Type-Theoretic Model Toposes

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The HoTTEST Conference
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The Internet

The Ecosystem & Our Niche

∞-categories

- Objects of semantic interest
- Everything defined up to equivalence

Coherence theorems

Strict structures that model type theory and present ∞-categories

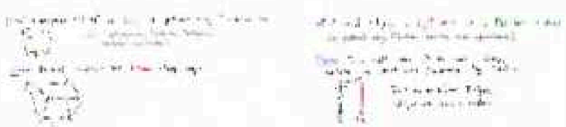
type theory

- Convenient, syntactic, implementable
- Operations defined up to strict equality.

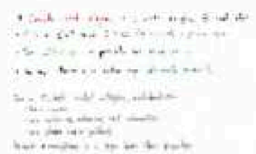
Idea: Type-theoretic model toposes have all the structure necessary to model HoTT (with UA + HITs), but are general enough to present all the desired ∞-categorical models.

A type-theoretic model topos is a

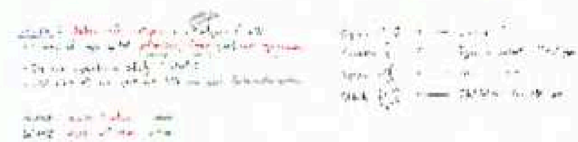
right proper



Cisinski

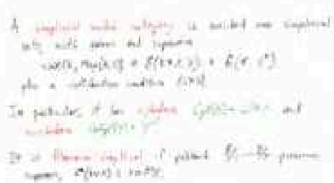


model category



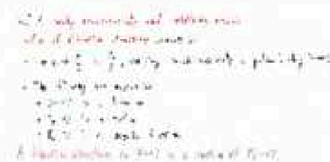
with

fiberwise enrichment



and

structured fibrations.



model category

<u>Definition</u> A Quillen model category is a complete + cocomplete category $\mathcal{E}$ with

- Classes of maps called cofibrations $\rightarrowtail$, fibrations $\twoheadrightarrow$, and weak equivalences $\xrightarrow{\sim}$.
- The weak equivalences satisfy 2-out-of-3.
- $(\mathrm{Cof}, \mathrm{Fib} \cap \mathrm{WE})$ and $(\mathrm{Cof} \cap \mathrm{WE}, \mathrm{Fib})$ are weak factorization systems

$\mathrm{Fib} \cap \mathrm{WE}$ = acyclic fibrations $\overset{\sim}{\twoheadrightarrow}$

$\mathrm{Cof} \cap \mathrm{WE}$ = acyclic cofibrations $\overset{\sim}{\rightarrowtail}$

Objects of $\mathcal{E}$	$\longleftrightarrow$	Contexts Γ
Fibrations $\begin{tikzcd} A \arrow[d, two heads] \\ \Gamma \end{tikzcd}$	$\longleftrightarrow$	Types in context $\Gamma \vdash A$ type
Sections $\begin{tikzcd} A \arrow[d, two heads] \\ \Gamma \arrow[u, bend left, "s"] \end{tikzcd}$	$\longleftrightarrow$	Terms $\Gamma \vdash s : A$
Pullback $\begin{tikzcd} \cdot \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, two heads] \\ \Delta \arrow[r, "\sigma"'] & \Gamma \end{tikzcd}$	$\longleftrightarrow$	Substitution $\Delta \vdash A[\sigma]$ type

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A type-theoretic model topos is a

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Cisinski

model category

with

fiberwise enrichment

and

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Cisinski

A Cisinski model category is a model category $\mathcal{E}$ such that

- $\mathcal{E}$ is a Grothendieck 1-topos. (For us, usually a presheaf topos.)
- The cofibrations are precisely the monomorphisms.
- The weak factorization systems are cofibrantly generated.

In a Cisinski model category, cofibrations

- have unions
- are extensive, adhesive, and exhaustive
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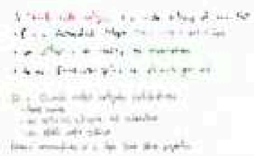

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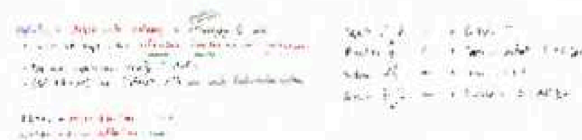

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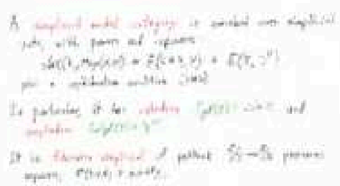

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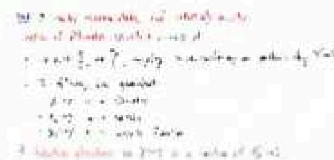

right proper

(Rezk) A morphism $f : A \to B$ is sharp if pullback along f preserves w.e.

$$\begin{tikzcd} f^*C \arrow[r] \arrow[d] & C \arrow[d] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

(a.k.a. "right proper map", "H-fibration", "W-fibration", "fibrillation", "weak fibration")

Theorem: Pullback preserves w.e. between sharp maps.

$$\begin{tikzcd} g^*X \arrow[rr] \arrow[dr] \arrow[ddr] & & X \arrow[dr] \arrow[ddr] & \\ & g^*Y \arrow[rr] \arrow[d] & & Y \arrow[d] \\ & A \arrow[rr, "g"'] & & B \end{tikzcd}$$

Def A model category is right proper if all fibrations are sharp
(i.e. pullback along fibrations preserves weak equivalences)

Theorem In a right proper Cisinski model category, fibrations are closed under pushforward along fibrations.

