Errata for "Univalence for inverse diagrams and homotopy canonicity"

Michael Shulman

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1 A redundant axiom

I originally included the following property in the definition of a type-theoretic fibration category, but Joyal has pointed out that it follows from the other axioms. The following simple proof is due to the referee and Tamara von Glehn.

Lemma 1 (The referee; see Remark 3.8)**.** *Assuming conditions (1)–(5) for a type-theoretic fibration category, condition (6) also holds: given a commutative diagram:*

$$\begin{array}{ccccc} X & \xrightarrow{j} & Y & \xrightarrow{f} & Z \\ \downarrow{\scriptstyle r} & & \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle p} \\ A & \xrightarrow[i]{} & B & \xrightarrow[g]{} & C \end{array}$$

if $g : B \to C$ and $gi : A \to C$ are fibrations, $i : A \to B$ is an acyclic cofibration, and both squares are pullbacks (hence $f : Y \to Z$ and $fj : X \to Z$ are fibrations), then $j : X \to Y$ is also an acyclic cofibration.

Proof. If p is a fibration, so is its pullback q, and thus pullback along q preserves acyclic cofibrations. Thus, factoring p according to (5), we may assume it is an acyclic cofibration. But then, since g and gi are fibrations, q and r are again acyclic cofibrations, and thus so is the composite $ir = qj$. Thus it suffices to prove the following lemma. □

Lemma 2 (von Glehn)**.** *Assuming conditions (1)–(5), if gf and g are acyclic cofibrations, so is f.*

Proof. Suppose given a commutative square as on the top below, where p is a fibration; we must construct a lift.

$$\begin{array}{ccc} X & \xrightarrow{s} & A \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle p} \\ Y & \xrightarrow{t} & B \\ {\scriptstyle g}\downarrow & & \downarrow \\ Z & \longrightarrow & 1 \end{array} \tag{1}$$

Since $B \to 1$ is also a fibration and g is an acyclic cofibration, we have a lift $h : Z \to B$ such that $hg = t$. Now since gf is also an acyclic cofibration, we have a lift $k : Z \to A$ such that $pk = h$ and $k(gf) = s$. Therefore, the composite kg has the properties that $(kg)f = s$ and $p(kg) = hg = t$, as desired. (Note that this argument applies to the left class of any weak factorization system for which all maps to 1 lie in the right class.) □

2 A small gap in Proposition 2.13

The definition of a type-theoretic model category implies that acyclic cofibrations are preserved by pullback along fibrations, which is also part of the definition of type-theoretic fibration category. However, the two notions of "acyclic cofibration" are not *a priori* the same! Both are defined by left lifting against fibrations, but in different categories: in the whole model category or in the subcategory of fibrant objects.

Thus, to complete the proof we need to show that if a map between fibrant objects in a type-theoretic model category has the left lifting property with respect to all fibrations *between fibrant objects*, then it is an acyclic cofibration. Fortunately, this is not difficult: if $f : A \to B$ is such a map, then factoring it as an acyclic cofibration followed by a fibration, $f = pi$, then p is a fibration between fibrant objects. Thus, by the retract argument, f is a retract of i, hence also an acyclic cofibration.