Magnitude homology of enriched categories and metric spaces

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AMS Sectional Meeting
Special Session on Homotopy Theory
UC Riverside, November 4, 2017

Outline

What is magnitude homology?

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Magnitude homology is a categorification of magnitude.

What is magnitude homology?

What is magnitude homology?

Magnitude homology is a categorification of magnitude.

Questions

1. What is a “categorification”?
2. What is “magnitude”?

Categorification of invariants

Definition

A homology theory H_* categorifies a numerical invariant α if

$$\alpha(X) = \sum_n (-1)^n \operatorname{rk}(H_n(X))$$

Euler characteristic	$\rightsquigarrow$	ordinary homology
Jones polynomial	$\rightsquigarrow$	Khovanov homology
magnitude	$\rightsquigarrow$	magnitude homology

What kind of thing is magnitude?

Invariant	applies to
Euler characteristic / ordinary homology	spaces
Jones polynomial / Khovanov homology	knots
magnitude / magnitude homology	metric spaces

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Where do they live?

Object	“number” is a	homology is a
space	integer	abelian group
knot	Laurent polynomial	$\mathbb{Z}$-graded group
metric space	function $\mathbb{R} \rightharpoonup \mathbb{R}$	??

Where does magnitude come from?

$$\begin{array}{c}
\text{ordinary categories} \\
\Big\downarrow{\scriptstyle \text{nerve}} \\
\text{spaces} \\
\Big\downarrow{\scriptstyle \text{homology}} \\
\text{groups} \\
\Big\downarrow{\scriptstyle \sum_n (-1)^n \operatorname{rk} H_n} \\
\text{Euler characteristic}
\end{array}$$

Where does magnitude come from?

ordinary categories

↓ nerve

spaces

↓ homology

groups

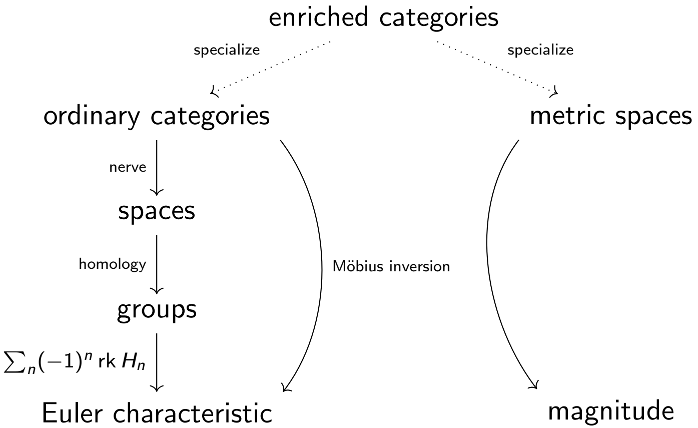
↓ $\sum_n (-1)^n \operatorname{rk} H_n$

Euler characteristic

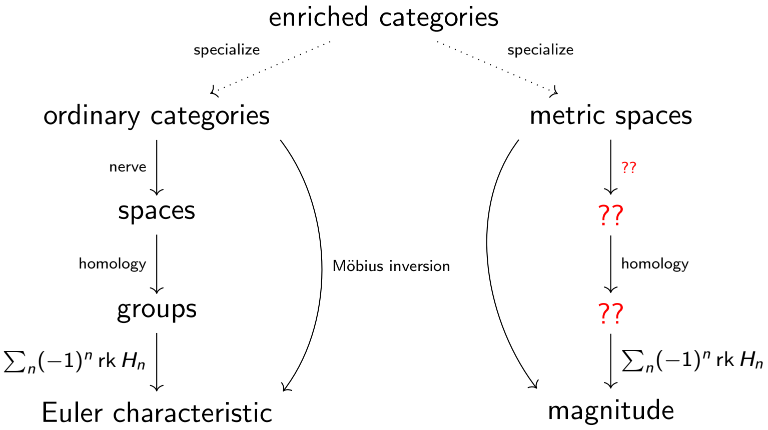
metric spaces

magnitude

Where does magnitude come from?



Where does magnitude come from?



