

# Type-Theoretic Model Toposes

Mike Shulman

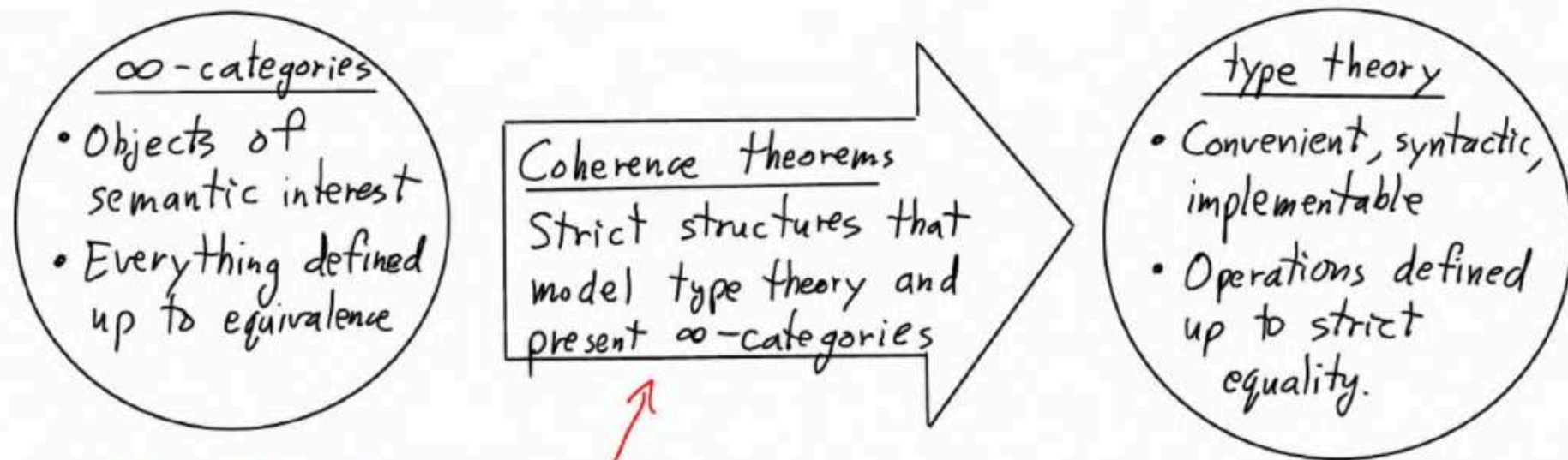
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The Internet

## The Ecosystem & Our Niche



Idea: **Type-theoretic model toposes** have all the structure necessary to model HoTT (with UA+HITs), but are general enough to present all the desired  $\infty$ -categorical models.

# A type-theoretic model topos is a

right proper

The category of  $\mathcal{C}$ -valued presheaves  $\mathcal{P}(\mathcal{C})$  is right proper. This is because  $\mathcal{C}$  is a right proper category.

Let  $\mathcal{C}$  be a right proper category. Then the category of  $\mathcal{C}$ -valued presheaves  $\mathcal{P}(\mathcal{C})$  is right proper.

Cisinski

A model category  $\mathcal{M}$  is Cisinski if it is a right proper model category and its cofibrations are monomorphisms.

model category

A model category  $\mathcal{M}$  is a right proper model category with cofibrations monomorphisms.

Let  $\mathcal{M}$  be a model category. Then  $\mathcal{M}$  is a right proper model category if and only if its cofibrations are monomorphisms.

with

fiberwise enrichment

A right proper model category  $\mathcal{M}$  is enriched over a right proper category  $\mathcal{C}$  with respect to the monoidal structure  $(\mathcal{C}, \otimes, I)$  if and only if  $\mathcal{M}$  is a right proper model category.

and

structured fibrations.

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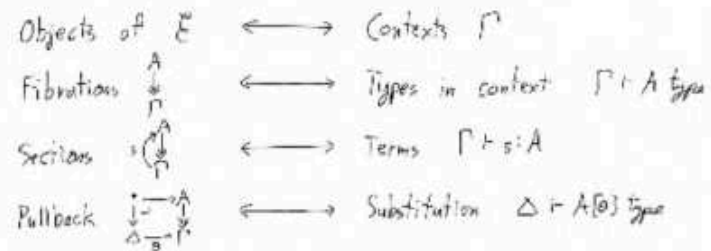
# model category

Definition A Quillen model category is a <sup>complete cocomplete</sup> category  $\mathcal{E}$  with

- Classes of maps called *cofibrations*, *fibrations*, and *weak equivalences*.
- The weak equivalences satisfy 2-out-of-3.
- $(\text{Cof}, \text{Fib} \cap \text{WE})$  and  $(\text{Cof} \cap \text{WE}, \text{Fib})$  are weak factorization systems

$\text{Fib} \cap \text{WE} = \text{acyclic fibrations} \xrightarrow{\sim}$

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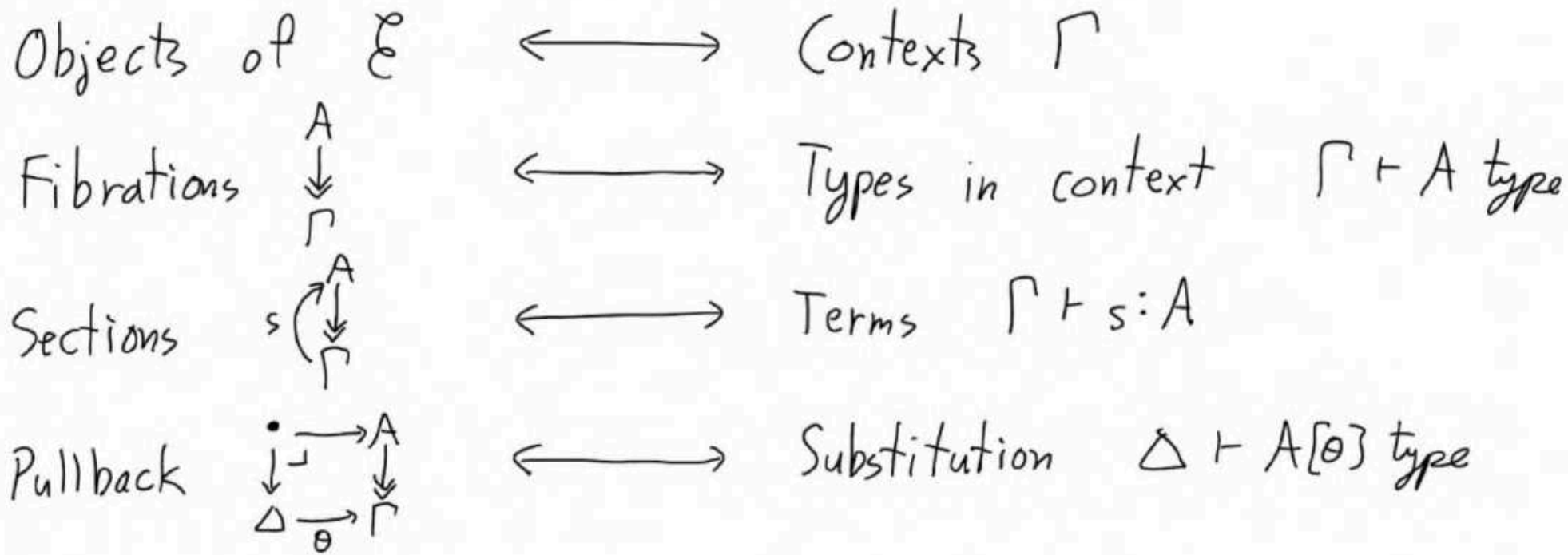


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# A type-theoretic model topos is a

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This is equivalent to saying that fibrations are right proper and monomorphisms are stable under pushouts.

if  $f$  is a fibration and  $g$  is a monomorphism, then  $f \circ g$  is a fibration.

Cisinski

A Cisinski model category is a model category  $\mathcal{M}$  with a set of cofibrations  $\mathcal{C}$  and a set of fibrations  $\mathcal{F}$  such that:

- $\mathcal{C}$  and  $\mathcal{F}$  are closed under isomorphisms.
- $\mathcal{C}$  is closed under cofibrations.
- $\mathcal{F}$  is closed under fibrations.
- $\mathcal{C}$  and  $\mathcal{F}$  are closed under filtered colimits.
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A model category is enriched over a monoidal model category  $(\mathcal{V}, \otimes, I)$  if:

- $\mathcal{M}$  is a  $\mathcal{V}$ -enriched category.
- $\mathcal{C}$  and  $\mathcal{F}$  are closed under  $\otimes$ .
- $\mathcal{C}$  and  $\mathcal{F}$  are closed under filtered colimits.
- $\mathcal{C}$  and  $\mathcal{F}$  are closed under filtered limits.

and

structured fibrations.

A structured fibration is a fibration  $f: E \rightarrow B$  in a model category  $\mathcal{M}$  such that:

- $f$  is a fibration.
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# Cisinski

A *Cisinski model category* is a model category  $\mathcal{E}$  such that

- $\mathcal{E}$  is a Grothendieck 1-topos. (For us, usually a presheaf topos.)
- The *cofibrations* are precisely the *monomorphisms*.
- The weak factorization systems are *cofibrantly generated*.

In a Cisinski model category, cofibrations

- have unions
- are extensive, adhesive, and exhaustive
- are stable under pullback

because monomorphisms in a topos have these properties.