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d, "ev"'] & \Pi_g C \arrow[dd] \\ C \arrow[d] & \\ B \arrow[r] & A \end{tikzcd}$$

This is how we interpret Π-types.
(Σ-types are composite fibrations)

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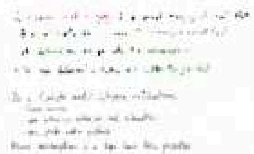

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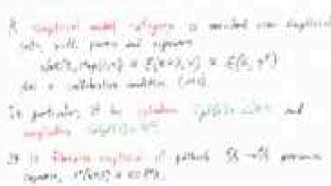

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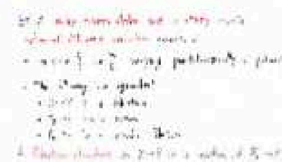

fiberwise enrichment

A simplicial model category is enriched over simplicial sets, with powers and copowers:

$$sSet(K, Map(X,Y)) \cong \mathcal{E}(K \odot X, Y) \cong \mathcal{E}(X, Y^K).$$

plus a co/fibration condition (SM7).

In particular, it has cylinders $Cyl(X) = \Delta^1 \odot X$ and cocylinders $CoCyl(Y) = Y^{\Delta^1}$.

It is fiberwise-simplicial if pullback $\mathcal{E}/Y \to \mathcal{E}/X$ preserves copowers, $f^*(K \odot X) \cong K \odot f^*X$.

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- For each $\begin{smallmatrix} X \\ \downarrow \\ Y \end{smallmatrix}$, an $\begin{smallmatrix} \mathbb{F}_X \\ \downarrow \\ Y \end{smallmatrix}$, varying pseudofunctorially in pullback along $Y' \to Y$.
- The following are equivalent:
  - $X \to Y$ is a fibration
  - $\mathbb{F}_X \to Y$ has a section
  - $\mathbb{F}_X \to Y$ is an acyclic fibration

A fibration structure on $X \to Y$ is a section of $\mathbb{F}_X \to Y$.

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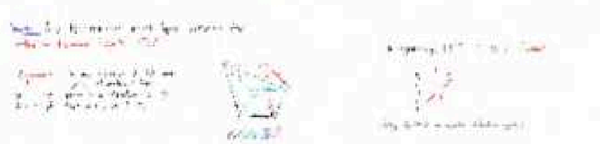

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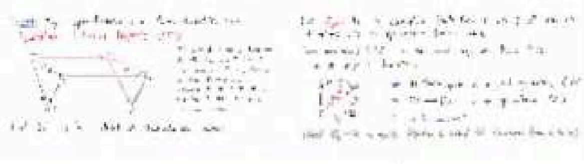

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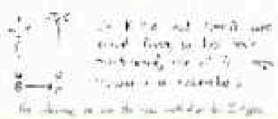

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universe

Suppose for simplicity $\mathcal{E} = [\mathcal{C}^{op}, \mathrm{Set}]$ is a presheaf topos.

We have a universe of all κ-small maps "defined" for $c \in \mathcal{C}$ by

$$\begin{array}{ccl} \tilde{V}_c & = & \{ \kappa\text{-small maps having codomain } よ_c \text{ with a section} \} \\ \downarrow & & \\ V_c & = & \{ \kappa\text{-small maps having codomain } よ_c \} \end{array}$$

(w/ a trick to make it a set & strictly functorial – Hofmann-Streicher, Voevodsky, ...)

<u>Define</u> $\mathcal{U} = \mathbb{F}_{V}$ and

$$\begin{tikzcd} \tilde{\mathcal{U}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \tilde{V} \arrow[d] \\ \mathcal{U} \arrow[r] & V \end{tikzcd}$$

"$\mathcal{E}(X, \mathcal{U})$ = κ-small structured fibrations over X"

<u>Theorem</u> $\tilde{\mathcal{U}} \to \mathcal{U}$ is a (κ-small) fibration.

$$\begin{tikzcd} \tilde{\mathcal{U}} \arrow[rr] \arrow[dd] \arrow[ddrr, phantom, "\lrcorner", very near start] & & \tilde{V} \arrow[dd] & \\ & \mathbb{F}_{\tilde{\mathcal{U}}} = \mathcal{U} \times_V \mathcal{U} \arrow[rr] \arrow[dl] & & \mathbb{F}_V = \mathcal{U} \arrow[dl] \\ \mathcal{U} \arrow[rr] \arrow[ur, bend left, "\Delta"] & & V & \end{tikzcd}$$

<u>Theorem</u> Every κ-small fibration is a pullback of $\tilde{\mathcal{U}} \twoheadrightarrow \mathcal{U}$

$$\begin{tikzcd} X \arrow[rr] \arrow[dd] \arrow[ddrr, phantom, "\lrcorner", very near start] & & \tilde{V} \arrow[dd] & \\ & \mathbb{F}_X \arrow[rr] \arrow[dl] & & \mathbb{F}_V = \mathcal{U} \arrow[dl] \\ Y \arrow[rr] \arrow[ur, bend left] & & V & \end{tikzcd}$$