Outline

Euler characteristic of categories

Observation

The Euler characteristic of the nerve of a (finite) category doesn't depend on its composition operation.

$$\begin{aligned}
\chi(NX) &= \sum_n (-1)^n \#(\text{nondegenerate } n\text{-simplices in } NX) \\
&= \sum_n (-1)^n \#(n\text{-strings of composable nonidentity arrows in } X) \\
&= \sum_n (-1)^n \sum_{x_0,\ldots,x_n} \prod_{k=1}^n \#(\text{non-id arrows } x_{k-1} \to x_k) \\
&= \sum_n (-1)^n \sum_{x_0,\ldots,x_n} \prod_{k=1}^n (\#X(x_{k-1}, x_k) - \delta_{x_{k-1},x_k})
\end{aligned}$$

Euler characteristic via matrices

Define a square matrix $Z_{xy} = \#X(x, y)$, with rows and columns labeled by the objects of X.

$$\begin{aligned}
\chi(NX) &= \sum_n (-1)^n \sum_{x_0,\ldots,x_n} \prod_{k=1}^{n} (\#X(x_{k-1}, x_k) - \delta_{x_{k-1},x_k}) \\
&= \sum_n (-1)^n \sum_{x_0,x_n} \sum_{x_1,\ldots,x_{n-1}} \prod_{k=1}^{n} (Z - \mathrm{Id})_{x_{k-1},x_k} \\
&= \sum_n (-1)^n \sum_{x_0,x_n} ((Z - \mathrm{Id})^n)_{x_0,x_n} \\
&= \sum_n (-1)^n (\text{sum of all the entries in } (Z - \mathrm{Id})^n) \\
&= \text{sum of all the entries in } \sum_n (-1)^n (Z - \mathrm{Id})^n
\end{aligned}$$

Euler characteristic via Möbius inversion

By formally summing a geometric series,

$$\sum_n (-1)^n (Z - \mathrm{Id})^n = \sum_n (\mathrm{Id} - Z)^n$$
$$= \frac{1}{\mathrm{Id} - (\mathrm{Id} - Z)} = \frac{1}{Z} = Z^{-1}$$

Being a little more careful about convergence, we get:

Theorem (Leinster)

If NX has only finitely many nondegenerate simplices, then its Euler characteristic is the sum of the entries in the inverse of the matrix $Z_{xy} = \#X(x, y)$.

The matrix Z is called the zeta function of X, and its inverse Z^{-1} is called its Möbius function.

How is a metric space like a category?

Example

A category X has hom-sets $X(x, y) \in \mathbf{Set}$ with unit and composition

$$1_x \in X(x, x)$$
$$X(x, y) \times X(y, z) \to X(x, z)$$

Example (Lawvere)

A metric space X has distances $d(x, y) \in [0, \infty)$ with unit and composition inequalities

$$0 \geq d(x, x)$$
$$d(x, y) + d(y, z) \geq d(x, z)$$

Enriched categories

Let $(\mathbf{V}, \otimes, I)$ be a monoidal category.

Definition

A **V**-enriched category has a set of objects, hom-objects $X(x, y) \in \mathbf{V}$, and unit and composition maps (plus axioms)

$$I \to X(x, x)$$
$$X(x, y) \otimes X(y, z) \to X(x, z)$$

Examples

- $\mathbf{V} = (\mathbf{Set}, \times, 1)$ – ordinary categories
- $\mathbf{V} = (\mathbf{Top}, \times, 1)$ – topological categories
- $\mathbf{V} = (\mathbf{AbGp}, \otimes, \mathbb{Z})$ – pre-additive categories
- $\mathbf{V} = (([0, \infty), \geq), +, 0)$ – (quasi-pseudo-)metric spaces

Magnitude of enriched categories

For X a $\mathbf{V}$-enriched category with finitely many objects, define

$$Z_{xy} = \#X(x, y)$$

where $\# : (\mathrm{ob}\,\mathbf{V}/\cong) \to \mathbb{k}$ is a homomorphism to the multiplicative monoid of a ring.

Definition (Leinster, $\sim$ 2008)

The magnitude of X is the sum of the entries of the inverse of the matrix Z over $\mathbb{k}$, if it exists:

$$\mathrm{Mag}(X) = \sum_{x,y} (Z^{-1})_{x,y}$$

We often complete $\mathbb{k}$ to a field, to invert more matrices.

The magnitude of a metric space

We need a "size" homomorphism $\# : ([0, \infty), +) \to (\mathbb{k}, \cdot)$.

Original answer

$$\mathbb{k} = \mathbb{R}, \qquad \#(\ell) = e^{-\ell}.$$

$\Rightarrow$ most finite metric spaces have a magnitude that is a real number.