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# A type-theoretic model topos is a

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A right proper topos is a topos where the pushout of a monomorphism and an epimorphism is again an epimorphism.

A topos is right proper if and only if it is a Grothendieck topos.

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A fiberwise enriched topos is a topos where the pushout of a monomorphism and an epimorphism is again an epimorphism.

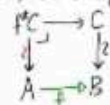
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A structured fibration is a topos where the pushout of a monomorphism and an epimorphism is again an epimorphism.

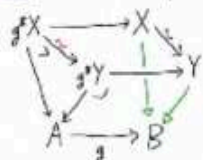
# right proper

(Rezk) A morphism  $f:A \rightarrow B$  is **sharp** if pullback along  $f$  preserves W.E.



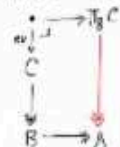
(e.g. "right proper map", "H-fibration", "W-fibration", "fibrant", "weak fibration")

Theorem: Pullback preserves W.E. **between** sharp maps.



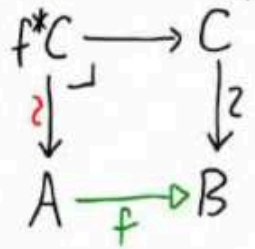
Def A model category is **right proper** if all fibrations are sharp (i.e. pullback along fibrations preserves weak equivalences)

Theorem In a right proper Cisinski model category, fibrations are closed under pushforward along fibrations.



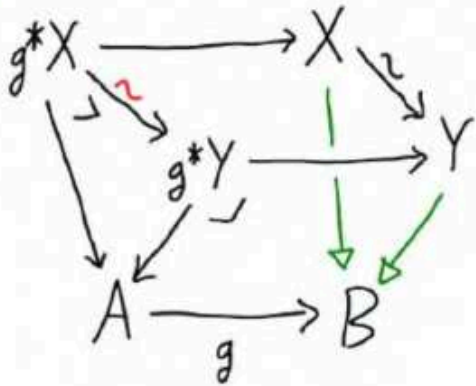
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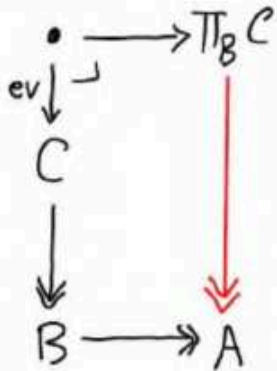
(a.k.a. "right proper map", "H-fibration", "W-fibration",  
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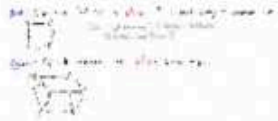
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right proper: if  $f$  is a fibration and  $g$  is a fibration, then  $g \circ f$  is a fibration.

Cisinski

Cisinski model category: a model category where the cofibrations are the monomorphisms and the fibrations are the fibrations in the sense of the theory of fibrations.

model category

Model category: a category with three distinguished classes of morphisms: cofibrations, fibrations, and weak equivalences, satisfying certain axioms.

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fiberwise enrichment

Fiberwise enrichment: a model category where the hom-objects are objects in a monoidal category, and the composition is given by a multiplication map.

and

structured fibrations,

Structured fibrations: a model category where the fibrations are equipped with a structure of a certain type, such as a monoidal structure.

# fiberwise enrichment

A simplicial model category is enriched over simplicial sets, with powers and copowers:

$$\mathbf{sSet}(K, \mathbf{Map}(X, Y)) \cong \mathcal{E}(K \otimes X, Y) \cong \mathcal{E}(X, Y^K).$$

plus a cofibration condition (SM7).

In particular, it has cylinders  $\mathit{Cyl}(X) = \Delta^1 \otimes X$  and cocylinders  $\mathit{CoCyl}(Y) = Y^{\Delta^1}$ .

It is fiberwise-simplicial if pullback  $\mathcal{E}/_Y \rightarrow \mathcal{E}/_X$  preserves copowers,  $f^*(K \otimes X) \cong K \otimes f^*X$ .

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# A type-theoretic model topos is a

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$\mathcal{C}$  is a right proper topos if it is a topos and for every monomorphism  $m: A \rightarrow B$  and every epimorphism  $e: C \rightarrow D$ , the image of  $m \circ e$  is the image of  $m$ .  
 $\mathcal{C}$  is right proper if for every monomorphism  $m: A \rightarrow B$  and every epimorphism  $e: C \rightarrow D$ , the image of  $m \circ e$  is the image of  $m$ .

Cisinski

A Cisinski topos is a topos  $\mathcal{C}$  with a fixed object  $\mathbb{I}$  (the interval) and a fixed monomorphism  $i: \mathbb{I} \rightarrow \mathbb{I}$  (the interval inclusion).  
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# structured fibrations.

Def. A locally representable and relatively acyclic  
notion of fibration structure consists of

- For each  $\begin{array}{c} X \\ \downarrow \\ Y \end{array}$ , an  $\begin{array}{c} F_X \\ \downarrow \\ Y \end{array}$ , varying pseudofunctorially in pullback along  $Y' \rightarrow Y$ .
- The following are equivalent:
  - $X \rightarrow Y$  is a fibration
  - $F_X \rightarrow Y$  has a section
  - $F_X \rightarrow Y$  is an acyclic fibration

A fibration structure on  $X \rightarrow Y$  is a section of  $F_X \rightarrow Y$ .

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# A type-theoretic model topos is a

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A right proper topos is a topos  $\mathcal{T}$  such that the fibration of points  $\pi_0: \mathcal{T} \rightarrow \mathbf{Set}$  is a right fibration.

A topos is right proper if and only if the fibration of points  $\pi_0: \mathcal{T} \rightarrow \mathbf{Set}$  is a right fibration.

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Theorem For <sup>(large enough)</sup> inaccessible  $\kappa$ , any type-theoretic model topos has a

fibrant

Let  $\mathcal{C}$  be a topos with a point  $*$ .  
 A fibration  $p: E \rightarrow \mathcal{C}$  is fibrant if

- 1.  $p$  is a fibration
- 2.  $p$  is a monomorphism
- 3.  $p$  is a regular epimorphism

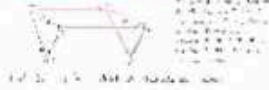


Equivalently,  $p$  is fibrant if



univalent

Let  $\mathcal{C}$  be a topos with a point  $*$ .  
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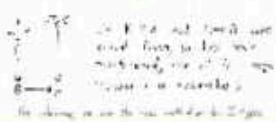


for fibrations with  $\kappa$ -small fibers, closed under

$\Sigma$ -types,



$\Pi$ -types,



and

Id-types.



# universe

Suppose for simplicity  $\mathcal{E} = [C^{op}, \text{Set}]$  is a presheaf topos.  
 We have a universe of all  $\kappa$ -small maps "defined" for  $c \in \mathcal{C}$  by

$$\tilde{V}_c = \{ \kappa\text{-small maps having codomain } \mathcal{C}_c \text{ with a section} \}$$

$$V_c = \{ \kappa\text{-small maps having codomain } \mathcal{C}_c \}$$

(w/ a trick to make it a set & strictly functorial - Hofmann-Streicher, Voevodsky, ...)

Define  $U = \mathbb{F}_V$  and

$$\begin{array}{ccc} \tilde{U} & \longrightarrow & \tilde{V} \\ \downarrow & \lrcorner & \downarrow \\ U & \longrightarrow & V \end{array}$$

" $\mathcal{E}(X, U) = \kappa$ -small structured fibrations over  $X$ "

Theorem  $\tilde{U} \rightarrow U$  is a ( $\kappa$ -small) fibration.

$$\begin{array}{ccc} \tilde{U} & \longrightarrow & \tilde{V} \\ \downarrow & \lrcorner & \downarrow \\ U & \longrightarrow & V \end{array} \quad \begin{array}{c} \mathbb{F}_U = U \circ U \\ \mathbb{F}_V = U \end{array}$$

Theorem Every  $\kappa$ -small fibration is a pullback of  $\tilde{U} \rightarrow U$

$$\begin{array}{ccc} X & \longrightarrow & \tilde{V} \\ \downarrow & \lrcorner & \downarrow \\ Y & \longrightarrow & V \end{array} \quad \begin{array}{c} \mathbb{F}_X \\ \mathbb{F}_V = U \end{array}$$

