All $(\infty, 1)$-toposes have strict univalent universes

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One model is not enough

A (Grothendieck–Rezk–Lurie) $(\infty, 1)$-topos is:

- The category of objects obtained by "homotopically gluing together" copies of some collection of "model objects" in specified ways.
- The free cocompletion of a small $(\infty, 1)$-category preserving certain well-behaved colimits.
- An accessible left exact localization of an $(\infty, 1)$-category of presheaves.

They are a powerful tool for studying all kinds of "geometry" (topological, algebraic, differential, cohesive, etc.).

It has long been expected that $(\infty, 1)$-toposes are models of HoTT, but coherence problems have proven difficult to overcome.

Main Theorem

Theorem (S.)

Every $(\infty, 1)$-topos can be given the structure of a model of "Book" HoTT with strict univalent universes, closed under Σs, Πs, coproducts, and identity types.

Caveats for experts:

1. Classical metatheory: ZFC with inaccessible cardinals.
2. We assume the initiality principle.
3. Only an interpretation, not an equivalence.
4. HITs also exist, but remains to show universes are closed under them.

Towards killer apps

Example

1. Hou–Finster–Licata–Lumsdaine formalized a proof of the Blakers–Massey theorem in HoTT.
2. Later, Rezk and Anel–Biedermann–Finster–Joyal unwound this manually into a new $(\infty, 1)$-topos-theoretic proof, with a generalization applicable to Goodwillie calculus.
3. We can now say that the HFLL proof already implies the $(\infty, 1)$-topos-theoretic result, without manual translation. (Modulo closure under HITs.)

Outline

Review of model-categorical semantics

We can interpret type theory in a well-behaved model category $\mathscr{E}$:

Type theory	Model category
Type $\Gamma \vdash A$	Fibration $\Gamma.A \twoheadrightarrow \Gamma$
Term $\Gamma \vdash a : A$	Section $\Gamma \to \Gamma.A$ over Γ
Id-type	Path object
$\vdots$	$\vdots$
Universe	Generic small fibration $\pi : \widetilde{U} \twoheadrightarrow U$

To ensure U is closed under the type-forming operations, we choose it so that every fibration with "κ-small fibers" is a pullback of π, where κ is some inaccessible cardinal.

Universes in presheaves

Let $\mathscr{E} = [\mathcal{C}^{\mathrm{op}}, \mathsf{Set}]$ be a presheaf model category.

Definition

Define a presheaf $U \in \mathscr{E} = [\mathcal{C}^{\mathrm{op}}, \mathsf{Set}]$ where

$$U(c) = \left\{ \kappa\text{-small fibrations over } よc = \mathcal{C}(-, c) \right\}$$

with functorial action by pullback along $よ\gamma : よc_1 \to よc_2$.
(Plus standard cleverness to make it strictly functorial.)

Similarly, define $\widetilde{U}$ using fibrations equipped with a section.
We have a κ-small map $\pi : \widetilde{U} \to U$.

Theorem

Every κ-small fibration is a pullback of π.

But π may not itself be a fibration!

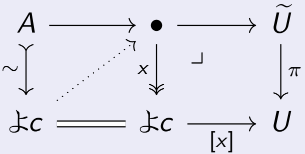
Universes via representability

Theorem

If the generating acyclic cofibrations in $\mathscr{E} = [\mathcal{C}^{\mathrm{op}}, \mathsf{Set}]$ *have representable codomains, then* $\pi : \widetilde{U} \to U$ *is a fibration.*

Proof.

To lift in the outer rectangle, instead lift in the left square.



□

Example (Voevodsky)

In simplicial sets, the generating acyclic cofibrations are $\Lambda^{n,k} \to \Delta^n$, where Δ^n is representable.

Universes via structure

In cubical sets, the fibrations have a *uniform choice* of liftings against generators $\sqcap^{n,k} \to \square^n$. Since $\square^n$ is representable, our π lifts against these generators, but not uniformly.