First refined answer

We can scale a metric space X to get tX where

$$d_{tX}(x, y) = t \cdot d_X(x, y).$$

Then tX has a magnitude for almost all t, giving a magnitude function $\mathrm{Mag}(X) : \mathbb{R} \rightharpoonup \mathbb{R}$ defined except at some singularities.

Applications of magnitude of metric spaces

The magnitude of metric spaces has been well-studied by Leinster, Meckes, Willerton, and others.

1. The magnitude function is increasing for $t \gg 0$ and converges to the cardinality as $t \to \infty$. Thus it is the "effective cardinality of X at scale t".
2. Magnitude can be extended to infinite but compact spaces, by approximating them with finite ones, or by "replacing addition with integration".
3. For compact $A \subseteq \mathbb{R}^n$, the magnitude function is asymptotic to a known constant times $\mathrm{Vol}(A) \cdot t^n$ as $t \to \infty$.
4. For compact $A \subseteq \mathbb{R}^n$, the magnitude function has growth $\Theta(t^{\mathrm{rk}\, A})$, where $\mathrm{rk}\, A$ is the Minkowski dimension of A.
5. etc.

Outline

Categorifying magnitude

Problem

We want to categorify the magnitude (function). But what sort of thing has an Euler characteristic that is a partial function $\mathbb{R} \rightharpoonup \mathbb{R}$?

Solution

Some partial functions $\mathbb{R} \rightharpoonup \mathbb{R}$ are defined by more algebraic/combinatorial data.

Magnitude of graphs

Definition

If X is a combinatorial graph (vertices and edges), we make it a metric space where $d(x, y) =$ length of shortest path from x to y.

Theorem (Leinster, 2014)

The magnitude function of a graph is a rational function of $q = e^{-t}$.

Proof.

Instead of $\# : [0, \infty) \to \mathbb{R}$ defined by $\#(\ell) = e^{-\ell}$, use $\# : [0, \infty) \to \mathbb{Q}(q)$ defined by $\#(\ell) = q^{\ell}$, with q a formal variable. Then evaluate at $q = e^{-t}$, since $e^{-t \cdot d(x,y)} = (e^{-t})^{d(x,y)}$. □

Magnitude homology for graphs

1. Rational functions $\mathbb{Q}(q)$ embed in formal Laurent series $\mathbb{Q}((q))$ by long division.

$$\frac{1}{q(1-q)} = q^{-1} + 1 + q + q^2 + q^3 + \cdots$$

2. A formal Laurent series $\sum_{n \in \mathbb{Z}} a_n q^n$ is a family of coefficients $\{a_n\}_{n \in \mathbb{Z}}$, which can be the Euler characteristic of a $\mathbb{Z}$-graded homology theory.

Theorem (Hepworth–Willerton, 2015)

There is a $\mathbb{Z}$-graded magnitude homology theory for graphs whose Euler characteristic is (the coefficients of the formal Laurent series representation of) the magnitude function.

The magnitude chain complex

For $n, \ell \in \mathbb{Z}$,

$$MC(X)_n^\ell = \mathbb{Z}\big[\{(x_0, \ldots, x_n) \mid x_0 \neq x_1 \neq \cdots \neq x_n \text{ and } d(x_0, x_1) + \cdots + d(x_{n-1}, x_n) = \ell\}\big]$$

The boundary map $d_n : MC(X)_n^\ell \to MC(X)_{n-1}^\ell$ is

$$d_n = \sum_{i=0}^{n} (-1)^i d_n^i$$

$$d_n^i(x_0, \ldots, x_n) = \begin{cases} (x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) & \text{if that makes sense} \\ 0 & \text{otherwise} \end{cases}$$

where "makes sense" means that it lies in $MC(X)_n^\ell$, i.e. its total distance is still ℓ.

The magnitude nerve

$$MC(X)_n^\ell = \mathbb{Z}\big[\{(x_0,\ldots,x_n) \mid x_0 \neq x_1 \neq \cdots \neq x_n \text{ and } d(x_0,x_1)+\cdots+d(x_{n-1},x_n)=\ell\}\big]$$

This is the (ℓ-graded) normalized Dold–Kan complex associated to a simplicial abelian group, the magnitude nerve of X:

$$N(X)_n^\ell = \mathbb{Z}\big[\{(x_0,\ldots,x_n) \mid d(x_0,x_1)+\cdots+d(x_{n-1},x_n)=\ell\}\big]$$

The face maps are the d_n^i, the degeneracies s_n^i duplicate $x_i = x_{i+1}$, so normalizing forces all $x_i \neq x_{i+1}$.

