PSEUDOTOPOLOGICAL SPACES AND THE STONE-ČECH COMPACTIFICATION

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1. Filters

Our slogan is ***topology is about convergence***. Mostly we are familiar with convergence of sequences. Recall the following definition.

Definition 1.1. If X is a topological space, (a_n) a sequence in X, and $x \in X$, we say that (a_n) **converges to** x if for any open set $U \ni x$, there exists an N such that if $n > N$, then $a_n \in U$.

We observe a few things about this definition.

- For some spaces, knowing which sequences converge is enough to determine the topology, but not for all spaces. So we'd like a more generalized notion of convergence.
- The definition only requires knowledge of the sets $A_N = \{a_n : n > N\}$ as N varies.
- More precisely, it only requires knowledge of which sets *contain* at least one of the A_N. For example, if we alter the sequence at finitely many places, it doesn't change its convergence properties because it doesn't change which sets contain some A_N.

With this in mind, we make the following definition.

Definition 1.2. Let X be a set. A **filter on** X is a subset $\mathcal{F} \subset \mathcal{P}X$ of the power set of X such that

(i) $X \in \mathcal{F}$;
(ii) $\emptyset \notin \mathcal{F}$;
(iii) If $A \in \mathcal{F}$ and $A \subset B$, then $B \in \mathcal{F}$; and
(iv) If $A, B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$.

Example 1.3. If (a_n) is a sequence in X, the **elementary filter** associated to (a_n) is the set

$$\mathcal{E}_{(a_n)} = \{A \subset X : \exists N \text{ s.t. if } n > N \text{ then } a_n \in A\}$$

Note that if we change finitely many terms of a sequence, its elementary filter is unchanged. Moreover, the elementary filter contains exactly the information required to talk about convergence.

Definition 1.4. Let X be a topological space. We say that a filter $\mathcal{F}$ in X **converges to** a point $x \in X$, and write $\mathcal{F} \to x$, if every open set containing x is in $\mathcal{F}$.

Clearly a sequence converges to a point x if and only if its associated elementary filter does; thus the elementary filter captures everything we need to know about the sequence to discuss its convergence properties. We think of an arbitrary filter as a generalization of a sequence; for example, if $A \in \mathcal{F}$, we say that $\mathcal{F}$ is **eventually in** A. If $A \cap B \neq \emptyset$ for all

$B \in \mathcal{F}$, we say that $\mathcal{F}$ is **frequently in** A. More philosophically, we may say that ***a filter is the extensional essence of a limiting process***.

Example 1.5. If X is any set and $x \in X$, the **principal filter** generated by x is

$$\eta_x = \{A \subset X : x \in A\}.$$

Clearly $\eta_x \to x$ in any topology.

Example 1.6. If X is a topological space and $x \in X$, its **neighborhood filter** is

$$\mathcal{N}_x = \{A \subset X : \exists U \text{ s.t. } x \in U \subset A \text{ and } U \text{ is open in } X\}.$$

The elements of $\mathcal{N}_x$ are called *neighborhoods* of x. Again, we have $\mathcal{N}_x \to x$ in any topology. Note also that a filter $\mathcal{F}$ converges to x if and only if $\mathcal{N}_x \subset \mathcal{F}$.

Often it is convenient to generate filters from smaller sets. We say that $\mathcal{A} \subset \mathcal{P}X$ has the **finite intersection property (FIP)** if the intersection of any finite subset of $\mathcal{A}$ is nonempty. In this case we define the filter **generated by** $\mathcal{A}$ to be

$$[\mathcal{A}] = \{B \subset X : \exists A_1, \dots, A_n \in \mathcal{A}\ A_1 \cap \cdots \cap A_n \subset B\}.$$

For example:

- The elementary filter $\mathcal{E}_{(a_n)}$ of a sequence is generated by the sets $A_N = \{a_n : n > N\}$.
- The principal filter η_x is generated by $\{\{x\}\}$.
- The neighborhood filter $\mathcal{N}_x$ is generated by the open sets containing x.

If $\mathcal{F}$ is a filter, we say that A is **consistent** with $\mathcal{F}$ if $X \setminus A \notin \mathcal{F}$. In this case $\mathcal{F} \cup \{A\}$ has the FIP and generates a filter, called the **extension** of $\mathcal{F}$ by A.

Example 1.7. A partially ordered set D is said to be *directed* if whenever $d_1, d_2 \in D$ there is a $d_3 \in D$ with $d_1 \le d_3$ and $d_2 \le d_3$. A function from a directed set D into a set X is called a *net* in X. If X is a topological space, a net $a \colon D \to X$ is said to *converge* to a point $x \in X$ if for any open set $U \ni x$, there exists a $d \in D$ such that whenever $d \le e$, we have $a(e) \in U$.

To every net $a \colon D \to X$ we can associate an elementary filter generated by the sets $A_e = \{a(e) : d \le e\}$ as $e \in D$ varies. Clearly a net converges to x if and only if its associated elementary filter does.

Example 1.8. Let $f \colon [a, b] \to \mathbb{R}$ be a function. Then for any partition P of $[a, b]$, we have upper and lower sums $U(f, P)$ and $L(f, P)$. Let $\mathcal{F}$ be the filter on $\mathbb{R}$ generated by

$$\Big\{ \big[L(f, P), U(f, P)\big] : P \text{ partitions } [a, b] \Big\},$$

which has the FIP because any two partitions have a common refinement. A limit of this filter (necessarily unique because $\mathbb{R}$ is Hausdorff) is a Riemann integral of f.

Unlike convergence of sequences, convergence of filters is sufficient to determine the topology of a space.