Instead one defines (BCH, CCHM, ABCFHL, etc.)

$$U(c) = \left\{ \text{small fibrations over } \text{よ}c \text{ with specified uniform lifts} \right\}.$$

Then the lifts against the generators $\sqcap^{n,k} \to \square^n$ cohere under pullback, giving π also a uniform choice of lifts.

Let's put this in an abstract context.

Notions of fibred structure

Definition

A notion of fibred structure $\mathbb{F}$ on a category $\mathscr{E}$ assigns to each morphism $f : X \to Y$ a set (perhaps empty) of "$\mathbb{F}$-structures", which vary functorially in pullback squares: given a pullback

$$\begin{array}{ccc} X' & \longrightarrow & X \\ {\scriptstyle f'}\downarrow & \lrcorner & \downarrow{\scriptstyle f} \\ Y' & \longrightarrow & Y \end{array}$$

any $\mathbb{F}$-structure on f induces one on f', functorially.

Definition

A notion of fibred structure $\mathbb{F}$ is locally representable if for any $f : X \to Y$, the functor $\mathscr{E}/Y \to \mathrm{Set}$, sending $g : Z \to Y$ to the set of $\mathbb{F}$-structures on $g^*X \to Z$, is representable.

Notions of fibration structure

Examples

The following notions of fibred structure on a map $f : X \to Y$ are locally representable:

1. The property of lifting against a set of maps with representable codomains (e.g. simplicial sets).
2. The *structure* of liftings against a *category* of maps with representable codomains (e.g. as in Emily's talk).
3. A G_Y-algebra structure for a fibred pointed endofunctor G (e.g. the partial map classifier, as in Steve's talk).
4. A section of $F_Y(X)$, for any fibred endofunctor F.
5. The combination of two or more locally representable notions of fibred structure.
6. The property of having κ-small fibers.
7. A square exhibiting f as a pullback of some $\pi : \widetilde{U} \to U$.

Universes from fibration structures

For a notion of fibred structure $\mathbb{F}$, define

$$U(c) = \left\{ \text{small maps into } \text{よ}c \text{ with specified } \mathbb{F}\text{-structures} \right\}.$$

and similarly $\pi : \widetilde{U} \to U$.

Theorem

If $\mathbb{F}$ is locally representable, then π also has an $\mathbb{F}$-structure, and every $\mathbb{F}$-structured map is a pullback of it.

Proof.

Write U as a colimit of representables. All the coprojections factor coherently through the representing object for $\mathbb{F}$-structures on π, so the latter has a section. □

(Can also use the representing object for $\mathbb{F}$-structures on the classifier $\widetilde{V} \to V$ of all κ-small morphisms, as Steve did yesterday.)

Type-theoretic model toposes

Definition (S.)

A type-theoretic model topos is a model category $\mathscr{E}$ such that:

- $\mathscr{E}$ is a right proper Cisinski model category.
- $\mathscr{E}$ has a well-behaved, locally representable, notion of fibred structure $\mathbb{F}$ such that the maps admitting an $\mathbb{F}$-structure are precisely the fibrations.
- $\mathscr{E}$ has a well-behaved enrichment (e.g. over simplicial sets).

It is not hard to show:

1. Every type-theoretic model topos interprets Book HoTT with univalent universes. (FEP+EEP $\Rightarrow$ U is fibrant and univalent.)
2. The $(\infty, 1)$-category presented by a type-theoretic model topos is a Grothendieck $(\infty, 1)$-topos. (It satisfies Rezk descent.)

The hard part is the converse of (2): are there enough ttmts?

The Plan

An $(\infty, 1)$-topos is, by one definition, an accessible left exact localization of a presheaf $(\infty, 1)$-category. Thus it will suffice to:

1. Show that simplicial sets are a type-theoretic model topos. ✓
2. Show that type-theoretic model toposes are closed under passage to presheaves.
3. Show that type-theoretic model toposes are closed under accessible left exact localizations.

We take the last two in reverse order.

Outline

Localization

Let S be a set of morphisms in a type-theoretic model topos $\mathscr{E}$.

Definition

A fibrant object $Z \in \mathscr{E}$ is (internally) S-local if $Z^f : Z^B \to Z^A$ is an equivalence in $\mathscr{E}$ for all $f : A \to B$ in S.