NB

Similarly, the normalized nerve of an ordinary category involves strings of nonidentity morphisms, which occurred in our Möbius calculation of Euler characteristic.

Graphs → metric spaces → enriched categories

- The definitions of $MC(X)_*^\ell$ and $N(X)_*^\ell$ don't depend on X being a graph or ℓ being an integer! So there is a "magnitude homology" for any metric space, which is naturally $\mathbb{R}$-graded.
- But the identification of the magnitude function as a rational function, hence a Laurent series, does depend at least on all distances in X being integers.

So what we want to do is:

1. Represent the magnitude of an arbitrary metric space "algebro/combinatorially" to relate it to an $\mathbb{R}$-graded Euler characteristic.
2. Generalize $N(X)$, $MC(X)$, and this comparison to more general enriched categories.

We'll do (2) first and then (1).

Outline

Enriched nerves and Hochschild homology

Let **A** be a monoidal category, Y an **A**-enriched category, and M an **A**-enriched functor $Y^{\mathrm{op}} \otimes Y \to \mathbf{W}$ (a Y-Y-bimodule).

Definition

The two-sided bar simplicial construction is the simplicial object

$$B_n(Y;M) = \sum_{x_0,x_1,\ldots,x_n} Y(x_0,x_1) \otimes \cdots \otimes Y(x_{n-1},x_n) \otimes M(x_n,x_0).$$

Definition

If **A** is abelian, the Hochschild homology $HH_*(Y;M)$ is the homology groups of the chain complex associated to $B_\bullet(Y;M)$.

(More commonly Y has only one object, hence is an "algebra".)

Semicartesian Hochschild homology

Problem

In $B_\bullet(Y; M)$,

- The degeneracies duplicate objects and insert identities,
- The inner face maps delete objects and compose,
- The outer face maps delete objects and act on M.

But in the ordinary nerve and magnitude nerve, there is no M: the outer face maps just delete objects.

Solution

- We'd like to take M to be constant at the unit object, but in general this doesn't make sense.
- It does if $\mathbf{A}$ is semicartesian, i.e. the unit object is terminal.
- More generally, it makes sense if $Y = \Sigma X$, for $\Sigma : \mathbf{V} \to \mathbf{A}$ monoidal and $\mathbf{V}$ semicartesian.

Magnitude homology

- **V** semicartesian, **A** abelian
- $\Sigma : \mathbf{V} \to \mathbf{A}$ monoidal
- $\mathbb{1}$ the "constant ΣX-ΣX bimodule at the unit"

Definition

The magnitude nerve of X is the simplicial **A**-object

$$N_\bullet^\Sigma(X) = B(\Sigma X, \mathbb{1}).$$

Definition

The magnitude homology of X is the Hochschild homology

$$H_*^\Sigma(X) = HH_*(\Sigma[X]; \mathbb{1})$$

Homology of ordinary categories

Example

$$\mathbf{V} = \mathbf{Set} \qquad \mathbf{A} = \mathbf{Ab} \qquad \Sigma Y = \mathbb{Z}[Y]$$

$$\begin{aligned} B_n(\Sigma X; \mathbb{1}) &= \bigoplus_{x_0, x_1, \ldots, x_n} \mathbb{Z}[X(x_0, x_1)] \otimes \cdots \otimes \mathbb{Z}[X(x_{n-1}, x_n)] \\ &= \mathbb{Z}\left[\coprod_{x_0, x_1, \ldots, x_n} X(x_0, x_1) \times \cdots \times X(x_{n-1}, x_n)\right] \end{aligned}$$

This is the free simplicial abelian group on the nerve of X, so

$$H_*^{\Sigma}(X) = HH_*(\Sigma[X]; \mathbb{1}) = H_*(|NX|; \mathbb{Z})$$

Magnitude homology of metric spaces

Example

$$\mathbf{V} = [0, \infty) \qquad \mathbf{A} = \mathbf{Ab}^{[0,\infty)} \qquad \Sigma(\ell)_m = \begin{cases} \mathbb{Z} & \ell = m \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{aligned} B_n(\Sigma X; \mathbb{1})^\ell &= N(X)_n^\ell \\ &= \mathbb{Z}\big[\{(x_0, \ldots, x_n) \mid d(x_0, x_1) + \cdots + d(x_{n-1}, x_n) = \ell\}\big] \end{aligned}$$

Thus each metric space X has a magnitude homology $H_*^\Sigma(X)$, which is $[0, \infty)$-graded, $H_*^{\Sigma,\ell}(X)$.