Proposition 1.9. *Let X be a topological space. A set $U \subset X$ is open if and only if whenever $\mathcal{F} \to x$ with $x \in U$, we have $U \in \mathcal{F}$.*

Proof. The "only if" direction is by definition of filter convergence. The converse follows since $\mathcal{N}_x \to x$ for any x. $\square$

Moreover, convergence of filters also determines when a function is continuous. If $\mathcal{F}$ is a filter on X and $f\colon X \to Y$ is a function, we write $f_*\mathcal{F}$ for the filter on Y generated by $\{f(A) : A \in \mathcal{F}\}$. Equivalently, we have

$$f_*\mathcal{F} = \{B \subset Y : f^{-1}(B) \in \mathcal{F}\}.$$

Proposition 1.10. *Let X and Y be topological spaces. A function $f\colon X \to Y$ is continuous if and only if whenever $\mathcal{F} \to x$ in X, we have $f_*\mathcal{F} \to f(x)$ in Y.*

Proof. For "if", note that $f_*\mathcal{N}_x$ is the set of all subsets of Y whose preimage is a neighborhood of x. Since $\mathcal{N}_x \to x$, we conclude that the preimage of any neighborhood of $f(x)$ is a neighborhood of x, hence f is continuous.

For "only if", the continuity of f implies that $\mathcal{N}_{f(x)} \subset f_*\mathcal{N}_x$. Therefore, if $\mathcal{F} \to x$, then $\mathcal{N}_x \subset \mathcal{F}$, hence $\mathcal{N}_{f(x)} \subset f_*\mathcal{N}_x \subset f_*\mathcal{F}$, so $f_*\mathcal{F} \to f(x)$. □

2. Ultrafilters

While powerful, the machinery of converging filters contains a good deal of redundancy. For example, if $\mathcal{F} \to x$, then any filter containing $\mathcal{F}$ also converges to x. This suggests restricting to those filters which are not properly contained in any other.

Definition/Proposition 2.1. *A filter $\mathcal{F}$ on X is an* ***ultrafilter*** *if it satisfies the following equivalent conditions.*

(i) For any $A \subset X$, either $A \in \mathcal{F}$ or $(X \setminus A) \in \mathcal{F}$.
(ii) $\mathcal{F}$ is not properly contained in any filter.

Proof. If we had an A such that $A \notin \mathcal{F}$ and $(X \setminus A) \notin \mathcal{F}$, then A would be consistent with $\mathcal{F}$ and we could extend $\mathcal{F}$ by A to obtain a larger filter. Conversely, if $\mathcal{F}$ were properly contained in a filter $\mathcal{G}$, then for any $A \in \mathcal{G} \setminus \mathcal{F}$ we would have $(X \setminus A) \notin \mathcal{F}$ either. □

Clearly the principal filters η_x are ultrafilters. Just as clearly, the explicit examples such as elementary filters $\mathcal{E}_{(a_n)}$ are *not* ultrafilters. Unfortunately, it is impossible to explicitly describe any non-principal ultrafilters, but we know (assuming the axiom of choice) that they exist.

Proposition 2.2. *Any filter is contained in an ultrafilter.*

Proof. Zornify. □

In less technical language, "Zornify" means that we start from a non-ultra filter $\mathcal{F}$, pick a set A such that neither A nor $X \setminus A$ is in $\mathcal{F}$, and extend $\mathcal{F}$ by either A or $X \setminus A$. We then repeat, probably transfinitely many times. The need to choose either A or $X \setminus A$ transfinitely many times is why we need the axiom of choice.

If $\mathcal{F}$ and $\mathcal{G}$ are filters with $\mathcal{F} \subset \mathcal{G}$, we say that $\mathcal{G}$ **refines** $\mathcal{F}$. Thus a filter is an ultrafilter if and only if it has no proper refinements, and every filter is refined by an ultrafilter.

Exercise 2.3. Show that any filter is the intersection of all the ultrafilters refining it. In particular, two filters are equal if and only if they are refined by the same set of ultrafilters.

Since a filter converges to x in a topological space if and only if it refines the neighborhood filter of x, any filter refining a filter converging to x must also converge to x. The purpose of ultrafilters is that they allow us to reverse this implication and thereby capture all the information about convergence of filters in a topological space.

Proposition 2.4. *A filter $\mathcal{F}$ converges to a point x in a topological space X if and only if all ultrafilters refining $\mathcal{F}$ converge to x.*

Proof. The "only if" direction is clear. For the "if" direction, suppose $\mathcal{F}$ does not converge to x. Then there is an open set $U \ni x$ such that $U \notin \mathcal{F}$, and therefore we can add $X \smallsetminus U$ to $\mathcal{F}$ and generate a larger filter. Any ultrafilter refining this filter will also refine $\mathcal{F}$, but cannot converge to x. □

Exercise 2.5. Show that if $\mathcal{F}$ is an ultrafilter and $f\colon X \to Y$ is a function, then $f_*\mathcal{F}$ is an ultrafilter.

Exercise 2.6. Show that if $f\colon X \to Y$ is surjective, then any ultrafilter $\mathcal{G}$ on Y is equal to $f_*\mathcal{F}$ for some (generally non-unique) ultrafilter $\mathcal{F}$ on X.

Exercise 2.7. Show that a function $f\colon X \to Y$ between topological spaces is continuous if and only if whenever $\mathcal{F}$ is an ultrafilter in X and $\mathcal{F} \to x$, we have $f_*\mathcal{F} \to f(x)$.

Many more topological concepts can be expressed directly in terms of (ultra)filters, without explicit reference to open sets. Here is a very important example.