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<u>Define</u> $U = \mathbb{F}_{\widetilde{V}}$ and

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Theorem For (large enough) inaccessible κ, any type-theoretic model topos has a

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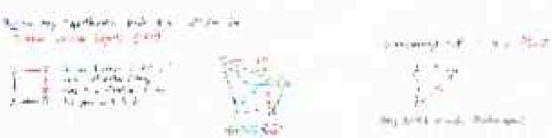

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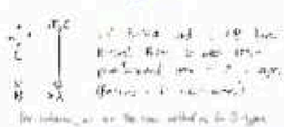

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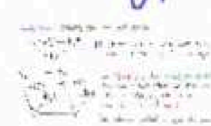

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<u>Theorem</u> Any type-theoretic model topos satisfies the Equivalence Extension Property (EEP).

$$\begin{tikzcd}
D_1 \arrow[rrr] \arrow[dr, "\tilde{v}", "\sim"'] \arrow[ddr, two heads] & & & E_1 \arrow[dr, "\tilde{w}"] \arrow[dd, two heads] & \\
& D_2 \arrow[rrr] \arrow[d, two heads] & & & E_2 \arrow[dl, two heads] \\
& A \arrow[rr, tail] & & B &
\end{tikzcd}$$

If $D_2 \twoheadrightarrow A$ is the pullback of $E_2 \twoheadrightarrow B$ along a cof. $A \rightarrowtail B$, any equivalence $D_1 \xrightarrow{\sim} D_2$ from a fibration $D_1 \twoheadrightarrow A$ can be extended to $E_1 \xrightarrow{\sim} E_2$ for a fibration $E_1 \twoheadrightarrow B$ that pulls back to $D_1 \twoheadrightarrow A$.

<u>Proof</u>: Just like for simplicial sets (Kapulkin-Lumsdaine-Voevodsky)

Let Equiv be the classifier (built from $\mathcal{U}$ + LCCC) of pairs of fibrations with an equivalence between them.

(Size-preserving) EEP $\Rightarrow$ the second projection $\mathrm{Equiv} \xrightarrow{\pi_2} \mathcal{U}$ is an acyclic fibration.

$$\begin{tikzcd}
A \arrow[r, "{(D_1, D_2, v)}"] \arrow[d, tail] & \mathrm{Equiv} \arrow[d, two heads, "\pi_2"] \\
B \arrow[r, "E_2"'] \arrow[ur, "{(E_1, E_2, w)}"] & \mathcal{U}
\end{tikzcd}$$

$\Rightarrow$ $\mathcal{U} \xrightarrow{\mathrm{idequiv}} \mathrm{Equiv}$ is a weak equivalence (2/3)

$\Rightarrow$ $P\mathcal{U} \longrightarrow \mathrm{Equiv}$ is an equivalence (2/3)

i.e. $\mathcal{U}$ is univalent.

(Need $\mathbb{F}_{\tilde{\mathcal{U}}} \to \mathcal{U}$ an acyclic fibration to extend fib. structures from A to B.)

Theorem Any type-theoretic model topos satisfies the **Equivalence Extension Property (EEP)**:

$$\begin{tikzcd}
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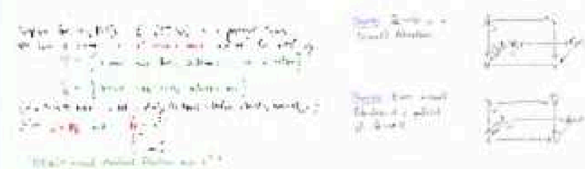

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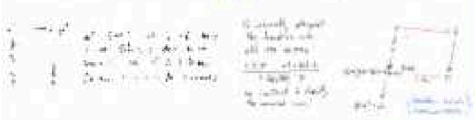

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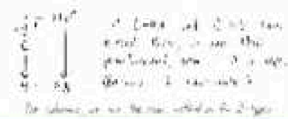

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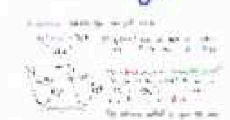
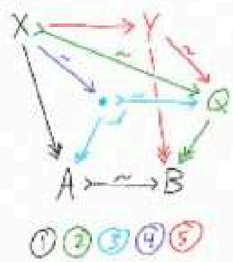


fibrant

Theorem Any type-theoretic model topos satisfies the Fibration Extension Property (FEP):

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow\!\!\downarrow & \lrcorner & \downarrow\!\!\downarrow \\ A & \overset{\sim}{\rightarrowtail} & B \end{array}$$

For any fibration $X \twoheadrightarrow A$ and acyclic cofibration $A \overset{\sim}{\rightarrowtail} B$, there is a fibration $Y \twoheadrightarrow B$ that pulls back to X.