Magnitude homology categorifies magnitude

With $\mathbf{V}, \mathbf{A}, \Sigma$ as before, and a well-behaved function $\mathrm{rk} : \mathbf{A} \to \Bbbk$, we can use $\# = \mathrm{rk} \circ \Sigma : \mathrm{ob}\,\mathbf{V} \to \Bbbk$ to define magnitude of $\mathbf{V}$-categories.

Theorem

If X has "only finitely many nondegenerate simplices", then its magnitude is the Euler characteristic of its magnitude homology:

$$\mathrm{Mag}(X) = \sum_{n=0}^{\infty} (-1)^n \,\mathrm{rk}\, H_n^{\Sigma}(X)$$

the apparently infinite sum being actually a finite one.

But this assumption fails even for finite metric spaces!

Outline

Generalized polynomials

Idea

There is no *a priori* reason why the exponents in a polynomial, rational function, or power series can only be integers.

Definition

A generalized polynomial is a finite formal linear combination of positive real-number powers of a variable, e.g.

$$7 + 3q^{\frac{1}{2}} - \frac{7}{3}q^{\sqrt{2}} + q^{20}$$

A generalized rational function is a formal quotient of generalized polynomials, e.g.

$$\frac{3 + 2q^{\frac{1}{3}} - q}{q^2 - q^{\pi}}$$

Formal magnitude

The generalized polynomials $\mathbb{Z}[q^{[0,\infty)}]$ are the ring freely generated by a monoid homomorphism $([0,\infty),+) \to (\mathbb{Z}[q^{[0,\infty)}],\cdot)$, and the generalized rational functions $\mathbb{Q}(q^{\mathbb{R}})$ are their field of fractions. So they are the "universal" choice of $\# : [0,\infty) \to \mathbb{k}$ for defining magnitude.

Theorem

Every finite metric space has a magnitude defined over $\mathbb{Q}(q^{\mathbb{R}})$.

A generalized rational function can be evaluated at any positive real value for q (except singularities), yielding the magnitude function $\mathbb{R} \rightharpoonup \mathbb{R}$.

Hahn series

Definition

A Hahn series (with coefficients $\mathbb{Q}$ and value group $\mathbb{R}$) is a possibly-infinite formal sum of monomials aq^{ℓ}, for $a \in \mathbb{Q}$ and $\ell \in \mathbb{R}$, such that the exponents ℓ are well-ordered.

Examples

- Any generalized polynomial
- Any generalized rational function, by long division
- Any formal power series, even any formal Laurent series
- $1 + q^{0.9} + q^{0.99} + q^{0.999} + q^{0.9999} + \cdots + q^{1}$

Hahn series form a complete non-Archimedean ordered field $\mathbb{Q}((q^{\mathbb{R}}))$, where positive powers of q are infinitesimal.

Magnitude homology categorifies magnitude, bis

Note

If an $\mathbb{R}$-graded group is finitely generated, and nonzero for only a well-ordered set of gradings, then its rank is a Hahn series.

Let $\mathbf{V} = [0, \infty)$, $\mathbf{A} = \mathbf{Ab}^{[0,\infty)}$, Σ as before.

Theorem

If X is a finite metric space, its magnitude is the Euler characteristic of its magnitude homology:

$$\mathrm{Mag}(X) = \sum_{n=0}^{\infty} (-1)^n \sum_{\ell} (\mathrm{rk}\, H_n^{\Sigma,\ell}(X)) q^{\ell}$$

where $\mathrm{Mag}(X) \in \mathbb{Q}(q^{\mathbb{R}})$ is embedded into $\mathbb{Q}((q^{\mathbb{R}}))$, and the infinite sum over n converges in the topology of $\mathbb{Q}((q^{\mathbb{R}}))$.

Magnitude homology detects convexity

A metric space is Menger convex if for any $x \neq z$ there exists y with $x \neq y \neq z$ and $d(x,y) + d(y,z) = d(x,z)$.