Theorem 2.8. *A topological space X is compact if and only if each ultrafilter on X converges (to at least one point).*

Proof. Recall that X is compact if and only if every family of closed sets with the FIP has nonempty intersection. Suppose first that every ultrafilter converges and say we are given a family $\mathcal{A}$ of closed sets with the FIP. Then $\mathcal{A}$ generates a filter, which is contained in an ultrafilter $\mathcal{F}$, which by assumption must converge to some point x. In particular, every open set containing x is in $\mathcal{F}$ and thus has nonempty interesction with each element of $\mathcal{A}$. Since each element of $\mathcal{A}$ is closed, this implies that x is in each element of $\mathcal{A}$, and hence in their intersection.

Conversely, suppose X is compact, and let $\mathcal{F}$ be an ultrafilter. Then the collection of closed elements of $\mathcal{F}$ has the FIP and thus its intersection contains a point x. Since x is in every closed element of $\mathcal{F}$, every open set $U \ni x$ must intersect every closed element of $\mathcal{F}$. Therefore $X \smallsetminus U$ cannot be in $\mathcal{F}$, and so (since $\mathcal{F}$ is an ultrafilter) we must have $U \in \mathcal{F}$; thus $\mathcal{F}$ converges to x. □

Here are some more examples, left as exercises. An introduction to topology making use of filters can be found in [Jam99].

Exercise 2.9. Let X be a topological space and $A \subset X$ with inclusion map $i\colon A \hookrightarrow X$. Show the following.

(i) X is T_0 iff no two points are converged to by exactly the same sets of ultrafilters.
(ii) X is T_1 iff no principal filter η_x converges to any point other than x.
(iii) X is Hausdorff (T_2) iff no ultrafilter converges to more than one point.
(iv) $A \subset X$ is closed iff for any (ultra)filter $\mathcal{F}$ on A, every limit of $i_*\mathcal{F}$ is in A.
(v) $A \subset X$ is open iff whenever $\mathcal{F}$ is an (ultra)filter on X and $\mathcal{F} \to x \in A$, then $A \in \mathcal{F}$.
(vi) An (ultra)filter $\mathcal{F}$ on A converges to $x \in A$ in the subspace topology iff $i_*\mathcal{F}$ converges to x in X.
(vii) If $\{X_\alpha\}$ is a family of topological spaces, then an ultrafilter $\mathcal{F}$ on the product $\prod_\alpha X_\alpha$ converges to $x = (x_\alpha)$ in the product topology iff $(p_\alpha)_*\mathcal{F}$ converges to x_α in X_α for each α (here p_α is the projection $\prod_\alpha X_\alpha \to X_\alpha$).

(viii) Conclude Tychonoff's Theorem as a corollary.

The next two results are due to [CH02].

Exercise 2.10. Recall that a continuous map $f\colon X \to Y$ is *open* if $f(U)$ is open whenever $U \subset X$ is open. Show that f is open if and only if for any $x \in X$ and any ultrafilter $\mathcal{G}$ which converges to $f(x)$ in Y, there exists an ultrafilter $\mathcal{F}$ on X which converges to x and such that $f_*\mathcal{F} = \mathcal{G}$.

Exercise 2.11. Recall that a continuous map $f\colon X \to Y$ is *closed* if $f(C)$ is closed whenever $C \subset X$ is closed, and *proper* if it is closed and the fiber $f^{-1}(y)$ is compact for all $y \in Y$. Show that f is proper if and only if for any ultrafilter $\mathcal{F}$ on X and any point $y \in Y$ such that $f_*\mathcal{F}$ converges to y, there exists a point $x \in X$ such that $\mathcal{F}$ converges to x and $f(x) = y$.

3. Pseudotopological spaces

The fact that so many aspects of topology can be captured by convergence naturally makes us wonder whether convergence could be taken to be more fundamental than open sets.

Definition 3.1. A **pseudotopological space**, or **pspace** (the first "p" is silent, as in "pseudo"), is a set X together with a relation between ultrafilters and points, called "convergence", with the property that the principal ultrafilter η_x converges to x for every x.

The definition refers only to ultrafilters, but it can easily be extended to all filters: if $\mathcal{F}$ is an arbitrary filter on a pspace X, we say that $\mathcal{F}$ converges to x if every ultrafilter refining $\mathcal{F}$ converges to x.

Exercise 3.2. Show that it would be equivalent to define a pspace by a convergence relation between filters and points with the following three properties:

(i) Each principal filter η_x converges to x.
(ii) If $\mathcal{F}$ converges to x and $\mathcal{G}$ refines $\mathcal{F}$, then $\mathcal{G}$ converges to x.
(iii) If $\mathcal{F}$ is a filter and x a point such that for any filter $\mathcal{G}$ refining $\mathcal{F}$, there is another filter $\mathcal{H}$ refining $\mathcal{G}$ such that $\mathcal{H}$ converges to x, then $\mathcal{F}$ converges to x.

All our theorems about topological spaces can now be extended to definitions about pseudotopological ones. For example:

Definitions 3.3. Let X and Y be pspaces.

(i) X is **Hausdorff** if no ultrafilter converges to more than one point.
(ii) X is **compact** if every ultrafilter converges to at least one point.
(iii) A function $f\colon X \to Y$ is **continuous** if for any ultrafilter $\mathcal{F}$ on X, $\mathcal{F} \to x$ implies $f_*\mathcal{F} \to f(x)$.
(iv) If $\{X_\alpha\}$ is a family of pspaces, then we define an ultrafilter $\mathcal{F}$ on the product $\prod_\alpha X_\alpha$ to converge to $x = (x_\alpha)$ if and only if $(p_\alpha)_*\mathcal{F}$ converges to x_α in X_α for each α.[1]

Clearly every topological space gives rise to a pseudotopological one. Conversely, if X is a pspace, we can define a subset $U \subset X$ to be **open** if whenever $\mathcal{F} \to x$ and $x \in U$, we have $U \in \mathcal{F}$. This collection of open sets clearly defines a topology on X such that every filter converging to x in the original pseudotopology also converges in this topology. The converse, however, is not necessarily true; we say that the pspace is **topological** if it is.