(size-preserving) FEP $\Rightarrow$ $\mathcal{U}$ is fibrant

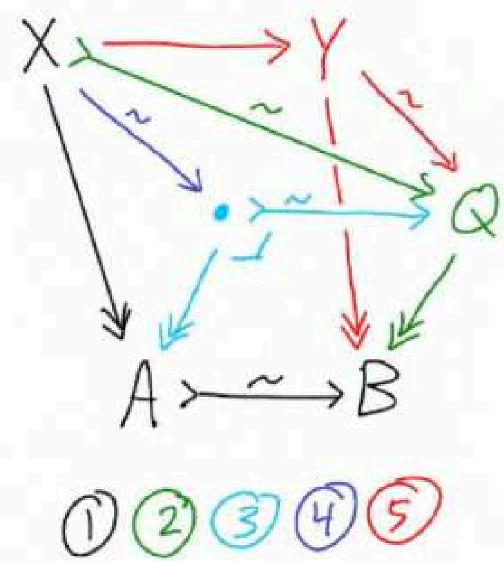
$$\begin{array}{ccc} A & \xrightarrow{x} & \mathcal{U} \\ \wr\downarrow & \nearrow_{Y} & \\ B & & \end{array}$$

(Using $F_{\simeq} \twoheadrightarrow \mathcal{U}$ an acyclic fibration again)

Theorem Any type-theoretic model topos satisfies the Fibration Extension Property (FEP):

$$\begin{tikzcd} X \arrow[r, red] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, red, two heads] \\ A \arrow[r, tail, "\sim"] & B \end{tikzcd}$$

For any fibration $X \twoheadrightarrow A$ and acyclic cofibration $A \overset{\sim}{\rightarrowtail} B$, there is a fibration $Y \twoheadrightarrow B$ that pulls back to X.



(size-preserving) FEP $\Longrightarrow$ $\mathcal{U}$ is fibrant

$$
\begin{tikzcd}
A \arrow[r, "x"] \arrow[d, tail, "\wr"] & \mathcal{U} \\
B \arrow[ur, "y"'] &
\end{tikzcd}
$$

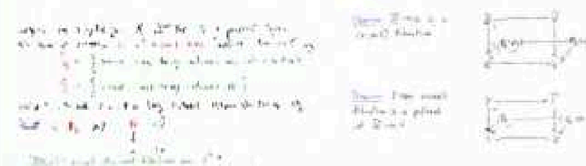
(Using $\mathbb{F}_{\tilde{\mathcal{U}}} \overset{\sim}{\twoheadrightarrow} \mathcal{U}$ an acyclic fibration again)

Theorem For inaccessible (large enough) κ, any type-theoretic model topos has a

fibrant

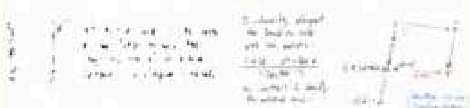
univalent

universe



for fibrations with κ-small fibers, closed under

Σ-types,



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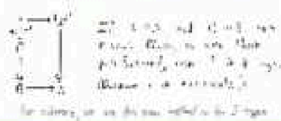
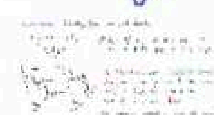


and

Id-types.



Σ-types,

$$\begin{tikzcd}
C \arrow[r, equal] \arrow[d, two heads] & \Sigma_B C \arrow[dd, two heads] \\
B \arrow[d, two heads] & \\
A \arrow[r, equal] & A
\end{tikzcd}$$

If $B \twoheadrightarrow A$ and $C \twoheadrightarrow B$ have κ-small fibers, so does their composite, even if A is large. (Because κ is a regular cardinal.)

To coherently interpret the formation rule with the universe:

$$\frac{\vdash A : \mathcal{U} \qquad x : A \vdash B(x) : \mathcal{U}}{\vdash \Sigma_{(x:A)} B(x) : \mathcal{U}}$$

we construct & classify the universal case:

$$\begin{tikzcd}
Y \arrow[rr] \arrow[d, two heads] \arrow[drr, phantom, "\lrcorner", very near start] & & \widetilde{\mathcal{U}} \arrow[dd, two heads] \\
X \arrow[d, two heads] & & \\
(X : \mathcal{U},\ Y : X \to \mathcal{U}) = (\mathcal{U}\times\mathcal{U} \to \mathcal{U})^{(\widetilde{\mathcal{U}} \to \mathcal{U})} \arrow[rr, "\Sigma\text{-form}"'] \arrow[d] & & \mathcal{U} \\
(X : \mathcal{U}) = \mathcal{U} & &
\end{tikzcd}$$

(Voevodsky; see also Lumsdaine-Warren)

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$$\begin{tikzcd} \bullet \arrow[r] \arrow[d, "ev"'] \arrow[dr, phantom, "\lrcorner", very near start] & \Pi_B C \arrow[dd, two heads] \\ C \arrow[d, two heads] & \\ B \arrow[r, two heads] & A \end{tikzcd}$$

If $B \twoheadrightarrow A$ and $C \twoheadrightarrow B$ have κ-small fibers, so does their pushforward, even if A is large. (Because κ is inaccessible.)

For coherence, we use the same method as for Σ-types.

$$
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Id-types.