Suppose for simplicity  $\mathcal{E} = [\mathcal{C}^{\text{op}}, \text{Set}]$  is a presheaf topos.  
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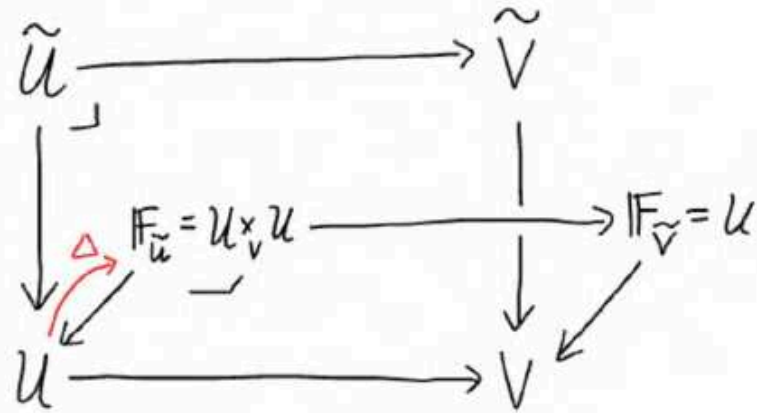
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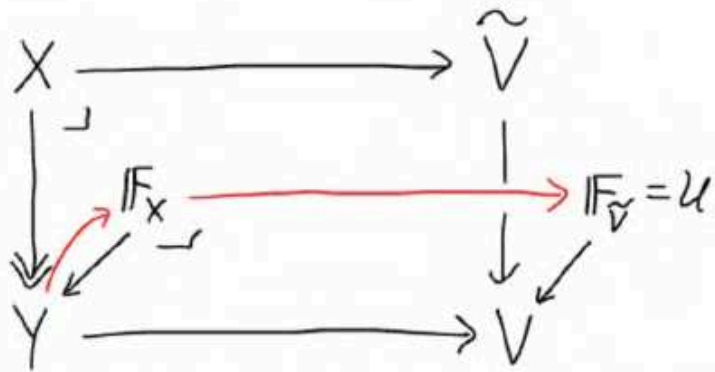
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Theorem Every  $k$ -small  
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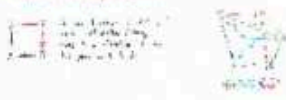
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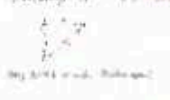
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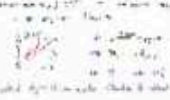
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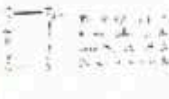
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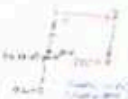
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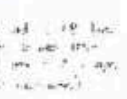
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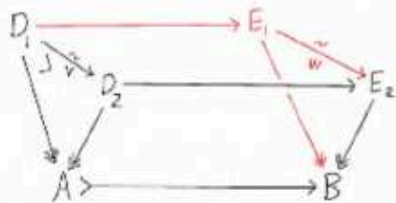


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# univalent

Theorem Any type-theoretic model topos satisfies the **Equivalence Extension Property (EEP)**:



If  $D_2 \rightarrow A$  is the pullback of  $E_2 \rightarrow B$  along a ref.  $A \rightarrow B$ , any equivalence  $D_1 \xrightarrow{\sim} D_2$  from a fibration  $D_1 \rightarrow A$  can be extended to  $E_1 \rightarrow E_2$  for a fibration  $E_1 \rightarrow B$  that pulls back to  $D_1 \rightarrow A$ .

Proof: Just like for simplicial sets (Equival-Luminaire-Vorodsky)

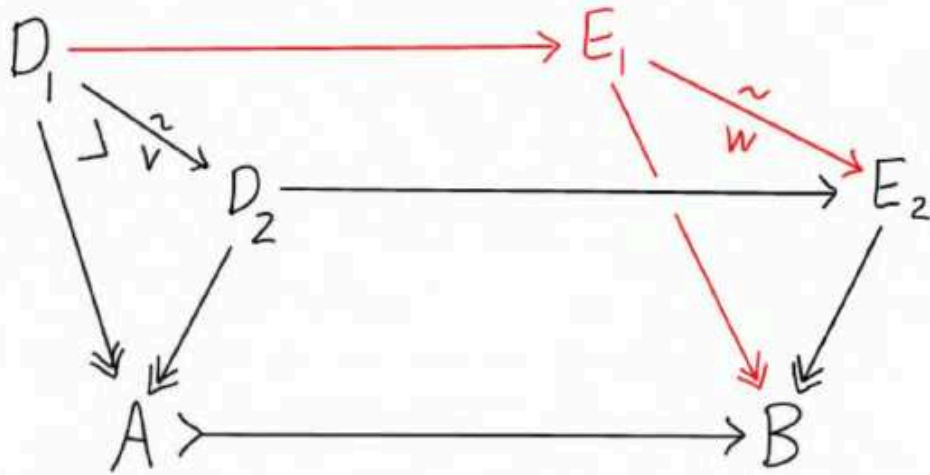
Let  $\mathbf{Equiv}$  be the classifier (built from  $U + LCCC$ ) of pairs of fibrations with an equivalence between them.

(Size-preserving)  $EEP \Leftrightarrow$  the second projection  $\mathbf{Equiv} \rightarrow U$  is an acyclic fibration.

$$\begin{array}{ccc}
 A & \xrightarrow{(j, w)} & \mathbf{Equiv} \\
 \downarrow & \searrow^{(j, w)} & \downarrow \\
 B & \xrightarrow{E_2} & U
 \end{array}
 \quad \begin{array}{l}
 \Rightarrow U \xrightarrow{\text{id}_{\mathbf{Equiv}}} \mathbf{Equiv} \text{ is a weak equivalence (2/3)} \\
 \Rightarrow PU \rightarrow \mathbf{Equiv} \text{ is an equivalence (2/3)} \\
 \text{i.e. } U \text{ is univalent.}
 \end{array}$$

(Need  $\mathbb{F}_U \rightarrow U$  an acyclic fibration to extend fib. structures from  $A$  to  $B$ )

Theorem Any type-theoretic model topos satisfies the  
 Equivalence Extension Property (EEP):



If  $D_2 \twoheadrightarrow A$  is the pullback of  $E_2 \twoheadrightarrow B$  along a cof.  $A \hookrightarrow B$ , any equivalence  $D_1 \xrightarrow{\sim} D_2$  from a fibration  $D_1 \twoheadrightarrow A$  can be extended to  $E_1 \xrightarrow{\sim} E_2$  for a fibration  $E_1 \twoheadrightarrow B$  that pulls back to  $D_1 \twoheadrightarrow A$ .

Proof: Just like for simplicial sets (Kapulkin-Lumsdaine-Voevodsky).

Let  $\mathit{Equiv}$  be the classifier (built from  $U + LCCC$ ) of pairs of fibrations with an equivalence between them.

(Size-preserving)  $EEP \Rightarrow$  the second projection  $\mathit{Equiv} \xrightarrow{\pi_2} U$  is an acyclic fibration.

$$\begin{array}{ccc}
 A & \xrightarrow{(P_1, P_2, V)} & \mathit{Equiv} \\
 \downarrow Y & \nearrow (E_1, E_2, W) & \downarrow \pi_2 \\
 B & \xrightarrow{E_2} & U
 \end{array}$$

$\Rightarrow U \xrightarrow{\text{id}_{\mathit{Equiv}}} \mathit{Equiv}$  is a weak equivalence (2/3)

$\Rightarrow PU \rightarrow \mathit{Equiv}$  is an equivalence (2/3)

i.e.  $U$  is univalent.

(Need  $\mathbb{F}_{\tilde{U}} \rightarrow U$  an acyclic fibration to extend fib. structures from  $A$  to  $B$ .)

Theorem For <sup>(large enough)</sup> inaccessible  $\kappa$ , any type-theoretic model topos has a

fibrant

There are fibrations with non-fibrant fibers

Example:  $\text{Set} \rightarrow \text{Set}$  with  $\text{fib}(f) = \text{pt}$



Example:  $\text{Set} \rightarrow \text{Set}$  with  $\text{fib}(f) = \text{pt}$

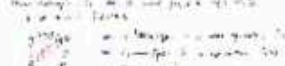


univalent

Example:  $\text{Set} \rightarrow \text{Set}$  with  $\text{fib}(f) = \text{pt}$

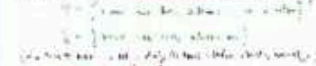


Example:  $\text{Set} \rightarrow \text{Set}$  with  $\text{fib}(f) = \text{pt}$



universe

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for fibrations with  $\kappa$ -small fibers, closed under

$\Sigma$ -types,



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$\Pi$ -types,

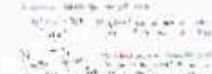


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The above are not the case for  $\Pi$ -types

and

Id-types.

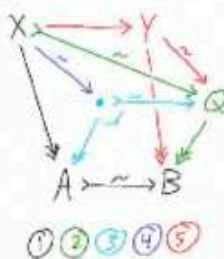


The above are not the case for Id-types

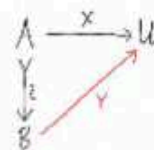
# fibrant

Theorem Any type-theoretic model topos satisfies the  
 Fibration Extension Property (FEP):

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & \lrcorner & \downarrow \\ A & \twoheadrightarrow & B \end{array}$$
 For any fibration  $X \twoheadrightarrow A$  and acyclic cofibration  $A \twoheadrightarrow B$ , there is a fibration  $Y \twoheadrightarrow B$  that pulls back to  $X$ .