These are the fibrant objects of a left Bousfield localization model structure $\mathcal{L}_S\mathscr{E}$ on the same underlying category $\mathscr{E}$. It is left exact if fibrant replacement in $\mathcal{L}_S\mathscr{E}$ preserves homotopy pullbacks in $\mathscr{E}$.

Example

If $\mathscr{E} = [\mathcal{C}^{\mathrm{op}}, \mathrm{Set}]$ and $\mathscr{C}$ is a site with covering sieves $R \rightarrowtail よc$, then Z^R is the object of local/descent data. Thus the local objects are the sheaves/stacks.

Left exact localizations as type-theoretic model toposes

Lemma

There is a loc. rep. notion of fibred structure whose $\mathbb{F}_S$-structured maps are the fibrations $X \to Y$ that are S-local in $\mathscr{E}/Y$.

Sketch of proof.

Define $\mathsf{isLocal}_S(X)$ using the internal type theory, and let an $\mathbb{F}_S$-structure be an $\mathbb{F}$-structure and a section of $\mathsf{isLocal}_S(X)$. □

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Theorem

If S-localization is left exact, $\mathcal{L}_S\mathscr{E}$ is a type-theoretic model topos.

Sketch of proof.

Using Rijke–S.–Spitters and Anel–Biedermann–Finster–Joyal (forthcoming), if we close S under homotopy diagonals, the above $\mathbb{F}_S$-structured maps also coincide with the fibrations in $\mathcal{L}_S\mathscr{E}$. □

Outline

Warnings about presheaf model structures

$\mathscr{E}$ = a type-theoretic model topos, $\mathscr{D}$ = a small (enriched) category, $[\mathscr{D}^{\mathrm{op}}, \mathscr{E}]$ = the presheaf category.

Warning #1

It's essential that we allow presheaves over $(\infty, 1)$-categories (e.g. simplicially enriched categories) rather than just 1-categories. But for simplicity here, let's assume $\mathscr{D}$ is unenriched.

Warning #2

In cubical cases, $[\mathscr{D}^{\mathrm{op}}, \mathscr{E}]$ has an "intrinsic" cubical-type model structure, which (when $\mathscr{D}$ is unenriched) coincides with the ordinary cubical model constructed in the internal logic of $[\mathscr{D}^{\mathrm{op}}, \mathsf{Set}]$. However, this generally does not present the correct $(\infty, 1)$-presheaf category, as discussed by Thierry yesterday.

Injective model structures

Theorem

The category $[\mathscr{D}^{\mathrm{op}}, \mathscr{E}]$ of presheaves has an injective model structure such that:

1. *The weak equivalences and cofibrations are pointwise.*
2. *It is right proper and Cisinski.*
3. *It presents the corresponding presheaf $(\infty, 1)$-category.*

Thus it lacks only a suitable notion of fibred structure to be a type-theoretic model topos.

Injective model structures

Theorem

The category $[\mathscr{D}^{\mathrm{op}}, \mathscr{E}]$ of presheaves has an injective model structure such that:

1. *The weak equivalences and cofibrations are pointwise.*
   - *The fibrations are . . . ?????*
2. *It is right proper and Cisinski.*
3. *It presents the corresponding presheaf $(\infty, 1)$-category.*

Thus it lacks only a suitable notion of fibred structure to be a type-theoretic model topos.

Why pointwise isn't enough

When is $X \in [\mathscr{D}^{\mathrm{op}}, \mathscr{E}]$ injectively fibrant? We want to lift in

$$\begin{array}{ccc} A & \xrightarrow{g} & X \\ {\scriptstyle i}\big\downarrow{\scriptstyle \sim} & \nearrow & \\ B & & \end{array}$$

where $i : A \to B$ is a pointwise acyclic cofibration.
If X is pointwise fibrant, then for all $d \in \mathscr{D}$ we have a lift

$$\begin{array}{ccc} A_d & \xrightarrow{g_d} & X_d. \\ {\scriptstyle i_d}\big\downarrow{\scriptstyle \sim} & \nearrow{\scriptstyle h_d} & \\ B_d & & \end{array}$$

These may not fit together into a natural transformation $B \to X$, but they do form a homotopy coherent natural transformation.