Awodey-Warren: Identity types are path objects.

$$\begin{tikzcd} B \arrow[r, tail, "\sim"] \arrow[dr, two heads] & P_A B \arrow[r, two heads] \arrow[d, two heads] & B \times_A B \arrow[dl, two heads] \\ & A & \end{tikzcd}$$

NB $P_A B \twoheadrightarrow A$ may not have small fibers even if $B \twoheadrightarrow A$ does, if A is large.

$$\begin{tikzcd}[column sep=small] B_a \arrow[rrr] \arrow[dr, "\sim"] \arrow[dddr, two heads] & & & B \arrow[dr, "\sim"] \arrow[dddr, two heads] & \\ & \mathrm{CoCyl}(B_a) \arrow[rrr] \arrow[dr, two heads] \arrow[dd, two heads] & & & \mathrm{CoCyl}_A(B) \arrow[dr, two heads] \arrow[dd, two heads] & \\ & & B_a \times_{\text{よ}_c} B_a \arrow[rrr] \arrow[dl, two heads] & & & B \times_A B \arrow[dl, two heads] \\ & \text{よ}_c \arrow[rrr, "a"'] & & & A & \end{tikzcd}$$

The fibered cocylinder $\mathrm{CoCyl}_A(B) = (B \twoheadrightarrow A)^{\Delta^1}$ does preserve small fibers: its fiber over $a \in A_c$ is $\mathrm{CoCyl}_{\text{よ}_c}(B_a)$, which is small since よ_c is a small object.

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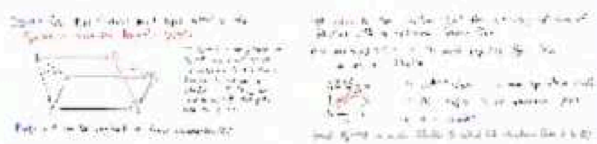
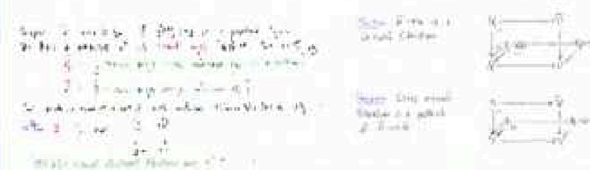


universe



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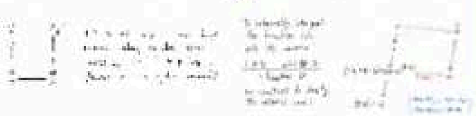

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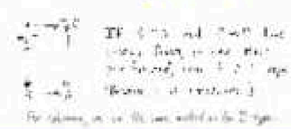

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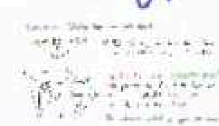

Theorem Type-theoretic model toposes include

simplicial sets

diagram categories

and are closed under

and **left exact localizations.**

Therefore, they model all Grothendieck-Lurie $(\infty,1)$-toposes.

Simplicial sets

The Quillen model structure on simplicial sets is a type-theoretic model topos.

- right proper, Cisinski, simplicial
- every fibration has a unique fibration structure.

This works because the generating acyclic cofibrations $\Lambda^n_k \hookrightarrow \Delta^n$ have representable codomains.

$\rightsquigarrow$ Voevodsky's original model in Kan complexes

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$\mathcal{E}$ a model category, S a set of maps.
A fibrant object $X \in \mathcal{E}$ is internally S-local if
$\mathrm{Map}(B,X) \xrightarrow{\sim} \mathrm{Map}(A,X)$ for all $(A \to B) \in S$.
These are the fibrant objects in a local model structure.

$\mathcal{E}$ a model of type theory, S a set of maps.
A fibration $X \twoheadrightarrow Y$ is internally S-local if
$Y \vdash \mathrm{IsEquiv}(\ldots)$ for all $(A \xrightarrow{f} B) \in S$.
These form a reflective subuniverse.

Theorem: If $\mathcal{E}$ is a type-theoretic model topos and S-localization is left exact, then a fibration $X \twoheadrightarrow \Gamma$ is internally S-local iff it is a fibration in the local model structure.

Thus we can take $\ldots$

So if $\mathcal{E}$ is a type-theoretic model topos, so is any left exact localization of it.

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diagram categories

$\mathcal{E}$ a type-theoretic model topos, $\mathcal{C}$ a small (enriched) category

$[\mathcal{C},\mathcal{E}]$ = strict functors $\mathcal{C} \to \mathcal{E}$ and strict transformations (a 1-category)

$[\![\mathcal{C},\mathcal{E}]\!]$ = weak functors and weak transformations (an $(\infty,1)$-category)

The "pointwise" homotopy theory of $[\mathcal{C},\mathcal{E}]$ doesn't model $[\![\mathcal{C},\mathcal{E}]\!]$:

- Every weak functor is equivalent to a strict one, but
- Not every weak transformation between strict functors is equivalent to a strict one.

A strict $X \in [\mathcal{C},\mathcal{E}]$ is injectively fibrant (aka. flexible) if any weak transformation $A \rightsquigarrow X$ is equivalent to a strict one (by a natural operation that leaves strict transformations fixed).

Similarly, we have injective fibrations and a whole injective model structure on $[\mathcal{C},\mathcal{E}]$ that does present $[\![\mathcal{C},\mathcal{E}]\!]$, and inherits right proper, Cisinski, and enrichment from $\mathcal{E}$.