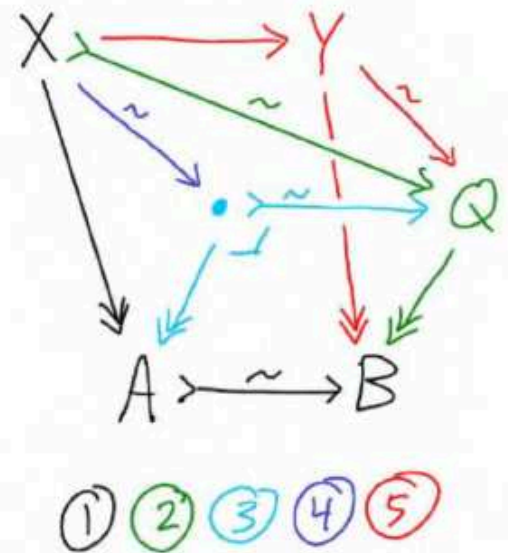
(size-preserving) FEP  $\Rightarrow$   $\mathcal{U}$  is fibrant



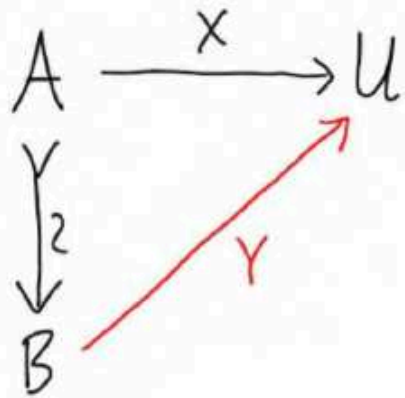
(Using  $F_B \twoheadrightarrow U$  an acyclic fibration again)

Theorem Any type-theoretic model topos satisfies the  
**Fibration Extension Property (FEP)**:

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(size-preserving) FEP  $\Rightarrow$   $\mathcal{U}$  is fibrant



(Using  $\mathbb{F}_{\mathcal{U}} \rightsquigarrow \mathcal{U}$  an acyclic fibration again)



Theorem For <sup>(large enough)</sup> inaccessible  $\kappa$ , any type-theoretic model topos has a

fibrant

From the fibration with the identity on the base (Lambert 2016, 2017)

A model of type theory with a fibration  $\mathcal{F}$  is a model of type theory with a fibration  $\mathcal{F}$ .



Proposition 1.1 (Lambert 2016)



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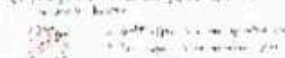
univalent

From the fibration with the identity on the base (Lambert 2016, 2017)



Proposition 1.1 (Lambert 2016)

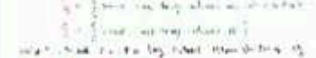
Proposition 1.1 (Lambert 2016)



Proposition 1.1 (Lambert 2016)

universe

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for fibrations with  $\kappa$ -small fibers, closed under

$\Sigma$ -types,



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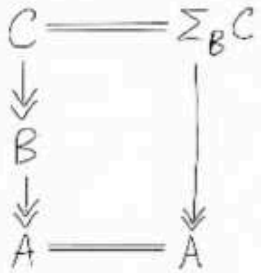
and

Id-types.



Proposition 1.1 (Lambert 2016)

# $\Sigma$ -types

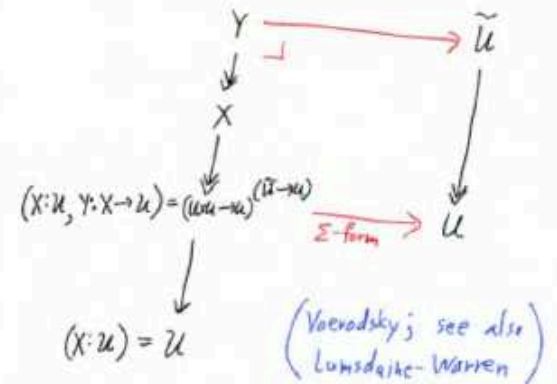


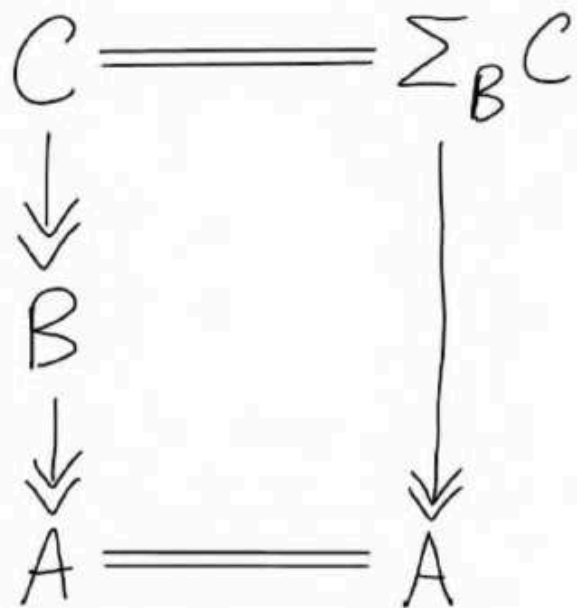
If  $B \twoheadrightarrow A$  and  $C \twoheadrightarrow B$  have  $\kappa$ -small fibers, so does their composite, even if  $A$  is large.  
(Because  $\kappa$  is a regular cardinal.)

To coherently interpret the formation rule with the universe:

$$\frac{\vdash A : \mathcal{U} \quad x:A \vdash B(x) : \mathcal{U}}{\vdash \Sigma_{(x:A)} B(x) : \mathcal{U}}$$

We construct & classify the universal case:



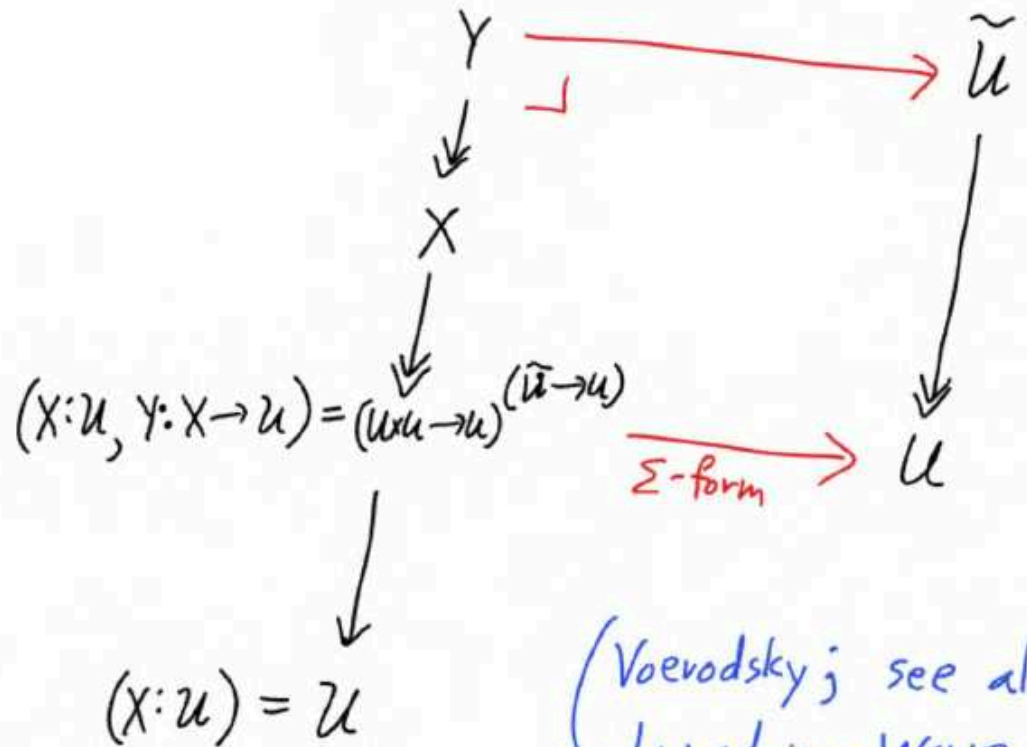


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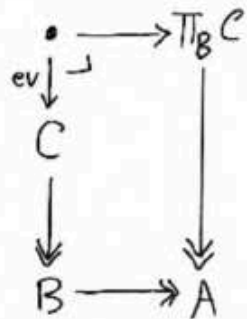
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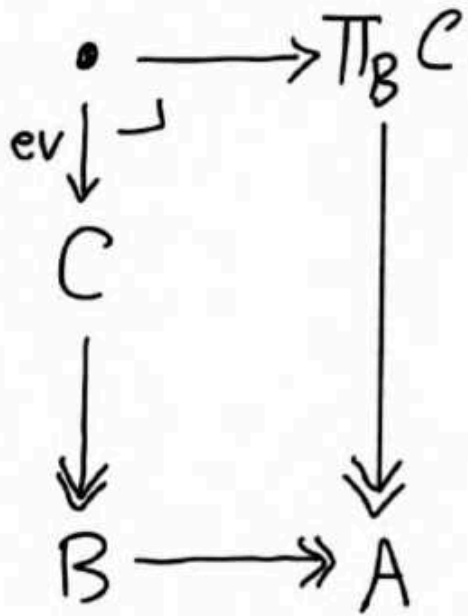


# $\Pi$ -types,



If  $B \twoheadrightarrow A$  and  $C \twoheadrightarrow B$  have  $k$ -small fibers, so does their push forward, even if  $A$  is large. (Because  $k$  is inaccessible.)

For coherence, we use the same method as for  $\Sigma$ -types.



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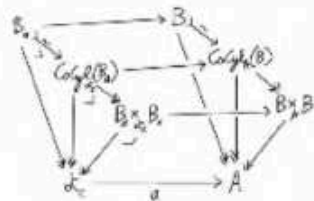
For coherence, we use the same method as for  $\Sigma$ -types.

# Id-types.

Aunty-Whore: Identity types are path objects.

$$B \rightrightarrows P_A B \rightrightarrows B \times_A B$$

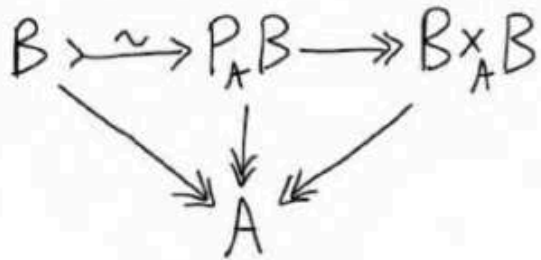
NB  $P_A B \rightarrow A$  may not have small fibers even if  $B \rightarrow A$  does, if  $A$  is large.