The coherent morphism coclassifier

Lemma

The notion of coherent natural transformation is representable. That is, there is a coherent transformation coclassifier $\mathsf{C}^{\mathscr{D}}(Y)$ *(classically called the cobar construction) with a natural bijection*

$$\frac{h : X \rightsquigarrow Y}{\overline{h} : X \to \mathsf{C}^{\mathscr{D}}(Y)}$$

- The (strictly natural) identity $X \rightsquigarrow X$ corresponds to a canonical map $\nu_X : X \to \mathsf{C}^{\mathscr{D}}(X)$.
- ν_X is always a pointwise acyclic cofibration!

Injective fibrancy

Theorem (S.)

$X \in [\mathscr{D}^{\mathrm{op}}, \mathscr{E}]$ is injectively fibrant if and only if it is pointwise fibrant and $\nu_X : X \to \mathsf{C}^{\mathscr{D}}(X)$ has a retraction $r : \mathsf{C}^{\mathscr{D}}(X) \to X$.

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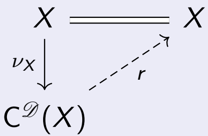
Proof of "only if".

If $X \in [\mathscr{D}^{\mathrm{op}}, \mathscr{E}]$ is injectively fibrant, then since ν_X is a pointwise acyclic cofibration we have a lift:



□

Injective fibrancy

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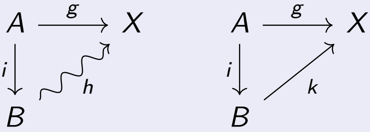
Proof of "if".

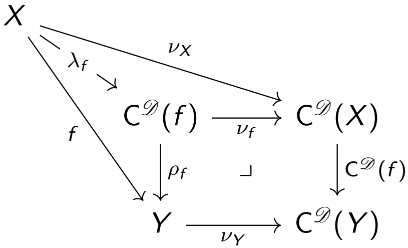
Given a pointwise acyclic cofibration $i : A \to B$ and a map $g : A \to X$, we construct a coherent $h : B \rightsquigarrow X$ with $h \circ i = g$.



We have $\overline{h} : B \to \mathsf{C}^{\mathscr{D}}(X)$; define $k = r \circ \overline{h} : B \to X$. Since $h \circ i = g$ is strict, $\overline{h} \circ i = \nu_X \circ g$, and $k \circ i = r \circ \overline{h} \circ i = r \circ \nu_X \circ g = g$. □

Injective fibrations

Given $f : X \to Y$, define a factorization by pullback:



Theorem (S.)

$f : X \to Y$ is an injective fibration if and only if it is a pointwise fibration and λ_f has a retraction $r : \mathsf{C}^{\mathscr{D}}(f) \to X$ over Y.

A notion of injective fibration structure

Note $\mathsf{C}^{\mathscr{D}}$ is a fibred pointed endofunctor of $[\mathscr{D}^{\mathrm{op}}, \mathscr{E}]$. Thus, if we define an $\mathbb{F}^{\mathscr{D}}$-structure to be a pointwise $\mathbb{F}$-structure and a $\mathsf{C}^{\mathscr{D}}$-algebra structure, we get a locally representable notion of fibred structure for the injective fibrations in $[\mathscr{D}^{\mathrm{op}}, \mathscr{E}]$.

Theorem

$[\mathscr{D}^{\mathrm{op}}, \mathscr{E}]$ *is a type-theoretic model topos with* $\mathbb{F}^{\mathscr{D}}$.

This completes the main result.

Outline

Future work

1. Are these universes closed under higher inductive types?
2. Do Grothendieck $(\infty, 1)$-toposes model *cubical* type theory? (Perhaps with cubically enriched type-theoretic model toposes?)
3. How much of this works in a constructive metatheory?
4. What about *elementary* $(\infty, 1)$-toposes? (E.g. by Yoneda?)