Cell complexes and inductive definitions

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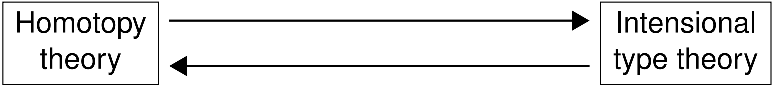
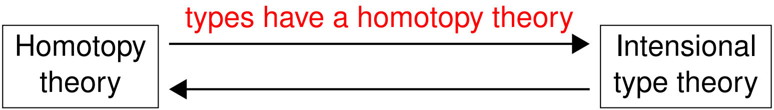
Homotopy type theory



Homotopy type theory



Homotopy type theory

$$\begin{tikzcd}[column sep=huge] \boxed{\text{Homotopy theory}} \arrow[r, shift left, "\text{types have a homotopy theory}"] & \boxed{\text{Intensional type theory}} \arrow[l, shift left] \end{tikzcd}$$

That means. . .

- Types form a model category (almost) with equivalences, fibrations, cofibrations
- We care about homotopically meaningful constructions
- . . .

Homotopy type theory

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That means. . .

- Type theory is a formal system, like ZFC
- Homotopy theory can be formalized in type theory

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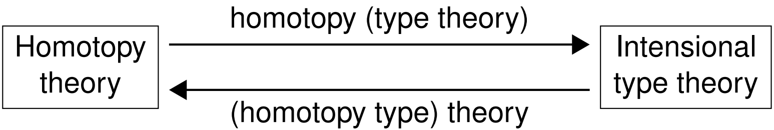
- Type theory is a formal system, like ZFC
- Homotopy theory can be formalized in type theory

What is this good for?

- A more direct formalization than in ZFC
- A more computer-friendly formal system than ZFC
- The same proof can apply to many homotopy theories (equivariant, parametrized, sheaves, . . .)

Homotopy type theory



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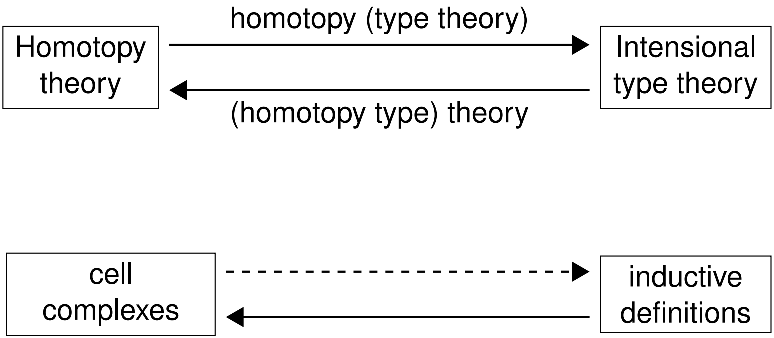
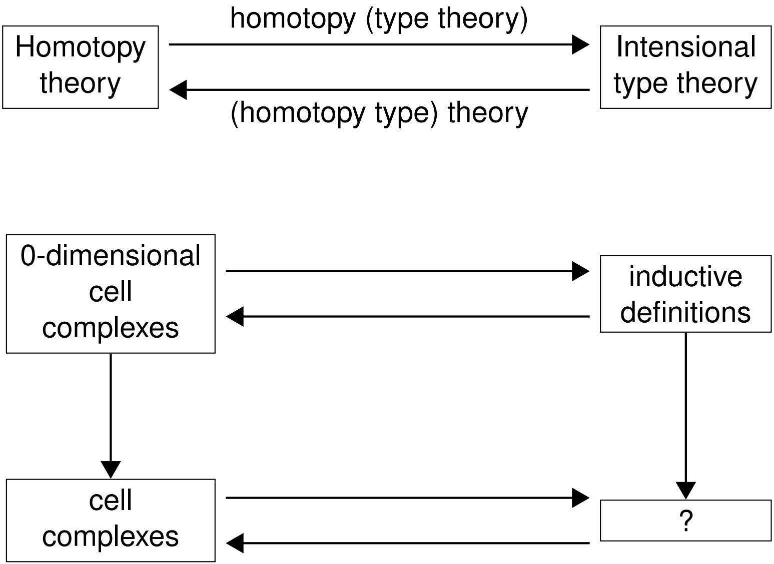