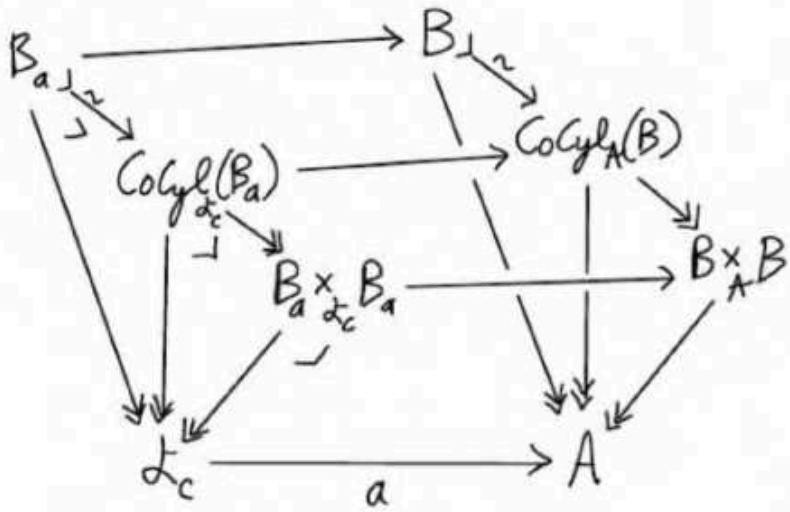
The **fibred** cocylinder  $CoGr_A(B) = (B \rightarrow A)^{\circ}$  does preserve small fibers: its fiber over  $a \in A_0$  is  $CoGr_A(B_a)$ , which is small since  $a_c$  is a small object.

The coherence method is again the same.

Awody-Warren: Identity types are path objects.



NB  $P_A B \twoheadrightarrow A$  may not have small fibers even if  $B \twoheadrightarrow A$  does, if  $A$  is large.



The **fibered** cocylinder  $CoCyl_A(B) = (B \rightarrow A)^{\Delta'}$  does preserve small fibers: its fiber over  $a \in A_c$  is  $CoCyl_{d_c}(B_a)$ , which is small since  $d_c$  is a small **object**.

The coherence method is again the same.



Theorem For <sup>(large enough)</sup> inaccessible  $\kappa$ , any type-theoretic model topos has a

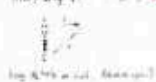
fibrant

From the definition of fibrant, it follows that

1.  $\text{Fib}(\text{Fib}(A)) = \text{Fib}(A)$   
 2.  $\text{Fib}(A) \subseteq \text{Fib}(B)$  if  $A \rightarrow B$  is a fibration



There are two ways to define fibrant



univalent

From the definition of univalent, it follows that

1.  $\text{Univ}(\text{Univ}(A)) = \text{Univ}(A)$   
 2.  $\text{Univ}(A) \subseteq \text{Univ}(B)$  if  $A \rightarrow B$  is a fibration

For fibrations with  $\kappa$ -small fibers, closed under

1.  $\Sigma$ -types

2.  $\Pi$ -types

3. Id-types

4.  $\text{Id}$ -types

universe

From the definition of universe, it follows that

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for fibrations with  $\kappa$ -small fibers, closed under

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$\Pi$ -types,



3.  $\Pi$ -types

4.  $\text{Id}$ -types

For fibrations with  $\kappa$ -small fibers, closed under

and

Id-types.



5. Id-types

# Theorem Type-theoretic model toposes include

simplicial sets

The standard model structure is cofibrant fib  
 - a fibrant object is fibrant  
 - right proper, fibrant cofibrant  
 - cofibrant fibrant for a right localization  
 The cofibrant fibrant model structure is cofibrant  
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and are  
closed under

diagram categories

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and

left exact localizations.

$\mathcal{E}$  a model category,  $S$  a set of maps  
 A fibrant object  $X \in \mathcal{E}$  is ultimately S-local if  
 $\text{Map}(B, X) \cong \text{Map}(A, X)$  for all  $(A \rightarrow B) \in S$   
 These are the fibrant objects in a local model structure  
 $\mathcal{E}$  a model of type theory,  $S$  a set of maps  
 A fibration  $X \rightarrow Y$  is ultimately S-local if  
 $Y = \text{Tot}(\text{fib}(X)) = \text{Tot}(\text{fib}(X))$  for all  $(A \rightarrow B) \in S$   
 These form a reflective subcategory ( $\text{Adm}(S)$ )

Theorem If  $\mathcal{E}$  is a type-theoretic model topos and  
 S-localization is left exact, then a fibration  $X \rightarrow Y$  is  
 ultimately S-local  $\iff$  it is a fibration in  
 the local model structure  
 (see the following section for a detailed construction of this structure)  
 Thus we can take  $\mathcal{E}^{\text{loc}} = \text{Tot}(\text{fib}(X))$   
 So if  $\mathcal{E}$  is a type-theoretic model topos  
 it is a left exact localization of  $\mathcal{E}$

Therefore, they  
model all  
Grothendieck-Lurie  
 $(\infty, 1)$ -toposes.

# simplicial sets

The Quillen model structure on simplicial sets  
is a type-theoretic model topos.

- right proper, Cisinski, simplicial
- every fibration has a unique fibration structure.

This works because the generating acyclic cofibrations  $\Lambda_n^* \hookrightarrow \Delta^n$   
have representable codomains.

→ Voevodsky's original model in Kan complexes

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→ Voevodsky's original model in Kan complexes

# Theorem Type-theoretic model toposes include

simplicial sets

The Quillen model structure on simplicial sets is a cofibrantly generated model structure with cofibrations the monomorphisms and fibrations the Kan fibrations. The cofibrations are the monomorphisms and the fibrations are the Kan fibrations. The model structure is cofibrantly generated with cofibrations the monomorphisms and fibrations the Kan fibrations.

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and are closed under

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left exact localizations.

$\mathcal{E}$  is a model category,  $S$  a set of maps.  
 A fibrant object  $X \in \mathcal{E}$  is relatively  $S$ -local if  
 $\text{Map}(B, X) \cong \text{Map}(A, X)$  for all  $(A \rightarrow B) \in S$ .  
 These are the fibrant objects in a local model structure.  
 $\mathcal{E}$  is a model of type theory,  $S$  a set of maps.  
 A fibration  $X \rightarrow Y$  is relatively  $S$ -local if  
 $\gamma^* \text{Map}(A, X) \cong \gamma^* \text{Map}(A, Y)$  for all  $(A \rightarrow B) \in S$ .  
 These form a reflection subcategory,  $(\text{Rel}^S) \rightarrow \text{Rel}^S$ .

Theorem If  $\mathcal{E}$  is a type-theoretic model topos and  $S$ -localization is left exact, then a fibration  $X \rightarrow Y$  is relatively  $S$ -local iff it is a fibration in the local model structure.  
 (See the previous section for the definition of left exact.)  
 Thus we can take  $\mathcal{E}_S^{\text{loc}} = \text{Rel}^S \cap \text{Rel}^S$  (the local model structure).  
 So if  $\mathcal{E}$  is a type-theoretic model topos, so is any left exact localization of it.

Therefore, they model all Grothendieck-Lurie  $(\infty, 1)$ -toposes.

# diagram categories

- $\mathcal{E}$  a type-theoretic model topos,  $\mathcal{C}$  a small (onlined) category
- $[\mathcal{C}, \mathcal{E}]$  = strict functors  $\mathcal{C} \rightarrow \mathcal{E}$  and strict transformations (a 1-category)
- $\llbracket \mathcal{C}, \mathcal{E} \rrbracket$  = weak functors and weak transformations (an  $(\infty, 0)$ -category)
- The "pointwise" homotopy theory of  $\llbracket \mathcal{C}, \mathcal{E} \rrbracket$  doesn't model  $\llbracket \mathcal{C}, \mathcal{E} \rrbracket$ :
- Every weak functor is equivalent to a strict one, but
  - Not every weak transformation between strict functors is equivalent to a strict one.

A strict  $X \in [\mathcal{C}, \mathcal{E}]$  is *injectively fibrant* (aka *fibrile*) if any weak transformation  $A \rightarrow X$  is equivalent to a strict one (by a natural operon that leaves strict transformations fixed).

Similarly, we have *injective fibrations* and a whole *injective model structure* on  $[\mathcal{C}, \mathcal{E}]$  that does present  $\llbracket \mathcal{C}, \mathcal{E} \rrbracket$ , and inherits right proper, Cisinski, and enrichment from  $\mathcal{E}$ .

There is a *weak morphism classifier*, known in classical homotopy theory as a *cube construction*.

$$[\mathcal{C}, \mathcal{E}](A, CX) \cong [\mathcal{C}, \mathcal{E}](A, X).$$

Theorem  $X$  is injectively fibrant  $\Leftrightarrow$  it is a retract of  $CX$ .

Similarly, we have a relative cube construction for fibrations, and we can define  $\mathbb{F}_X$  to be the "object of such retractions."

Thus, if  $\mathcal{E}$  is a type-theoretic model topos, so is the "injective model structure" on  $[\mathcal{C}, \mathcal{E}]$ .

$\mathcal{E}$  a type-theoretic model topos,  $\mathcal{C}$  a small (enriched) category.

