbb{R}^2
homotopy type theory	∞-groupoids à la Kan, Grothendieck, Batanin,
set theory $\begin{cases} ZFC \\ ETCS \leftarrow \end{cases}$	constructs the category of sets models

Homotopy type theory

Synthetic ∞ -groupoids 00000000

Category theory •00000000



Homotopy type theory

Synthetic ∞ -groupoids 00000000

Category theory •00000000



Grothendieck's problem 000000	Homotopy type theory	Synthetic ∞ -groupoids 00000000	Category theory

Definition

- An ∞-groupoid A is a subsingleton if the two projections A × A ⇒ A are naturally equivalent.
- A is a set if each $\underline{A}(a, b)$ is a subsingleton.

Theorem (Rijke-Spitters)

These sets satisfy (constructive) ETCS.

Grothendieck's problem	Homotopy type theory	Synthetic ∞ -groupoids	Category theory
			00000000

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We can use homotopy type theory as a foundation for mathematics.

Homotopy type theory

Synthetic ∞ -groupoids

Category theory

Mathematics in set theory



Homotopy type theory

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Category theory

Mathematics in homotopy type theory



Synthetic ∞ -groupoids 00000000

Precategories

Definition

- A precategory & consists of
 - **1** An ∞ -groupoid \mathscr{C}_0 of objects.
 - **2** A family $\underline{\mathscr{H}om}_{\mathscr{C}} : \mathscr{C}_0 \times \mathscr{C}_0 \to \infty$ -Gpd.
 - **3** Each $\underline{\mathscr{H}om}_{\mathscr{C}}(x, y)$ is a set (i.e. essentially discrete).
 - **4** Composition, identity, associativity, ...

 $\begin{array}{l} \text{Synthetic ∞-groupoids} \\ \text{00000000} \end{array}$

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Questions:

- Is \mathscr{C}_0 a set?
- How is $\underline{\mathscr{C}_0}(x, y)$ related to $\underline{\mathscr{H}om}_{\mathscr{C}}(x, y)$?

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Category theory 00000000

Categories and strict categories

Definition (Bartels)

A precategory \mathscr{C} is a strict category if \mathscr{C}_0 is a set.

Definition (Voevodsky, Ahrens-Kapulkin-S.)

A precategory \mathscr{C} is a category if $\underline{\mathscr{C}_0}(x, y)$ is equivalent to the subset of isomorphisms in $\underline{\mathscr{H}om}_{\mathscr{C}}(x, y)$.

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Category theory 00000000

Equivalences of categories

"A fully faithful and essentially surjective functor is an equivalence."

- 1 For strict categories: equivalent to the axiom of choice.
- 2 For precategories: just false.
- **3** For categories: just true.

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Univalence for categories

Everything is an ∞ -functor \Longrightarrow

• we cannot distinguish isomorphic objects in a category.

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Category theory

Univalence for categories

Everything is an ∞ -functor \Longrightarrow

- we cannot distinguish isomorphic objects in a category.
- we cannot distinguish equivalent categories!

 $\mathsf{categories}(\mathscr{C},\mathscr{D}) = \mathsf{Equivalences}(\mathscr{C},\mathscr{D})$

Synthetic ∞ -groupoids 00000000

Why ∞ -groupoids?

Why not synthetic ω -categories?

- 1 Harder to axiomatize.
- 2 Harder to work with.
- 3 Synthetic ∞ -groupoids give us most of what we want.

Why ∞ -groupoids?

Why not synthetic ω -categories?

- **1** Harder to axiomatize.
- 2 Harder to work with.
- **3** Synthetic ∞ -groupoids give us most of what we want.
- There are other important higher-categorical structures! (*n*-fold categories, equipments, enriched categories, multicategories, fibrations, ...)

 $\infty\mbox{-}{\rm groupoids}$ are the raw material of higher-dimensionality.

... the set-based mathematics we know and love is just the tip of an immense iceberg of n-categorical, and ultimately ω -categorical, mathematics.... The basic philosophy is simple: never mistake equivalence for equality. The technical details, however, are not so simple — at least not yet. To proceed efficiently it is crucial that we gain a clearer understanding of the foundations ...

– Baez and Dolan, 1998

[Homotopy type theory is] a new conception of foundations of mathematics, with intrinsic homotopical content, [and] an "invariant" conception of the objects of mathematics... – The HoTT Book, 2013

http://homotopytypetheory.org/book