Theorem

A metric space X is Menger convex if and only if $H_1^{\Sigma}(X) = 0$.

Proof.

- A generating 1-cochain of degree ℓ is a pair (x,y) with $d(x,y) = \ell$; they are all cycles.
- A generating 2-cochain of degree ℓ is a triple (x,y,z) with $x \neq y \neq z$ and $d(x,y) + d(y,z) = \ell$.
- The boundary of (x,y,z) is (x,z) if $d(x,z) = \ell$, otherwise 0.
- Thus, a 1-boundary is a pair (x,z) such that there exists a y with $x \neq y \neq z$ and $d(x,y) + d(y,z) = d(x,z)$.

□

Menger convexity vs. geodesy

Fact

If closed bounded subsets of X are compact, then X is Menger convex if and only if it is geodesic, i.e. for all x, y there is an isometric injection $[0, d(x, y)] \to X$ joining x to y.

Corollary

If $X \subseteq \mathbb{R}^n$ is closed, with the subspace metric, then it is convex (in the usual sense) if and only if $H_1^{\Sigma}(X) = 0$.

If $X \subseteq \mathbb{R}^n$ is not closed, then Menger convexity is much weaker than convexity. E.g. any open subset of $\mathbb{R}^n$ is Menger convex, as is any dense subset of $\mathbb{R}$.

Quantitative non-convexity

More generally, $H_1^{\Sigma}(X)$ measures the failure of convexity, along with the length scales at which it happens.

Example

If X is a closed annulus with inner diameter δ, then $H_1^{\Sigma,\ell}(X) = 0$ for $\ell > \delta$.

Example

If X is two disjoint closed convex bodies at a distance δ apart, then $H_1^{\Sigma,\ell}(X) = 0$ for $\ell < \delta$.

Higher magnitude homology

Intuitively, $H_2^{\Sigma}(X)$ measures the non-uniqueness of geodesics connecting distinct points.

Example

If $X \subseteq \mathbb{R}^n$ is convex, with the subspace metric, then $H_2^{\Sigma}(X) = 0$.

Example

If S^1 has its geodesic metric, then $H_2^{\Sigma}(S^1)$ is the free abelian group in degree π on ordered pairs of antipodal points in S^1.

Outline

Questions about magnitude homology of categories

1. What other enriching categories **V** support an interesting magnitude homology that categorifies their magnitude?
   A possibility: $([0,\infty], \max)$, *giving ultrametric spaces, whose magnitudes convey "entropy" or "capacity".*
2. If **V** is not semicartesian, can we find an M such that $HH_*(\Sigma[X]; M)$ categorifies the magnitude of X?
   Evidence: *for a sufficiently finite algebra A over an algebraically closed field, the magnitude of the* FDVect*-category of indecomposable projective A-modules is the Euler characteristic of some Hochschild cohomology $HH^*(\Sigma[X]; M)$, but only by calculation.*
3. What is the role of Cauchy completion?
4. (How) do Künneth, Mayer-Vietoris, etc. theorems for Hochschild homology decategorify to magnitude?
   Evidence: *Hepworth–Willerton proved Künneth and Mayer-Vietoris theorems for magnitude homology of graphs.*

Questions about magnitude homology of metric spaces

1. What exactly does $H_n^{\Sigma}(X)$ mean geometrically for $n > 1$?
2. Can the magnitude of an *infinite* metric space be recovered from its magnitude homology? Or can we define an "analytic" magnitude homology in such cases?
3. Relatedly, when magnitude homology groups of an infinite metric space are nonzero, they tend to be infinitely generated. Can we impose structure to make them more manageable?
4. Are there any metric spaces whose magnitude homology contains torsion? *(Hepworth–Willerton asked this for graphs.)*
5. Magnitude homology (and magnitude) only notices where the triangle inequality is a strict equality. Is there a "blurred" version that notices approximate equalities?
6. (How) is magnitude homology related to persistent homology?

Further reading and discussion

Preprint

Tom Leinster and Michael Shulman, *Magnitude homology of enriched categories and metric spaces*, https://arxiv.org/abs/1711.00802.

A couple of minor errors have been pointed out, and I'm looking for help naming some definitions. Join the conversation:

Blog

https://golem.ph.utexas.edu/category/2017/11/magnitude_homology_is_hochschi.html