There is a weak morphism coclassifier, known in classical homotopy theory as a cobar construction:

$$[\mathcal{C},\mathcal{E}](A, CX) \cong [\![\mathcal{C},\mathcal{E}]\!](A, X).$$

Theorem (S) X is injectively fibrant $\iff$ it is a retract of CX.

Similarly, we have a relative cobar construction for fibrations, and we can define F_X to be the "object of such retractions."

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<u>Theorem</u> If $\mathcal{E}$ is a type-theoretic model topos and S-localization is left exact, then a fibration $X \twoheadrightarrow \Gamma$ is

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(uses Anel-Biedermann-Finster-Joyal forthcoming characterization of left exactness)

Thus we can take $\mathbb{F}^S_X = \mathbb{F}_X \times_Y \prod_{f \in S} \mathrm{IsEquiv}(\lambda g.\, g\circ f)$.

So if $\mathcal{E}$ is a type-theoretic model topos,

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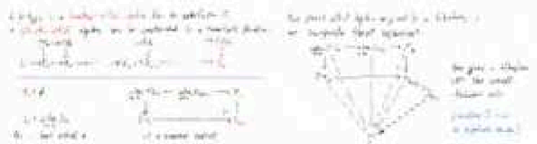
Theorem Any type-theoretic model topos has higher inductive

pushouts,

W-types,



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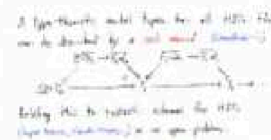
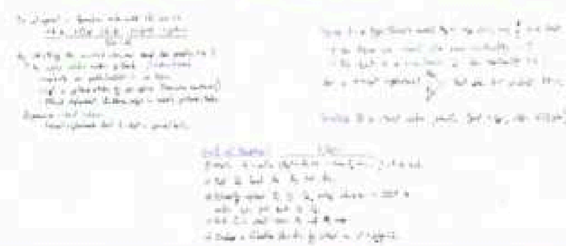
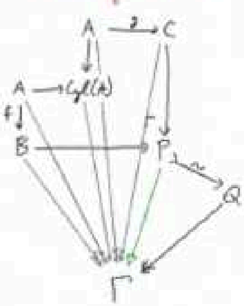


and the universes are closed under them.



pushouts,

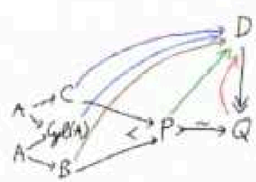
A homotopy pushout of $B \xleftarrow{f} A \xrightarrow{g} C$ is a fibrant replacement:



NB The explicit homotopy pushout $P \to \Gamma$ is not a fibration, but it is sharp!

(Sharp maps are closed under pullback-stable homotopy colimits, because weak equivalences are closed under homotopy colimits.)

This gives it the correct elimination rule:



$$\frac{\begin{array}{c} z:Q \vdash D(z)\ \mathsf{type} \\ x:B \vdash d(x) : D(\mathsf{inl}(x)) \qquad y:C \vdash e(y) : D(\mathsf{inr}(y)) \\ w:A \vdash f : \mathsf{Id}^{D}_{\mathsf{glue}(w)}(d(f(w)), e(g(w))) \end{array}}{z:Q \vdash Q\text{-}\mathsf{ind}(d,e,f,z) : D(z)}$$

A homotopy pushout of $\begin{tikzcd} B \arrow[dr, two heads] & A \arrow[l, "f"'] \arrow[r, "g"] \arrow[d, two heads] & C \arrow[dl, two heads] \\ & \Gamma & \end{tikzcd}$ is a fibrant replacement:

$$\begin{tikzcd}
 & A \arrow[r, "g"] \arrow[d] \arrow[ddddl, bend right=10] & C \arrow[dd] \arrow[ddddll] & \\
A \arrow[r] \arrow[d, "f"'] \arrow[dddr] & Cyl(A) \arrow[dddd] & & \\
B \arrow[rr] \arrow[ddr] & & P \arrow[ul, phantom, "\ulcorner", very near start] \arrow[dr, tail, "\sim"] \arrow[ddl, -open triangle] & \\
 & & & Q \arrow[dll] \\
 & \Gamma & &
\end{tikzcd}$$

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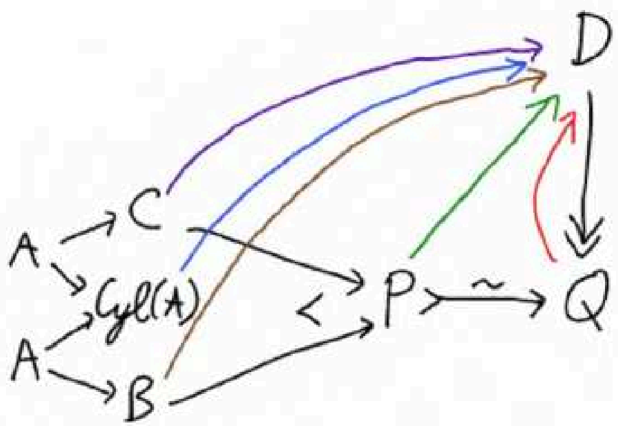