[1] The categorically minded reader can check that this is the categorical product in the category of pspaces and continuous maps.

Exercise 3.4. Show that a pseudotopology on a finite set is just a reflexive binary relation on that set. When is it topological?

In categorical language, topological spaces embed fully and faithfully in pseudotopological spaces, preserving limits and many other topological properties.[2]

Why would we want to make this generalization? It turns out that some natural constructions on pspaces, even when applied to topological ones, give back non-topological ones. The main example is *function spaces*. If X and Y are topological spaces, the problem of putting a suitable topology on the set Y^X of continuous maps from X to Y is important and nontrivial. However, if we allow pseudotopologies, it becomes much easier.

If $\mathcal{F}$ and $\mathcal{G}$ are filters on sets X and Y, we write $\mathcal{F} \times \mathcal{G}$ for the filter on $X \times Y$ *generated by* the sets $\{A \times B : A \in \mathcal{F}, B \in \mathcal{G}\}$. Note that $\mathcal{F} \times \mathcal{G}$ may not be an ultrafilter even if $\mathcal{F}$ and $\mathcal{G}$ are. Of course, not every filter on $X \times Y$ is of the form $\mathcal{F} \times \mathcal{G}$, but it is true that any filter $\mathcal{H}$ on $X \times Y$ *refines* $p_*\mathcal{H} \times q_*\mathcal{H}$, where $p\colon X \times Y \to X$ and $q\colon X \times Y \to Y$ are the projections.

Definition 3.5. Let Y and Z be pspaces, let Z^Y be the set of continuous maps $Y \to Z$, and let $\mathcal{F}$ be an ultrafilter on Z^Y. We say that $\mathcal{F}$ **converges continuously** to a function $f \in Z^Y$ if whenever $\mathcal{G} \to y$ in Y, we have $\varepsilon_*(\mathcal{F} \times \mathcal{G}) \to f(y)$ in Z, where $\varepsilon\colon Z^Y \times Y \to Z$ is the evaluation map.

Continuous convergence defines a pseudotopology on Z^Y. Not only does the definition of this pseudotopology look natural, but it has the following property which makes it the "correct" pseudotopology in practically all possible ways.

Theorem 3.6. *Let X, Y, Z be pspaces. A function $g\colon X \to Z^Y$ is continuous (with respect to the pseudotopology of continuous convergence on Z^Y) if and only if the function $\widehat{g}\colon X \times Y \to Z$ defined by $\widehat{g}(x, y) = g(x)(y)$ is continuous (with respect to the product pseudotopology on $X \times Y$).*

Proof. If $\mathcal{H}$ is an ultrafilter on $Z^Y \times Y$, then $\mathcal{H}$ refines $p_*\mathcal{H} \times q_*\mathcal{H}$ and converges to (f, y) if and only if $p_*\mathcal{H} \to f$ and $q_*\mathcal{H} \to y$. In this case, the definition of continuous convergence implies that $\varepsilon_*(p_*\mathcal{H} \times q_*\mathcal{H}) \to f(y)$, and therefore $\varepsilon_*\mathcal{H} \to f(y) = \varepsilon(f, y)$ as well. Thus $\varepsilon\colon Z^Y \times Y \to Z$ is continuous. Hence, if g is continuous, so is $\widehat{g}$, being the composite of ε with $g \times 1_Y$.

Conversely, suppose that $\widehat{g}$ is continuous. Then if $\mathcal{F} \to x$ in X and $\mathcal{G} \to y$ in Y, we have $\widehat{g}_*(\mathcal{F} \times \mathcal{G}) \to g(x)(y)$ in Z. But $\widehat{g}_*(\mathcal{F} \times \mathcal{G}) = \varepsilon_*(g_*\mathcal{F} \times \mathcal{G})$. Since this is true for all convergences $\mathcal{G} \to y$ in Y, we see that $g_*\mathcal{F}$ converges continuously to $g(x)$ by definition; thus g is continuous, as desired. □

In categorical language, the category of pseudotopological spaces is *cartesian closed*, which the category of topological spaces is not.

From this perspective, the "reason" that it's hard to find good topologies on function spaces is that even when Y and Z are topological spaces, continuous convergence on Z^Y may not be topological. In some cases, however, it is, and this explains why sometimes there do exist good function-space topologies, and why those topologies are given the odd definitions they are.

[2]Some colimits are also preserved, but not all of them.

Exercise 3.7. Show that if X and Y are topological spaces and X is locally compact Hausdorff, then the pseudotopology of continuous convergence on Y^X is topological, and is identical to the well-known "compact-open" topology.

Pseudotopological spaces have further nice properties. For instance, not only is the category **PsTop** cartesian closed, but so is the slice category **PsTop**$/X$ for any pspace X. More about pseudotopological spaces can be found in [Wyl91].

4. Discrete Stone-Čech compactification

Let's think about compact Hausdorff pspaces a little more. Observe that a pspace is compact and Hausdorff if and only if each ultrafilter converges to exactly one point; thus the convergence relation is actually a *function* from ultrafilters to points. In other words, if we write ϕX for the set of ultrafilters on X, a compact Hausdorff pseudotopology on X is given by a function $\theta\colon \phi X \to X$ such that $\theta \circ \eta = 1_X$ (where $\eta\colon X \to \phi X$ sends $x \in X$ to η_x).

Similarly, a function $f\colon X \to Y$ between compact Hausdorff pspaces is continuous if and only if $\theta_Y \circ \phi f = f \circ \theta_X$, where $\phi f\colon \phi X \to \phi Y$ sends $\mathcal{F}$ to $f_*\mathcal{F}$.

Exercise 4.1. Show that a continuous bijection between compact Hausdorff pspaces is actually a homeomorphism (that is, its inverse is also continuous).