Homotopy type theory

$$
\begin{tikzcd}
\boxed{\text{Homotopy theory}} \arrow[r, shift left, "\text{homotopy (type theory)}"] & \boxed{\text{Intensional type theory}} \arrow[l, shift left, "\text{(homotopy type) theory}"]
\end{tikzcd}
$$

$$
\begin{tikzcd}
\boxed{\text{0-dimensional cell complexes}} \arrow[r, shift left] & \boxed{\text{inductive definitions}} \arrow[l, shift left]
\end{tikzcd}
$$

Homotopy type theory



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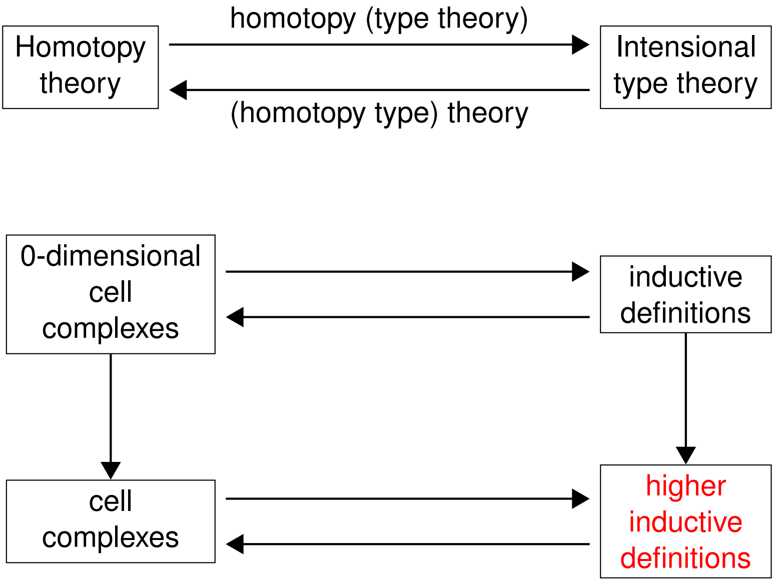

Outline

1. Inductive definitions

2. Higher inductive definitions

Outline

The natural numbers

Definition

The natural numbers $\mathbb{N}$ are inductively defined by

1. an element $0 \in \mathbb{N}$
2. an operation $s\colon \mathbb{N} \to \mathbb{N}$.

What does this mean?

The natural numbers

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What does this mean?

Answer #1

- 0 is a natural number;
- for any natural number x there is another called $s(x)$;
- and every natural number is constructed in exactly one of these ways.

The natural numbers

Definition

The natural numbers $\mathbb{N}$ are inductively defined by

1. an element $0 \in \mathbb{N}$
2. an operation $s \colon \mathbb{N} \to \mathbb{N}$.

What does this mean?

Answer #2

- $(\mathbb{N}, 0, s)$ is an initial object in the category whose objects are triples $(X,\ 0_x \in X,\ s_X \colon X \to X)$.

More inductive definitions

Example

For any set X, the set L_X of finite lists of elements of X is inductively defined by

1. the empty list $\epsilon \in L_X$
2. the "cons" operation $X \times L_X \to L_X$.

More inductive definitions

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Example

The set T of finite binary rooted trees is inductively defined by

1. a leaf node $\ell \in T$
2. a "branch node" operation $T \times T \to T$.

Inductive definitions

Definition

An inductive definition of a set A is a list of constructor types

$$F_i(A) \to A$$

where each F_i is an endofunctor of **Set**.

- For $\mathbb{N}$, we have $F_0(A) = 1$, $F_1(A) = A$.
- For L_X, we have $F_0(A) = 1$, $F_1(A) = X \times A$.
- For T, we have $F_0(A) = 1$, $F_1(A) = A \times A$.

The set being defined is the initial object of the category of sets equipped with such constructor maps.

Non-recursive inductive definitions

The domains of the constructors don't have to involve A at all.

Examples

- For any sets X and Y, their disjoint union $X \sqcup Y$ is inductively defined by
  1. a function $X \to X \sqcup Y$
  2. a function $Y \to X \sqcup Y$

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- A three-element set Z is inductively defined by
    1. An element $a \in Z$
    2. An element $b \in Z$
    3. An element $c \in Z$

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- A three-element set Z is inductively defined by
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  2. An element $b \in Z$
  3. An element $c \in Z$
- The empty set $\emptyset$ is inductively defined by

Constructing inductively defined sets

Theorem

Any inductive definition defines an essentially unique set.

Proof.