$[\mathcal{C}, \mathcal{E}]$  = strict functors  $\mathcal{C} \rightarrow \mathcal{E}$  and strict transformations (a 1-category)

$[[\mathcal{C}, \mathcal{E}]]$  = weak functors and weak transformations (an  $(\infty, 1)$ -category)

The "pointwise" homotopy theory of  $[\mathcal{C}, \mathcal{E}]$  *doesn't* model  $[[\mathcal{C}, \mathcal{E}]]$ :

- Every weak functor is equivalent to a strict one, but
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A strict  $X \in [\mathcal{C}, \mathcal{E}]$  is *injectively fibrant* (a.k.a. *flexible*) if any weak transformation  $A \rightsquigarrow X$  is equivalent to a strict one (by a natural operation that leaves strict transformations fixed).

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There is a weak morphism cocomma, known in classical homotopy theory as a cobar construction:

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Thus, if  $\mathcal{L}$  is a type-theoretic model topos, so is the injective model structure on  $[\mathcal{L}, \mathcal{E}]$ .

# Theorem Type-theoretic model toposes include

simplicial sets

The theory of simplicial sets is captured by a type-theoretic model theory.  
 - right adjoint to the forgetful functor  
 - more structure by a type-theoretic model theory  
 The theory of simplicial sets with the initial object is a model theory.  
 - fibrations are maps to the initial object

and are closed under

diagram categories

If  $\mathcal{E}$  is a type-theoretic model theory,  $\mathcal{I}$  is a model category, then the model category of  $\mathcal{I}$ -valued simplicial sets is a type-theoretic model theory.  
 The theory of  $\mathcal{I}$ -valued simplicial sets is a model theory.  
 - fibrations are maps to the initial object  
 - cofibrations are maps from the initial object  
 - monomorphisms are maps from the initial object  
 - epimorphisms are maps to the initial object  
 - isomorphisms are maps from the initial object to the initial object  
 - zero objects are the initial object  
 - kernels are the initial object  
 - cokernels are the initial object  
 - images are the initial object  
 - coimages are the initial object  
 - factorizations are the initial object  
 - extensions are the initial object  
 - retractions are the initial object  
 - sections are the initial object  
 - normal subobjects are the initial object  
 - quotient objects are the initial object  
 - subobjects are the initial object  
 - objects are the initial object  
 - morphisms are the initial object  
 - composites are the initial object  
 - identities are the initial object  
 - inverses are the initial object  
 - adjoints are the initial object  
 - limits are the initial object  
 - colimits are the initial object  
 - products are the initial object  
 - coproducts are the initial object  
 - equalizers are the initial object  
 - coequalizers are the initial object  
 - pullbacks are the initial object  
 - pushouts are the initial object  
 - fiber products are the initial object  
 - direct products are the initial object  
 - direct sums are the initial object  
 - tensor products are the initial object  
 - cotensor products are the initial object  
 - Hom objects are the initial object  
 - exponential objects are the initial object  
 - power objects are the initial object  
 - comonads are the initial object  
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 $\mathcal{E}$  is a model of type theory,  $S$  is a set of maps.  
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 These form a reflective subcategory,  $(\text{Rel}(S), \text{Isigma}(S, \text{pt}))$ .

Theorem If  $\mathcal{E}$  is a type-theoretic model theory and  $S$ -localization is left exact, then a fibration  $X \rightarrow Y$  is relatively  $S$ -local iff it is a fibration in the local model structure.  
 (see the book on Model Theory for a characterization of left exactness)  
 Then we can take  $\mathcal{E}_S = \text{Rel}(S)$  if  $\text{Isigma}(S, \text{pt})$ .  
 So if  $\mathcal{E}$  is a type-theoretic model theory, then  $\mathcal{E}_S$  is a left exact localization of  $\mathcal{E}$ .

Therefore, they model all Grothendieck-Lurie  $(\infty, 1)$ -toposes.

# left exact localizations.

$\mathcal{E}$  a model category,  $S$  a set of maps

A fibrant object  $X \in \mathcal{E}$  is *externally  $S$ -local* if

$$\text{Map}(B, X) \xrightarrow{\cong} \text{Map}(A, X) \text{ for all } (A \rightarrow B) \in S.$$

These are the fibrant objects in a local model structure.

$\mathcal{E}$  a model of type theory,  $S$  a set of maps

A fibration  $X \rightarrow Y$  is *internally  $S$ -local* if

$$Y \vdash \text{IsEquiv}(\lambda g. g \circ f : (B \rightarrow X) \rightarrow (A \rightarrow X)) \text{ for all } (A \xrightarrow{f} B) \in S.$$

These form a *reflective subuniverse*. (Rijke-S. Spitters)

Theorem If  $\mathcal{E}$  is a type-theoretic model topos and  $S$ -localization is *left exact*, then a fibration  $X \rightarrow Y$  is

*internally  $S$ -local*  $\iff$  it is a fibration in the local model structure

(uses Anel-Bicikman-Priest-Joyal forthcoming characterization of left exactness)

Thus we can take  $F_X^S = \text{Fib}_X \times_Y \prod_{f \in S} \text{IsEquiv}(\lambda g. g \circ f)$ .

So if  $\mathcal{E}$  is a type-theoretic model topos, so is any left exact localization of it.

$\mathcal{E}$  a model category,  $S$  a set of maps

A fibrant object  $X \in \mathcal{E}$  is **externally  $S$ -local** if

$$\text{Map}(B, X) \xrightarrow{\cong} \text{Map}(A, X) \quad \text{for all } (A \rightarrow B) \in S.$$

These are the fibrant objects in a **local model structure**.

---

$\mathcal{E}$  a model of type theory,  $S$  a set of maps

A fibration  $X \twoheadrightarrow Y$  is **internally  $S$ -local** if

$$Y \vdash \text{IsEquiv}(\lambda g. \text{gof} : (B \rightarrow X) \rightarrow (A \rightarrow X)) \quad \text{for all } (A \xrightarrow{f} B) \in S.$$

These form a **reflective subuniverse**. (Rijke-S.-Spitters)

Theorem If  $\mathcal{E}$  is a type-theoretic model topos and  $S$ -localization is left exact, then a fibration  $X \rightarrow \Gamma$  is internally  $S$ -local  $\iff$  it is a fibration in the local model structure

(uses Anel-Bicdermann-Finster-Joyal forthcoming characterization of left exactness)

Thus we can take  $\mathbb{F}_X^S = \mathbb{F}_X \times_Y \prod_{f \in S} \text{IsEquiv}(\lambda g. \text{gof})$ .

So if  $\mathcal{E}$  is a type-theoretic model topos, so is any left exact localization of it.

# Theorem Type-theoretic model toposes include

simplicial sets

*[Faint handwritten notes]*

and are closed under

diagram categories

*[Faint handwritten notes]*

and left exact localizations.

*[Faint handwritten notes]*

Therefore, they model all Grothendieck-Lurie  $(\infty, 1)$ -toposes.

# Theorem Any type-theoretic model topos has higher inductive

pushouts,

W-types,

and

other HITs,

**A pushout square**

Let  $A \rightarrow B$  and  $A \rightarrow C$  be maps in a topos. The pushout is the universal object  $D$  with maps  $B \rightarrow D$  and  $C \rightarrow D$  such that the square commutes.

**W-types**

A W-type is a sequence of objects  $A_0, A_1, A_2, \dots$  with arrows  $A_0 \rightarrow A_1$ ,  $A_1 \rightarrow A_2$ , etc., and diagonal arrows  $A_0 \rightarrow A_2$ ,  $A_1 \rightarrow A_3$ , etc.

**Higher Inductive Types (HITs)**

A HIT is a type with a set of constructors and a set of relations between them. For example, a HIT with two constructors  $c_1, c_2$  and a relation  $r$  between them.

and the universes are

closed under them.

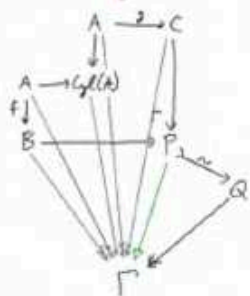
**Universes**

A universe is a type  $U$  with a map  $U \rightarrow \mathcal{U}$  such that  $U$  is closed under the operations of the topos.

Universes are closed under the operations of the topos, including pushouts, W-types, and HITs.

# pushouts,

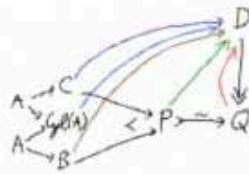
A *homotopy pushout* of  $B \xleftarrow{f} A \xrightarrow{g} C$  is a fibration replacement:



**NB** The explicit homotopy pushout  $P \rightarrow \Gamma$  is not a fibration, but it is sharp!

(Sharp maps are closed under pullback-stable homotopy colimits, because weak equivalences are closed under homotopy colimits.)