The world becomes even more interesting when we observe that for any set X, the set ϕX has a natural pseudotopology on it. Things can get a little hairy here, because we're now talking about ultrafilters on a set *of* ultrafilters, but bear with me. First some notation: for $A \subset X$ we write

$$A^* = \{\mathcal{G} \in \phi X : A \in \mathcal{G}\}.$$

We now define an ultrafilter Φ on ϕX to converge to the ultrafilter

$$\mu(\Phi) = \Big\{A \subset X : A^* \in \Phi\Big\}.$$

Theorem 4.2. *$\mu(\Phi)$, thus defined, is an ultrafilter, and $\mu\colon \phi\phi X \to \phi X$ defines a compact Hausdorff topology on ϕX.*

Proof. Given $A \subset X$, the sets $\{\mathcal{G} \in \phi X : A \in \mathcal{G}\}$ and $\{\mathcal{G} \in \phi X : (X \setminus A) \in \mathcal{G}\}$ are complements in ϕX, since the elements of ϕX are all ultrafilters. Thus exactly one of them is in the ultrafilter Φ, so exactly one of A and $X \setminus A$ is in $\mu(\Phi)$. Thus $\mu(\Phi)$ is an ultrafilter.

It is clear that $\mu(\eta_{\mathcal{F}}) = \mathcal{F}$, so μ defines a compact Hausdorff pseudotopology. Moreover, $\Phi \to \mathcal{F}$ if and only if $A^* \in \Phi$ for all $A \in \mathcal{F}$, so ϕX is topological and the sets A^* form a base for its open sets. ◻

Exercise 4.3. Show that the closed sets in ϕX are those of the form $\uparrow\mathcal{F} = \{\mathcal{G} : \mathcal{F} \subset \mathcal{G}\}$, where $\mathcal{F}$ is an arbitrary filter on X.

Exercise 4.4. Show that the image of $\eta\colon X \to \phi X$ is dense.

Exercise 4.5. Show that for any function $f\colon X \to Y$, the induced function $\phi f = f_*\colon \phi X \to \phi Y$ is continuous.

The natural question to ask now is whether θ is continuous.

Theorem 4.6. *The function θ is continuous (when ϕX has the above pseudotopology and X has the compact Hausdorff pseudotopology defined by θ) if and only if X is topological.*

Proof. First suppose that X is topological. Let $\Phi \in \phi(\phi X)$ with $\phi\theta(\Phi) = \mathcal{F}$ and $\theta(\mathcal{F}) = x$. We want to show that $\theta(\mu(\Phi)) = x$; that is, that $\mu(\Phi) \to x$. Since X is topological, this is equivalent to $U \in \mu(\Phi)$ for each open $U \ni x$. By definition of μ, this is equivalent to

$$U^* = \{\mathcal{G} \in \phi X : U \in \mathcal{G}\} \in \Phi.$$

Now, since $\mathcal{F} \to x$, we have $U \in \mathcal{F}$ for each open $U \ni x$. Since $\phi\theta(\Phi) = \theta_*(\Phi) = \mathcal{F}$, this implies that $\theta^{-1}(U) \in \Phi$. But since U is open, any ultrafilter converging to a point of U (that is, any $\mathcal{G} \in \theta^{-1}(U)$) must contain U, and hence $\theta^{-1}(U) \subset U^*$. This implies $U^* \in \Phi$, as desired.

Now suppose that θ is continuous, i.e. that $\theta \circ \phi\theta = \theta \circ \mu$. Recall that we define a set $U \subset X$ to be 'open' if whenever $\mathcal{H} \to x$ with $x \in U$, we have $U \in \mathcal{H}$. Let $\mathcal{F}$ be an ultrafilter containing every open $U \ni x$; we want to show that $\mathcal{F} \to x$. Our goal is to build a $\Phi \in \phi\phi X$ such that $\mu(\Phi) = \mathcal{F}$ and $\phi\theta(\Phi) = \eta_x$; the condition $\theta \circ \phi\theta = \theta \circ \mu$ will then tell us that $\theta(\mathcal{F}) = \theta(\eta_x) = x$.

Define

$$\mathcal{B} = \{\mathcal{G} \in \phi X : \mathcal{G} \to x\}$$

and for every $U \in \mathcal{F}$ define

$$\mathcal{B}_U = \{\mathcal{G} \in \phi X : U \in \mathcal{G}\}$$

If $U, V \in \mathcal{F}$, we have $U \cap V \neq \emptyset$, and any ultrafilter containing U and V must also contain $U \cap V$. Thus we have

$$\mathcal{B}_U \cap \mathcal{B}_V \supset \mathcal{B}_{U \cap V} \neq \emptyset,$$

so the collection $\{\mathcal{B}_U : U \in \mathcal{F}\}$ of subsets of ϕX has the FIP. The definition of μ, and the maximality of $\mathcal{F}$, then tell us that $\mu(\Phi) = \mathcal{F}$ for any ultrafilter Φ containing all the sets $\mathcal{B}_U$. If, moreover, the set $\mathcal{B}$ were consistent with all the sets $\mathcal{B}_U$, so that there existed an ultrafilter Φ containing all of them, then since $\theta(\mathcal{B}) = \{x\}$ we would have $\phi\theta(\Phi) = \eta_x$ as desired.