Uniqueness is easy (it is initial in some category). For existence, construct $A_0 \to A_1 \to A_2 \to \cdots$ as follows.

1. Let $A_0 = \emptyset$.
2. Let each A_{n+1} be A_n plus new images for all constructors acting on elements of A_n (that haven't been added yet).

This eventually converges. $\square$

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Example

For $\mathbb{N}$, we have $\emptyset \to \{0\} \to \{0, 1\} \to \{0, 1, 2\} \to \cdots$

A slogan

An inductive definition is a precise description of a universal property for a set, from which we can automatically extract an iterative construction of that set.

Outline

From sets to spaces

Observation

The iterative construction

$$A_0 \to A_1 \to A_2 \to \cdots$$

is a 0-dimensional cell complex. The constructors tell us when to glue in a new 0-cell.

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Question

Can we describe more general cell complexes with “inductive definitions”?

An “n-cell constructor” will need to specify, not just when to glue in a new n-cell, but what its attaching map should be.

Higher inductive definitions

Definition (Lumsdaine, S.)

A higher inductive definition of a space A is a list of constructor types, each of which has

- a domain $F_i(A)$; and
- a codomain which is one of
  1. A;
  2. the space of paths between two specified points of A;
  3. the space of homotopies between two specified paths in A;
  4. $\cdots$

Higher inductive definitions

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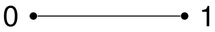
  4. $\cdots$

Remark

Instead of "homotopies between two specified paths in A" we could say "nullhomotopies of a specified loop in A", or any other way to describe a 1-sphere in A. This way just matches the type theory better.

Cell complexes as higher induction



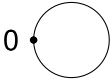
Example

The interval I is inductively defined by

1. a point $0 \in I$
2. a point $1 \in I$
3. a path $0 \rightsquigarrow 1$

Thus I is initial in the category of spaces X equipped with two points $0_X, 1_X \in X$ and a path $0_X \rightsquigarrow 1_X$.

Cell complexes as higher induction



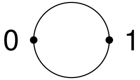
Example

The circle S^1 is inductively defined by

1. a point $0 \in S^1$
2. a path $0 \rightsquigarrow 0$

NB: the path $0 \rightsquigarrow 0$ is not the constant path!

Cell complexes as higher induction



Example

S^1 is also inductively defined by

1. two points $0, 1 \in S^1$
2. two paths $0 \rightsquigarrow 1$, $0 \rightsquigarrow 1$

Cell complexes as higher induction

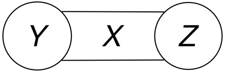
$$
\begin{tikzcd}
\bullet \arrow[r, dash, "\ell"] \arrow[d, dash, "m"'] \arrow[dr, phantom, "\simeq"] & \bullet \arrow[d, dash, "m"] \\
\bullet \arrow[r, dash, "\ell"'] & \bullet
\end{tikzcd}
$$

Example

The torus T^2 is inductively defined by

1. a point $0 \in T^2$
2. a path $\ell \colon 0 \rightsquigarrow 0$
3. a path $m \colon 0 \rightsquigarrow 0$
4. a path $\ell * m \rightsquigarrow m * \ell$ (where $*$ denotes path concatenation)

Homotopy colimits



The homotopy pushout of $f\colon X \to Y$ and $g\colon X \to Z$ is inductively defined by

1. a map $h\colon Y \to P$
2. a map $k\colon Z \to P$
3. for every $x \in X$, a path $h(f(x)) \rightsquigarrow k(g(x))$

Remark

Everything happens in the category of spaces, so all constructors are automatically continuous. In particular, $h(f(x)) \rightsquigarrow k(g(x))$ depends continuously on x.

Postnikov towers

$$\begin{tikzcd}
 & & & X \arrow[dll] \arrow[d] \arrow[dr] \arrow[drr] & & \\
\cdots \arrow[r] & X_n \arrow[r] & \cdots \arrow[r] & X_1 \arrow[r] & X_0 \arrow[r] & X_{-1}
\end{tikzcd}$$

The “very bottom” X_{-1} is

- empty if X is empty, and
- contractible if X is nonempty.

It is inductively defined by

1. a map $X \to X_{-1}$
2. for every $x, y \in X_{-1}$, a path $x \rightsquigarrow y$

Remark

Again, the path $x \rightsquigarrow y$ depends continuously on x and y.