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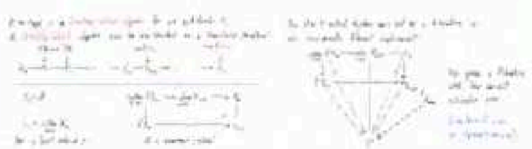

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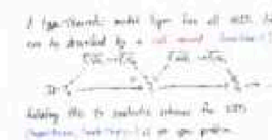

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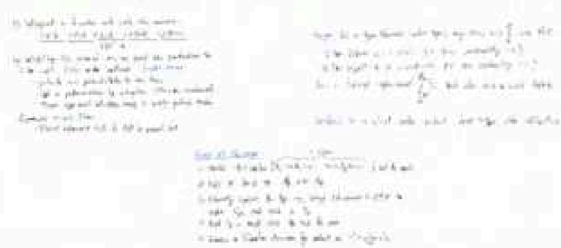

W-types,

A W-type is a homotopy-initial algebra for an endofunctor F.

A strictly initial algebra can be constructed as a transfinite iteration:

$$\begin{tikzcd}
 & FX_0 \arrow[r] \arrow[d] & FX_1 \arrow[d] & & & \arrow[r] & FX_\omega \arrow[d] & & & \arrow[r] & FX_\infty \arrow[d] \\
X_0 \arrow[r] & X_1 \arrow[r] & X_2 \arrow[r] & \cdots & \arrow[r] & X_\omega \arrow[r] & X_{\omega+1} \arrow[r] & \cdots & & \arrow[r] & X_\infty
\end{tikzcd}$$

$X_0 = \emptyset$

$X_n = \operatorname{colim}_{m<n} X_m$

for a limit ordinal n

$$\begin{tikzcd}
\operatorname{colim}_{m<n} FX_m \arrow[r] \arrow[d] & \operatorname{colim}_{m<n} X_{m+1} \arrow[r] & X_n \arrow[d] \\
FX_n \arrow[rr] & & X_{n+1}
\end{tikzcd}$$

at a successor ordinal.

The strict initial algebra may not be a fibration, so we incorporate fibrant replacement:

$$\begin{tikzcd}
\operatorname{colim}_{m<n} FX_m \arrow[r] \arrow[d] \arrow[ddrr] & \operatorname{colim}_{m<n} X_{m+1} \arrow[r] \arrow[ddr] & X_n \arrow[d] \arrow[dd] & \\
FX_n \arrow[rr] \arrow[drr] & & \bullet \arrow[r, "\sim"] \arrow[d] & X_{n+1} \arrow[dl] \\
 & & \Gamma &
\end{tikzcd}$$

This gives a fibration with the correct elimination rule.

(Lumsdaine-S uses an algebraic version)

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X_0 \arrow[r] & X_1 \arrow[r] & X_2 \arrow[r] & \cdots & X_\omega \arrow[r] \arrow[ur] & X_{\omega+1} \arrow[r] & \cdots \arrow[ur] & X_\infty
\end{tikzcd}$$

$X_0 = \emptyset$

$$X_n = \operatorname*{colim}_{m<n} X_m$$

for a limit ordinal n

$$\begin{tikzcd}
\operatorname*{colim}_{m<n} FX_m \arrow[r] \arrow[d] & \operatorname*{colim}_{m<n} X_{m+1} \arrow[r] & X_n \arrow[d, dash] \\
FX_n \arrow[rr] & & X_{n+1} \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

at a successor ordinal.

The strict initial algebra may not be a fibration, so we incorporate fibrant replacement:

$$\begin{tikzcd}
\operatorname{colim}_{m<n} FX_m \arrow[r] \arrow[d] \arrow[ddd] & \operatorname{colim}_{m<n} X_{m+1} \arrow[r] \arrow[ddd] & X_n \arrow[d] \arrow[dddl] & \\
FX_n \arrow[rr] \arrow[ddr] & & \bullet \arrow[ul, phantom, "\ulcorner", very near start] \arrow[dr, "\sim"] \arrow[ddl, two heads] & \\
& & & X_{n+1} \arrow[dll] \\
& \Gamma & &
\end{tikzcd}$$

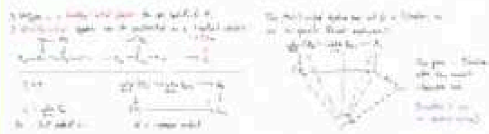
This gives a fibration with the correct elimination rule.