To show that $\mathcal{B}$ is consistent with $\mathcal{B}_U$ we argue as follows. Choose $U \in \mathcal{F}$; we want to find an ultrafilter $\mathcal{G}$ such that $U \in \mathcal{G}$ and $\mathcal{G} \to x$, as then $\mathcal{G} \in \mathcal{B}_U \cap \mathcal{B}$. If $x \in U$ then we can choose $\mathcal{G} = \eta_x$, so suppose $x \notin U$, and suppose, to the contrary, that such a $\mathcal{G}$ does not exist. Then $X \smallsetminus U$ is a set containing x such that every ultrafilter converging to x contains $X \smallsetminus U$. We would like to conclude that therefore $X \smallsetminus U$ is a neighborhood of x, and hence contained in $\mathcal{F}$ by assumption, giving a contradiction. This is the content of the following lemma, which completes the proof. □

Lemma 4.7. *Let X be a compact Hausdorff pspace such that θ is continuous. If $x \in U \subset X$ and U is contained in every ultrafilter converging to x, then U contains an open set V containing x.*

Proof. Recall that an open set is defined to be a set which is contained in any ultrafilter which converges to *any* of its points, so this is a statement about the compatibility of ultrafilters converging to different points. Let $A = X \smallsetminus U$, let B be the set of all points which are limits of ultrafilters containing A, and let $V = X \smallsetminus B$. Then $V \subset U$ since any point of A is a limit of its principal ultrafilter, and $x \in V$ by our assumption on U.

We claim that V is open. For suppose $\mathcal{F}$ were an ultrafilter containing B and converging to $y \in V$. By definition of B, the function $\theta\colon \phi A \to B$ is surjective, and therefore by Exercise 2.6 so is $\theta\colon \phi\phi A \to \phi B$. Thus $\mathcal{F} = \phi\theta(\Phi)$ for some ultrafilter Φ on ϕA, or equivalently on ϕX

and containing the set $A^* = \{\mathcal{G} \in \phi X : A \in \mathcal{G}\}$. Since θ is continuous, we have $\mu(\Phi) \to y$ as well; but since $A^* \in \Phi$ we have $A \in \mu(\Phi)$. Thus y is the limit of an ultrafilter containing A, hence $y \in B$, a contradiction. This shows that V is open. $\square$

Remark 4.8. Let us define the *closure* of a subset A of a pspace X to be the set of all points which are limits of ultrafilters containing A; note that a set is closed (in the sense that its complement is open) if and only if it is equal to its closure. Then Lemma 4.7 can equivalently be phrased as "the closure of a set is closed". This is how it is phrased in [Bar70], from which I took the essentials of this proof.

Therefore, a compact Hausdorff *topological* space is given by a set X together with a function $\theta\colon \phi X \to X$ such that $\theta \circ \eta = 1_X$ and $\theta \circ \phi\theta = \theta \circ \mu$.

The categorically sophisticated reader will have already recognized that ϕ is clearly a monad on the category of sets whose algebras are compact Hausdorff spaces, and that therefore compact Hausdorff spaces are an "algebraic" category in the categorical sense. For the reader who is not familiar with monads, rather than recount the definition here, we will content ourselves with giving another example of a monad, which will hopefully make it clear that there should be a common abstract structure underlying both.

For any set X, let FX denote the free group on X. Thus the elements of FX are words such as $xz^{-3}y^{-1}xy^2$ in elements of X. We have a function $\eta\colon X \to FX$ sending each $x \in X$ to the length-one word "x".

We define a *pseudogroup* to be a set X together with a function $\theta\colon FX \to X$ such that $\theta \circ \eta = 1_X$, and a *pseudogroup homomorphism* to be a function $f\colon X \to Y$ such that $\theta_Y \circ Ff = f \circ \theta_X$. Clearly every group is a pseudogroup; the map θ is given by multiplying words with the given group multiplication. In particular, FX itself is a pseudogroup, so we have a map $\mu\colon F(FX) \to FX$.

Exercise 4.9. Show that a pseudogroup is a group if and only if θ is a pseudogroup homomorphism.

In other words, a group is a set X together with a map $\theta\colon FX \to X$ such that $\theta \circ \eta = 1_X$ and $\theta \circ F\theta = \theta \circ \mu$. The analogy between ϕ and F should be clear. In both cases we describe a structure of some type (a topological space, resp. a group) as a set equipped with "operations". The functors ϕ and F, respectively, together with the natural transformations η and μ, describe all possible ways of applying these operations; they are called **monads**.

More about monads can be learned in [ML98, Awo06], among other places. The monadicity of compact Hausdorff spaces is shown in a more abstract way in [ML98], although of course the monad is the same.

Exercise 4.10. Show that a function $f\colon X \to Y$ between compact Hausdorff pspaces is continuous if and only if its graph is a closed subset of $X \times Y$.

Exercise 4.11. Show that an arbitrary (not necessarily compact Hausdorff) pspace X is topological if and only if the convergence relation is a closed subset of $\phi X \times X$. (This can also be found in [Bar70].)

Remark 4.12. If you did Exercise 3.4, you may recognize that this characterization of topological spaces can be regarded as a *transitivity* condition on convergence, to go along with the *reflexivity* imposed by $\eta_x \to x$. In fact, there is a sense in which a topological space can be regarded as a 'generalized preorder'; see [CT03, CHT04].

5. General Stone-Čech compactification

We now make the following observation.

Theorem 5.1. *Let X be a set, Y a compact Hausdorff space and $g\colon X \to Y$ an arbitrary function. Then there is a unique continuous map $\widetilde{g}\colon \phi X \to Y$ extending g.*

Proof. Since $\eta\colon X \to \phi X$ is dense (Exercise 4.4) and Y is Hausdorff, the extension is unique if it exists. But since Y is topological, $\theta_Y\colon \phi Y \to Y$ is continuous (Theorem 4.6), and thus so is $\widetilde{g} = \theta_Y \circ \phi g$, which is easily checked to extend g. □

Now let X be any topological space. A **Stone-Čech compactification** of X is a compact Hausdorff space βX with a continuous map $\eta\colon X \to \beta X$ such that any continuous map from X to a compact Hausdorff space Y factors uniquely through η. Standard categorical arguments show that such a βX is unique up to homeomorphism if it exists; in categorical language, it is a *reflection* of X into the category of compact Hausdorff spaces.