Postnikov towers, II

Similarly, X_0 is inductively defined by

1. a map $X \to X_0$
2. for any $x, y \in X_0$ and paths $\alpha, \beta \colon x \rightsquigarrow y$, a path $\alpha \rightsquigarrow \beta$.

X_1 is inductively defined by

1. a map $X \to X_1$
2. for any $x, y \in X_1$, any paths $\alpha, \beta \colon x \rightsquigarrow y$, and any paths $\mu, \nu \colon \alpha \rightsquigarrow \beta$, a path $\mu \rightsquigarrow \nu$.

and so on...

Constructing higher inductive types

Theorem

Any higher inductive definition defines an essentially unique space.

Proof.

Proceed as before, constructing $A_0 \to A_1 \to A_2 \to \cdots$.
This time, at each step we glue in cells of appropriate dimensions corresponding to all the constructors. $\square$

Localization

Given $f\colon A \to B$.

Definition

Z is *f-local* if $\mathrm{Map}(B, Z) \xrightarrow{f^*} \mathrm{Map}(A, Z)$ is an equivalence.
An *f-localization* of X is an (up-to-homotopy) reflection of X into f-local spaces.

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An f-localization of X is an (up-to-homotopy) reflection of X into f-local spaces.

First try

The f-localization X_f of X is inductively defined by

1. a map $X \to X_f$
2. for any $g\colon A \to X_f$ and $b \in B$, a point $e_g(b) \in X_f$
3. for any $g\colon A \to X_f$ and $a \in A$, a path $e_g(f(a)) \rightsquigarrow g(a)$
4. for any $h\colon B \to X_f$ and $b \in B$, a path $e_{h \circ f}(b) \rightsquigarrow h(b)$

Idea: $g \mapsto e_g$ defines $\mathrm{Map}(A, X_f) \to \mathrm{Map}(B, X_f)$, and the two path-constructors make it a homotopy inverse to f^*.

Localization

- This space X_f is f-local.
- But it is not the f-localization of X: it is (homotopy) initial among spaces under X equipped with a chosen homotopy inverse to f^*.
- We need "homotopy equivalence data" for f^* which lives in a contractible space.

Second try (this one works)

The f-localization X_f of X is inductively defined by

1. a map $X \to X_f$
2. for $g\colon A \to X_f$ and $b \in B$, a point $e_g(b) \in X_f$
3. for $g\colon A \to X_f$ and $a \in A$, a path $\sigma_g(a)\colon e_g(f(a)) \rightsquigarrow g(a)$
4. for $h\colon B \to X_f$ and $b \in B$, a path $\rho_h(b)\colon e_{h\circ f}(b) \rightsquigarrow h(b)$
5. for $h\colon B \to X_f$ and $a \in A$, a path $\rho_h(f(a)) \rightsquigarrow \sigma_{h\circ f}(a)$

Third try (this one works too)

The f-localization X_f of X is inductively defined by

1. a map $X \to X_f$
2. for $g\colon A \to X_f$ and $b \in B$, a point $e^1_g(b) \in X_f$
3. for $g\colon A \to X_f$ and $b \in B$, a point $e^2_g(b) \in X_f$
4. for $g\colon A \to X_f$ and $a \in A$, a path $\sigma_g(a)\colon e^1_g(f(a)) \rightsquigarrow g(a)$
5. for $h\colon B \to X_f$ and $b \in B$, a path $\rho_h(b)\colon e^2_{h\circ f}(b) \rightsquigarrow h(b)$

Spectrification

Definition

A prespectrum is a sequence of pointed spaces $\{X_n \mid n \in \mathbb{N}\}$ and maps $\gamma_n \colon X_n \to \Omega X_{n+1}$.

It is an (Ω-)spectrum if each γ_n is an equivalence.

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The spectrification $\{LX_n\}$ of $\{X_n\}$ is inductively defined by

1. maps $\ell_n \colon X_n \to LX_n$
2. for each $x \in LX_n$, a path $\ell_{n+1}(*) \rightsquigarrow \ell_{n+1}(*)$ (i.e. a map $L\gamma_n \colon LX_n \to \Omega(LX_{n+1})$)
3. for each $x \in X_n$, a path $L\gamma_n(\ell_n(x)) \rightsquigarrow (\Omega\ell_{n+1})(\gamma_n(x))$
4. data making each $L\gamma_n$ an equivalence, as for localization.

Concluding slogan

A higher inductive definition is a precise description of a universal property for a space, from which we can automatically extract an iterative construction of that space.

More Information

`http://www.homotopy type theory.org`