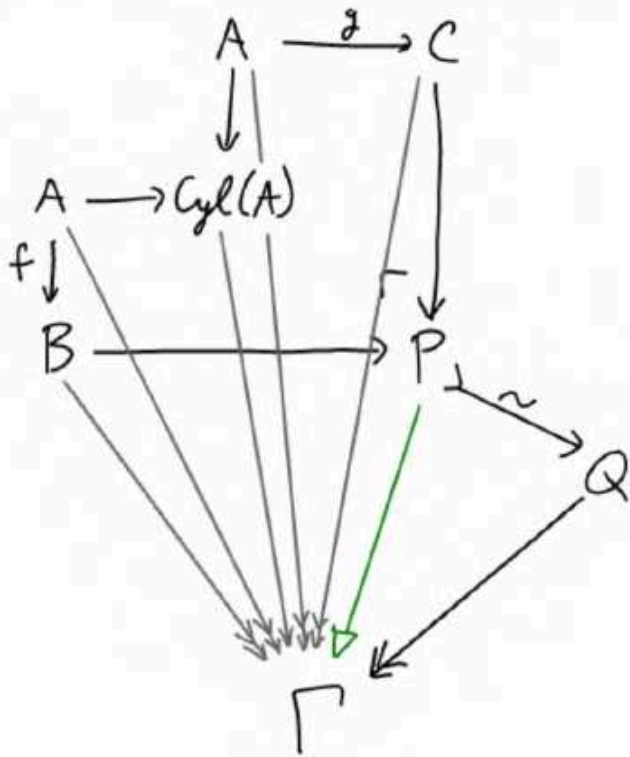
This gives it the correct elimination rule:



$$\frac{\begin{array}{l} z: Q \vdash D(z) \text{ type} \\ u: B \vdash d(u): D(d(u)) \quad y: C \vdash c(y): D(c(y)) \\ w: A \vdash f: \text{Id}_{\text{fib}(c)}(d(f \circ i), c(j \circ l)) \end{array}}{z: Q \vdash \text{Qwd}(d, c, f, z) : D(z)}$$



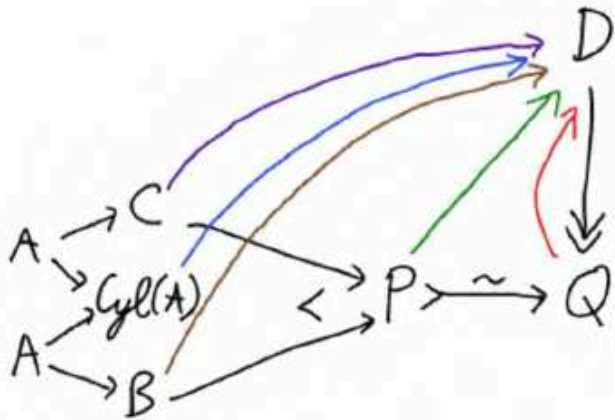
A homotopy pushout of  $B \xleftarrow{f} A \xrightarrow{g} C$  is a fibrant replacement:



NB The explicit homotopy pushout  $P \rightarrow \Gamma$  is not a fibration, but it is sharp!

(Sharp maps are closed under pullback-stable homotopy colimits, because weak equivalences are closed under homotopy colimits.)

This gives it the correct elimination rule:



$$\begin{array}{l}
 z:Q \vdash D(z) \text{ type} \\
 x:B \vdash d(x):D(\text{inl}(x)) \quad y:C \vdash e(y):D(\text{inr}(y)) \\
 w:A \vdash f: \text{Id}_{\text{glue}(w)}^D(d(f(w)), e(g(w))) \\
 \hline
 z:Q \vdash Q\text{-ind}(d, e, f, z):D(z)
 \end{array}$$

# Theorem Any type-theoretic model topos has higher inductive

pushouts,

W-types,

and

other HITs,

**A pushout**

Let  $A, B, C$  be types in a model  $\mathcal{M}$ . A pushout of  $A \rightarrow B$  and  $A \rightarrow C$  is a type  $D$  with maps  $B \rightarrow D$  and  $C \rightarrow D$  such that the square commutes. In a type-theoretic model, this is realized by the coequalizer of the two maps from  $A$  to  $B$  and  $C$ .

**W-types**

A W-type is a type  $W$  with a constructor  $\text{fold} : \prod_{x:A} (A \rightarrow B) \rightarrow W$ . In a type-theoretic model,  $W$  is the colimit of the chain  $A \rightarrow B \rightarrow B \rightarrow B \rightarrow \dots$  where each map is the constructor  $\text{fold}$ .

**Other HITs**

A higher inductive type (HIT) is a type  $X$  with a set of constructors  $\{c_i\}$  and a set of path constructors  $\{p_j\}$ . In a type-theoretic model,  $X$  is the colimit of the constructors, with the path constructors identifying the images of the constructors.

and the universes are

closed under them.

**Universes**

A universe  $\mathcal{U}$  is a type of types. In a type-theoretic model, the universes are closed under the operations of pushouts, W-types, and other HITs. This means that if  $A, B, C$  are in  $\mathcal{U}$ , then their pushout, W-type, and other HITs are also in  $\mathcal{U}$ .

# W-types

A W-type is a **homotopy-initial algebra** for an endofunctor  $F$ .

A **strictly initial algebra** can be constructed as a transfinite iteration:

$$\begin{array}{ccccccc}
 FX_0 & \rightarrow & FX_1 & & \rightarrow & FX_n & \rightarrow & FX_{n+1} \\
 \downarrow & & \downarrow & & & & & \downarrow \\
 X_0 & \rightarrow & X_1 & \rightarrow & X_2 & \rightarrow & \dots & \rightarrow X_{n+1}
 \end{array}$$

$$X_0 = \emptyset$$

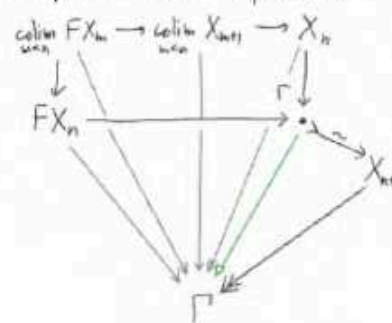
$$X_n = \operatorname{colim}_{m < n} X_m$$

for a limit ordinal  $n$

$$\begin{array}{ccccc}
 \operatorname{colim}_{m < n} FX_m & \rightarrow & \operatorname{colim}_{m < n} X_{m+1} & \rightarrow & X_n \\
 \downarrow & & & & \uparrow \\
 FX_n & \rightarrow & & & X_{n+1}
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at a successor ordinal.

The strict initial algebra may not be a fibration, so we incorporate fibrot replacement:



This gives a fibration with the correct elimination rule.

(Lumsdaine-S uses an algebraic version)

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 & \downarrow & & \downarrow & & & \downarrow & & & \downarrow \\
 X_0 & \longrightarrow & X_1 & \longrightarrow & X_2 & \longrightarrow & \dots & \longrightarrow & X_\omega & \longrightarrow & X_{\omega+1} & \longrightarrow & \dots & \longrightarrow & X_\infty
 \end{array}$$

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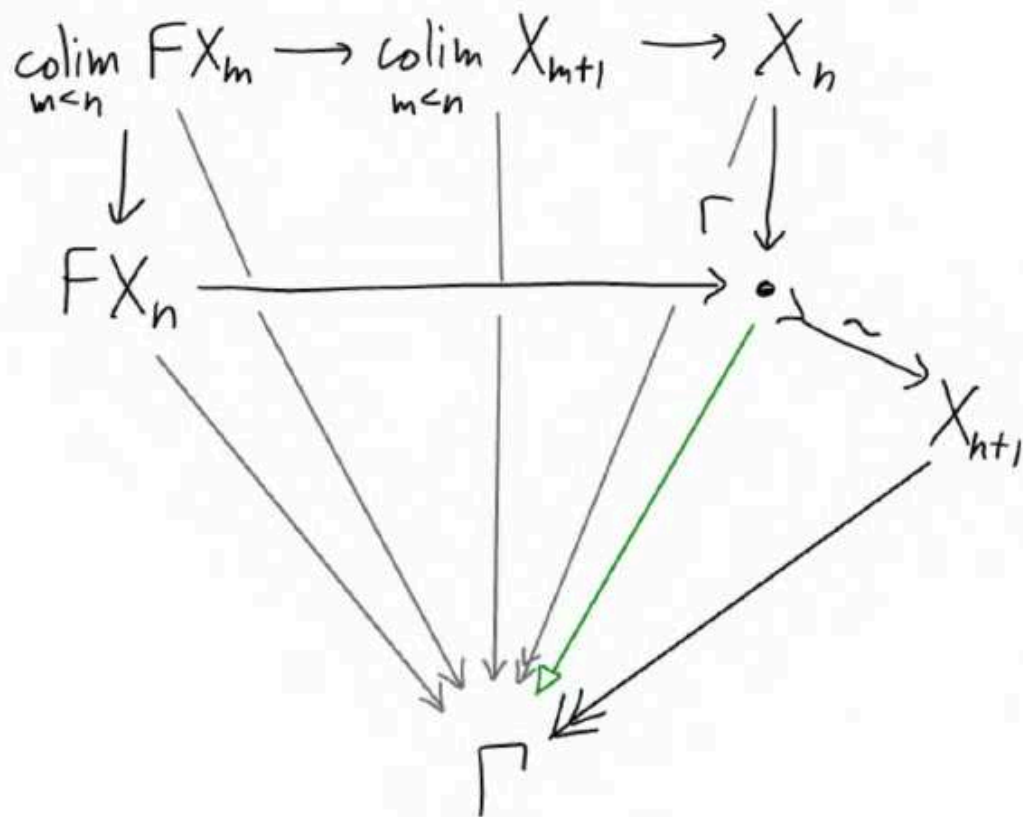
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Theorem Any type-theoretic model topos has higher inductive