Theorem 5.1 shows that the ultrafilter space ϕX we constructed in §4 is the Stone-Čech compactification of the discrete topology on X. In this section we will construct the Stone-Čech compactification of an arbitrary topological space X in an analogous way.

A natural way to start is the following. If X is a topological space, let ψX be be the set of ultrafilters on X with the topology generated by sets of the form U^*, where $U \subset X$ is open. This construction retains many of the nice properties of the space ϕX from §4.

Exercise 5.2. Show the following.

- (i) The map $\eta\colon X \to \psi X$ is a continuous dense embedding.
- (ii) If $f\colon X \to Y$ is continuous, then $\psi f\colon \psi X \to \psi Y$ is continuous.
- (iii) If X is compact Hausdorff, then $\theta\colon \psi X \to X$ is continuous.
- (iv) Every basic open set U^* is compact in ψX. In particular, ψX is compact and has a basis of compact open sets.

However, ψX is rarely Hausdorff. In fact, we already know all the cases in which it is.

Exercise 5.3. Show that ψX is Hausdorff if and only if X is discrete.

This is easily remedied; we define βX to be the maximal Hausdorff quotient (the 'Hausdorffification') of ψX. This is easily shown to exist; we simply take the quotient of ψX by the equivalence relation generated by the pairs (x, y) where x and y are not separable by disjoint open sets.

Theorem 5.4. *For any compact Hausdorff space Y and any continuous map $f\colon X \to Y$, there is a unique map $\widetilde{f}\colon \beta X \to Y$ extending f. Thus βX is the Stone-Čech compactification of X.*

Proof. Clearly βX is compact Hausdorff. Since X is dense in ψX, its image will be dense in βX; thus a map $f\colon X \to Y$ can extend to at most one map $\beta X \to Y$ as long as Y is Hausdorff. Now, the composite $\theta \circ \psi f\colon \psi X \to Y$ clearly extends f. But since Y is Hausdorff, $\theta \circ \psi f$ factors through the Hausdorff quotient βX of ψX, giving the desired map $\widetilde{f}$. □

This construction of the Stone-Čech compactification is due to [Sal00]. While nice, it is not maximally explicit; it would be nice to have a more concrete description of the points of βX than equivalence classes under an abstractly defined relation.

Consider first the following question: when is $X \to \beta X$ an embedding? Clearly X must be Hausdorff, since this would make it a subspace of a Hausdorff space, but this is not enough; it turns out that compact Hausdorff spaces automatically satisfy an extra separation axiom.

Definition 5.5. A topological space X is **completely regular** if for any closed set C and point $x \in X \smallsetminus C$, there is a continuous map $f \colon X \to [0,1]$ such that $f(x) = 1$ and $f|_C = 0$.

Clearly, if a completely regular space is T_1, then it is also Hausdorff. A completely regular Hausdorff space is called a $T_{3\frac{1}{2}}$**-space** or a **Tychonoff space**.

Theorem 5.6. *Every compact Hausdorff space is completely regular, hence Tychonoff.*

Proof. This is a version of Urysohn's lemma. □

Since a subspace of a completely regular space is completely regular, any space X which embeds in a compact Hausdorff space (such as βX) must also be completely regular, hence Tychonoff. It turns out that being Tychonoff is also sufficient for embeddability in a compact Hausdorff space. For if we define P_X to be the product of one copy of $[0,1]$ for each continuous map $X \to [0,1]$ and define $\iota \colon X \to P_X$ so that the component of $\iota(x)$ at $g \colon X \to [0,1]$ is $g(x)$, then clearly P_X is compact Hausdorff, and it is easy to show that ι is an embedding when X is Tychonoff.

Of course, P_X is much larger than βX, but it can be used to construct (a space homeomorphic to) βX. This construction is perhaps the most frequently seen one.

Theorem 5.7. *Let $\beta' X$ denote the closure of the image of $\iota \colon X \to P_X$. Then $\beta' X$ is a Stone-Čech compactification of X.*

Proof. Since $\beta' X$ is a closed subspace of a compact Hausdorff space, it is compact Hausdorff. It remains to show that for any compact Hausdorff space Y, any continuous map $f \colon X \to Y$ has an extension $\widetilde{f} \colon \beta' X \to Y$ (which will be unique because the image of X is dense in $\beta' X$ by construction). But since Y is compact, its image in P_Y is also compact, hence closed; thus its inverse image under $P_f \colon P_X \to P_Y$ is a closed set containing $\iota(X)$, and hence containing $\beta' X$. It follows that P_f maps $\beta' X$ into $\iota(Y)$, which is isomorphic to Y because Y is completely regular. Thus, the restriction of P_f to $\beta' X$ gives the desired $\widetilde{f}$. □

It follows that $\beta X \cong \beta' X$ in a canonical way. In fact, the induced map $\psi X \to \beta' X$ is easy to describe: it takes an ultrafilter $\mathcal{F} \in \beta X$ to the point of P_X whose component at $g \colon X \to [0,1]$ is the unique limit of the ultrafilter $g_*(\mathcal{F})$ on $[0,1]$.

The fact that this map induces an isomorphism $\beta X \cong \beta' X$ tells us that two ultrafilters $\mathcal{F}$ and $\mathcal{G}$ are identified in the quotient βX precisely when $g_*(\mathcal{F})$ and $g_*(\mathcal{G})$ have the same limit in $[0,1]$ for all continuous $g \colon X \to [0,1]$. Intuitively, this means that $\mathcal{F}$ and $\mathcal{G}$ 'look the same' to the eyes of real-valued functions.