(Lumsdaine–S uses an algebraic version)

Theorem Any type-theoretic model topos has higher inductive

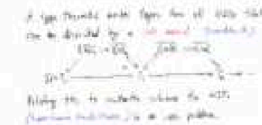
pushouts,

W-types,



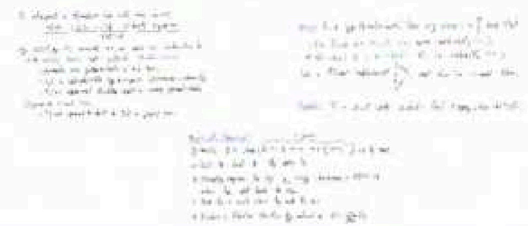
and

other HITs,



and the universes are

closed under them.



other HITs,

A type-theoretic model topos has all HITs that can be described by a cell monad (Lumsdaine-S.)

$$\begin{tikzcd}
 & \overline{F_0\times\partial G_0} \arrow[r] \arrow[dl] & \overline{F_0\times G_0} \arrow[dr] & & \overline{F_1\times\partial G_1} \arrow[r] \arrow[dl] & \overline{F_1\times G_1} \arrow[dr] & & \\
Id = T_0 \arrow[rrr] & & & T_1 \arrow[ul, phantom, "\ulcorner" very near start] \arrow[rrr] & & & T_2 \arrow[r] & \cdots
\end{tikzcd}$$

Relating this to syntactic schemas for HITs (Kaposi-Kovacs, Cavallo-Harper, ...) is an open problem.

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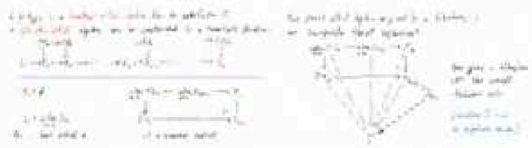

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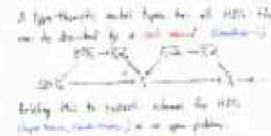

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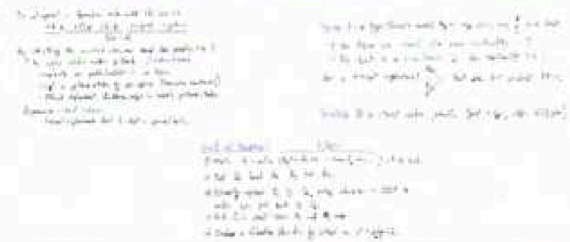

Variations on the definition

Simplicial

[illegible]

Cubical

[illegible]

Constructive

[illegible]

The original definition was a simplicial t.t.m.t.:

- Fiberwise enriched over simplicial sets

  $Cyl(X) = \Delta^1 \otimes X$

- Simplicial model categories are familiar in homotopy theory
- Suffices to model all G-L $(\infty,1)$-toposes
- Cobar constructions are naturally simplicial
- Doesn't include newer cubical models.

Variations on the definition

Simplicial

[illegible]

- [illegible]
- [illegible]
- [illegible]
- [illegible]
- [illegible]

Cubical

[illegible]

- [illegible]
- [illegible]
- Can cubical type theory be interpreted in every (∞,1)-topos?
- [illegible]

Constructive

In an abstract constructive logic, it seems the notion of Quillen model category must be relaxed: constructive model structures rarely allow all monomorphisms to be cofibrations.

[illegible]

Cartesian cubical set models are fiberwise self-enriched, giving cubical t.t.m.t.s:

$$\mathrm{Cyl}(X) = \Box^1 \times X$$

- Fibration structures $\mathbb{F}_X$ arise naturally from Orton-Pitts and give the same universe as in Licata-Orton-Pitts-Spitters (and help construct the model structure: Sattler, Awodey, ...)

- More generally, axiomatizing the abstract structure of $\mathrm{Cyl} \dashv \mathrm{CoCyl}$ includes both simplicial & cubical models.
- Can cubical type theory be interpreted in every $(\infty,1)$-topos?

Variations on the definition

Simplicial

The original definition was in simplicial sets:

- Fibrations specified over simplicial sets [illegible]
- Simplicial model categories are familiar in homotopy theory
- Suffices to model all $(\infty,1)$-toposes
- [illegible] constructions are [illegible] simplicial
- [illegible]

cubical

Cartesian cubical set models use fibrations with [illegible], giving cubical [illegible]:

[illegible]

- Fibration structures [illegible] (and help construct the model structure [illegible])
- More generally, axiomatizing the abstract structure [illegible] includes both simplicial & cubical models
- Can cubical type theory be interpreted in every $(\infty,1)$-topos?
- [illegible]

constructive

In an ambient constructive logic, it seems the notion of Quillen model category must be relaxed: constructive model structures rarely allow all monomorphisms to be cofibrations.

? ? ?

In an ambient constructive logic, it seems the notion of Cisinski model category must be relaxed: constructive model structures rarely allow all monomorphisms to be cofibrations.

? ? ?

Variations on the definition

Simplicial

The original definition was in simplicial sets:

- Fibrations [illegible] in simplicial sets
  [illegible]
- Simplicial model categories are familiar in homotopy theory
- Suffices to model all (∞,1)-toposes
- Colimit constructions are naturally simplicial
- Doesn't include some cubical models

cubical

[illegible] cubical set models may fibrancy [illegible] equivalent, giving cubical [illegible]:

[illegible]

- Fibration structures [illegible] and give the same answer [illegible] (and help construct the model structure [illegible])
- More generally, axiomatizing the abstract structure of [illegible] includes both simplicial & cubical models
- Can cubical type theory be interpreted in every (∞,1)-topos?
- If a fibrant object [illegible] is equivalent to [illegible] cubical sets with [illegible] model structure — [illegible]

constructive

In an ambient constructive logic, it seems the choice of Quillen model category must be relevant: constructive model structures rarely allow all monomorphisms to be cofibrations.

? ? ?