pushouts,

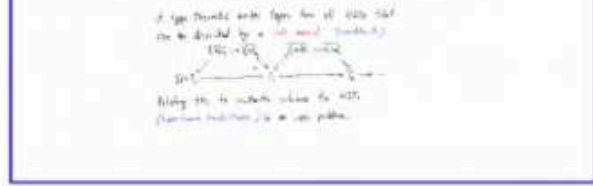


W-types,



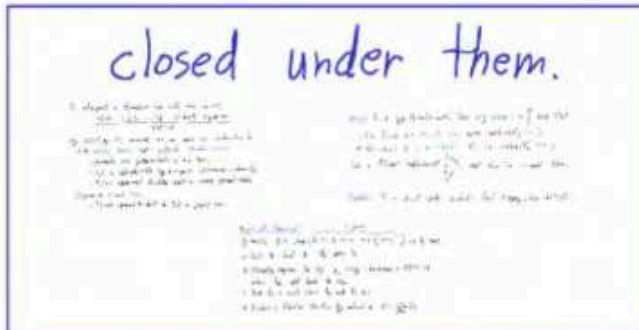
and

other HITs,



and the universes are

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# other HITs,

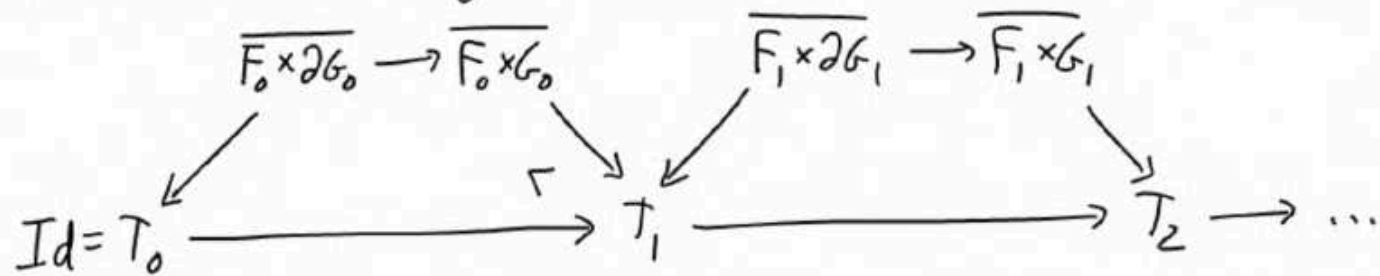
A type-theoretic model topos has all HITs that can be described by a **cell monad** (Lumsdaine-S.)

$$\begin{array}{ccccc} & \overline{F_0 \times \mathcal{G}_0} \rightarrow \overline{F_0 \times G_0} & & \overline{F_1 \times \mathcal{G}_1} \rightarrow \overline{F_1 \times G_1} & \\ & \swarrow & \searrow & \swarrow & \searrow \\ \text{Id} = T_0 & \xrightarrow{\quad \tau \quad} & T_1 & \xrightarrow{\quad} & T_2 \rightarrow \dots \end{array}$$

Relating this to syntactic schemas for HITs (Kopzi-Kouaca, Cavallo-Harper, ...) is an open problem.



A type-theoretic model topos has all HITs that can be described by a **cell monad** (Lumsdaine-S.)



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**W-types**

A W-type is a sequence of objects  $A_0, A_1, A_2, \dots$  with arrows  $A_i \rightarrow A_{i+1}$  and  $A_i \rightarrow A_{i+2}$  for  $i \geq 0$ .

**Higher Inductive Types (HITs)**

A HIT is a type with a set of constructors and a set of equations. For example, the natural numbers  $\mathbb{N}$  are a HIT with constructors  $0$  and  $s$ , and an equation  $s(0) = 1$ .

and the universes are

closed under them.

**Universes**

A universe is a type  $U$  with a map  $U \rightarrow \mathcal{U}$  such that  $U$  is closed under the operations of the topos.

Examples of universes include  $\mathcal{U}_0$ ,  $\mathcal{U}_1$ , and  $\mathcal{U}_2$ .

# Variations on the definition

## Simplicial

Do simplicial fibrations have a simplicial structure?

- Fibrations equipped with simplicial structure
- $\mathcal{S}plice = \mathcal{S}plice$
- Simplicial model categories are fibrations in homotopy theory
- Sufficient to model all the homotopy types
- Other constructions are naturally simplicial
- Quotient models have simplicial models

## cubical

Can we extend cubical models over fibrations?

not needed, going cubical models!

$$\mathcal{C}ube = \mathcal{C}ube$$

- Fibration structures  $\mathcal{C}ube$  also naturally fibrations
- and give the same structure as in  $\mathcal{S}plice$  (the same)
- and help extend the model structure:  $\mathcal{C}ube = \mathcal{C}ube$
- More precisely, extending the fibrations structure of  $\mathcal{C}ube$  to  $\mathcal{C}ube$
- includes both simplicial & cubical models
- Can we extend type theory to incorporate a more flexible type?

\* If I think about cubical, I should be in fibrations and  
 simplicial models with their associated model structure -  
 for simplicial models all the fibrations are fibrations!

## constructive

In an ambient constructive logic, it seems  
 the nature of  $\mathcal{C}ube$  under categories must be relevant:  
 constructive model structures really allow all assumptions  
 to be constructive.

8/17

The original definition was a **simplicial** t.t.m.t.:

- Fiberwise enriched over simplicial sets

$$\text{Cyl}(X) = \Delta^1 \otimes X$$

- Simplicial model categories are familiar in homotopy theory
- Suffices to model all G-L  $(\infty, 1)$ -toposes
- Cobar constructions are naturally simplicial
- Doesn't include newer cubical models.

# Variations on the definition

## Simplicial

The original definition was a simplicial set:

- Fibrewise pointed sets (pointed sets)
- $\mathcal{S}(n) = \mathcal{S}(n-1)$
- Simplicial model categories are finitary in homotopy theory
- Difficult to model all AC (algebraic)
- Like constructions are already explicit
- Don't really have cubical models

## cubical

Cubical model categories are fibrewise  
well-powered, group-coalgebra sets:

$$\mathcal{C}(n) = \mathcal{C}(n-1)$$

- Fibrewise structure  $\mathcal{C}$  sets already have structure  
and give the same answer as the simplicial case  
but only require the model structure: [10, 11, 12]
  - May provide something like abstract structure of [13] (e.g.  $\mathcal{C}$ )  
which both explicit & cubical models
  - In what type theory is interpreted in any [14] steps?
- If  $\mathcal{C}$  had a good model structure is possible to a model on  
simplicial sets with the same homotopy theory -  
is abstract, like and not being dependent?

## constructive

In an abstract constructive logic, it seems  
the notion of cubical model category need to be relaxed:  
constructive model structures need allow all assumptions  
to be constructive.

$$\mathcal{C} \rightarrow \mathcal{S}$$

Cartesian cubical set models are fiberwise self-enriched, giving cubical t.t.m.t.s:

$$\text{Cyl}(X) = \square^1 \times X$$

- Fibration structures  $\mathbb{F}_X$  arise naturally from Orton-Pitts and give the same universe as in Licata-Orton-Pitts-Spitters (and help construct the model structure: Sattler, Awodey, ...)
- More generally, axiomatizing the abstract structure of  $\text{Cyl} \dashv \text{CoCyl}$  includes both simplicial & cubical models.
- Can cubical type theory be interpreted in every  $(\infty, 1)$ -topos?

# Variations on the definition

## Simplicial

The original definition was a simplicial presheaf:

- Sites were assumed over simplicial sets
- $\mathcal{S}h(\mathcal{C}) = \mathcal{S}h(\mathcal{S})$
- Simplicial model categories are factor in bridge theory
- better to model all in first place
- Cubic constructions are already simplicial
- Don't add more cubical models

## cubical

Cartesian natural cell models are fibrant

with enriched, group cubical objects:

$$\mathcal{S}h(\mathcal{C}) = \mathcal{S}h(\mathcal{C})$$

- Flexibility structures of sets already show interesting and give the same answer as in cubical model theory (but help control the model structure - with more...)
  - More generally, according to cubical structures of  $\mathcal{S}h(\mathcal{C})$  include full enriched & cubical models
  - Is cubical type theory to be interpreted in some "cell-type"?
- Is a fixed cubical object  $\mathcal{C}$  possible to be enriched over enriched cubical sets with the enrichment model structure - let cubical sets all be at  $\mathcal{S}h(\mathcal{C})$  enriched!

## constructive

To be a cubical *constructive logic*, it seems the other of "Cubical model category" must be defined: constructive cubical structures only allow all assumptions to be constructive.

???

In an ambient **constructive logic**, it seems  
the notion of **Cisinski model category** must be relaxed:  
constructive model structures rarely allow **all** monomorphisms  
to be cofibrations.

? ? ?



# Variations on the definition

## Simplicial

The original definition was a complex that:

- Fibres are finite sets
- $\text{sgn}(\sigma) = \text{sgn}(\sigma)$
- Simplicial set categories are finitary in topology theory
- Sufficient to model all CW complexes
- Colours construction are naturally simplicial
- Don't include more natural models

## cubical

Sections related to models may fibration  
with associated group action results:

$$\text{sgn}(\sigma) = \text{sgn}(\sigma)$$

- Fibration structures. By using naturality from the fibration and give the same answer as the cubical case gives (but help understand the model structure - look notes...)
- More generally, connecting the abstract structure of fibrations models, but complex & natural models
- Can cubical type theory be interpreted in any model theory?
- If a third object exists is essential to an internal model, can we still use the cartesian model structure - do cubical sets use all theory of model?

## constructive

In an abstract *constructive logic*, it seems the notion of cubical model categories need to be replaced: constructive model structures would allow all assumptions to be reflexive.

???