This leads to yet a third construction of the Stone-Čech compactification; this one is taken from [SS95]. First we have to slightly generalize the notion of filter. Recall that a partially ordered set is called a **lattice** if it has a least element 0 and a greatest element 1, and every pair of elements a, b has a least upper bound (or *join*) $a \vee b$ and a greatest lower bound (or *meet*) $a \wedge b$. A **filter** in a lattice L is a subset $F \subset L$ such that (a) $1 \in F$, (b) $0 \notin F$, (c) if $a \in F$ and $a \le b$, then $b \in F$, and (d) if $a, b \in F$ then $a \wedge b \in F$. Thus a filter *on* a set X is a filter *in* the lattice $\mathcal{P}X$. A **maximal filter** is a filter not contained in any other filter.

Given a space X, let $\mathcal{Z}X$ denote the set of **zero-sets** in X; that is, sets of the form $\{x : g(x) = 0\}$ for some continuous $g\colon X \to [0,1]$ (or equivalently, $g\colon X \to \mathbb{R}$). Clearly $\mathcal{Z}X$ is a sublattice of $\mathcal{P}X$. Let $\beta''X$ denote the set of maximal filters in $\mathcal{Z}X$, with a topology whose basic *closed* sets are those the form $A^* = \{\mathcal{F} \in \beta''X : A \in \mathcal{F}\}$, where A is a zero-set.

Any point $x \in X$ determines a filter $\eta_x = \{A \in \mathcal{Z}X : x \in A\}$ which it is easy to see is maximal. Thus we have a map $\eta\colon X \to \beta''X$. Since $\eta^{-1}(A^*) = A$, we see that η is continuous. If X is completely regular, so that the zero-sets generate its topology, then η is an embedding.

Theorem 5.8. *$\beta''X$ is another Stone-Čech compactification of X.*

Proof. To see that $\beta''X$ is compact, consider a collection of closed sets F_i with the FIP. Each is an intersection of basic closed sets A^*_{ij}, so the whole collection $\{A^*_{ij}\}$ has the FIP and thus so does $\{A_{ij}\}$. By Zorn's lemma, this collection is contained in a maximal filter in $\mathcal{Z}X$, which is thus a point of $\beta''X$ contained in all the F_i; thus they have nonempty intersection.

To see that it is Hausdorff, suppose that $\mathcal{F}$ and $\mathcal{G}$ are distinct maximal filters in $\mathcal{Z}X$. Then there exists $A \in \mathcal{F} \smallsetminus \mathcal{G}$. Since $\mathcal{G}$ is maximal, A must be inconsistent with it; that is, there is $B \in \mathcal{G}$ with $A \cap B = \emptyset$. Then the disjoint open sets $\beta''X \smallsetminus A^*$ and $\beta''X \smallsetminus B^*$ separate $\mathcal{F}$ and $\mathcal{G}$.

Moreover, since the only basic closed set A^* containing $\eta(X)$ is X^*, $\eta(X)$ is dense in $\beta''X$. Thus it remains only to show that any continuous map $f\colon X \to Y$, where Y is compact Hausdorff, extends to $\beta''X$.

Consider, for any $\mathcal{F} \in \beta''X$, the collection of zero-sets A in Y such that $f^{-1}(A) \in \mathcal{F}$. This is a collection of closed sets with the FIP, so since Y is compact, it has a nonempty intersection $\emptyset \neq F = \bigcap\{A : f^{-1}(A) \in \mathcal{F}\}$. Moreover, if we have $x, y \in F$ with $x \neq y$, then since Y is itself completely regular (being compact Hausdorff), there exist zero-sets A and B with $x \notin B$, $y \notin A$, and $A \cup B = Y$. Thus $f^{-1}(A) \cup f^{-1}(B) = X$, so since $\mathcal{F}$ is maximal, one of the zero-sets $f^{-1}(A)$ and $f^{-1}(B)$ must be in $\mathcal{F}$, and so one of A or B must be in F. But this is impossible since x and y are both in all elements of F. Thus F consists of just one point, which we label $\widetilde{f}(\mathcal{F})$. It remains to check that $\widetilde{f}$ is continuous; this follows because $\widetilde{f}^{-1}(A) = (f^{-1}(A))^*$ for any zero-set $A \subset Y$. $\square$

Therefore, we have an isomorphism $\beta X \cong \beta''X$. The induced map $\psi X \to \beta''X$ is again easy to describe. Given an ultrafilter $\mathcal{F}$ on X, we can consider the set of all zero-sets in $\mathcal{F}$. This is a filter in $\mathcal{Z}X$, but not in general a maximal one. However, the above arguments essentially show that it is contained in a *unique* maximal filter, which we define to be the image of $\mathcal{F}$.

Therefore, we can now say that two ultrafilters $\mathcal{F}$ and $\mathcal{G}$ become identified in βX precisely when their intersections with $\mathcal{Z}X$ are contained in the same maximal filter. Clearly a necessary condition for this is that $A \cap B \neq \emptyset$ whenever $A \in \mathcal{F}$ and $B \in \mathcal{G}$ are zero-sets; this is also sufficient, since then $(\mathcal{F} \cup \mathcal{G}) \cap \mathcal{Z}X$ generates a filter in $\mathcal{Z}X$, which is contained in a maximal one that must contain both $\mathcal{F} \cap \mathcal{Z}X$ and $\mathcal{G} \cap \mathcal{Z}X$. This is about as concrete a condition as we could hope for, given how un-concrete non-principal ultrafilters are anyway.

There are still other ways to construct the Stone-Čech compactification. We can consider maximal completely regular filters of open sets, or maximal ideas of cozero-sets, or maximal ideals in the ring $C^*(X)$ of bounded continuous real-valued functions on X. See [Joh86].

Exercise 5.9. Show that a space X is compact Hausdorff if and only if $X \cong \beta X$, and in particular that $\beta(\beta X) \cong \beta X$ for any X.

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