Real-cohesion: from connectedness to continuity

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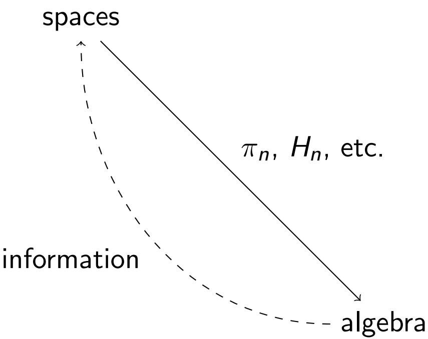
My hat today

- I am a mathematician: not a computer scientist.
- I am a categorical logician: type theory is a formal system for reasoning internally to categories. Good formal properties of type theory are valued but negotiable.
- I am a *pragmatic constructivist*: I use constructive logic when, and only when, I have good reasons to.

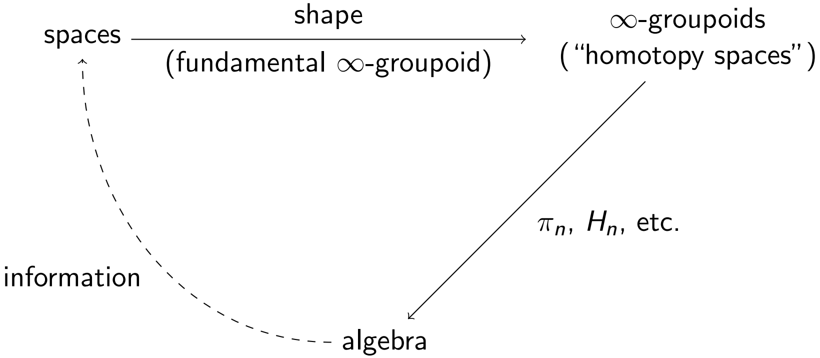
No disrespect is meant to the wearers of other hats, including myself on other days.

Outline

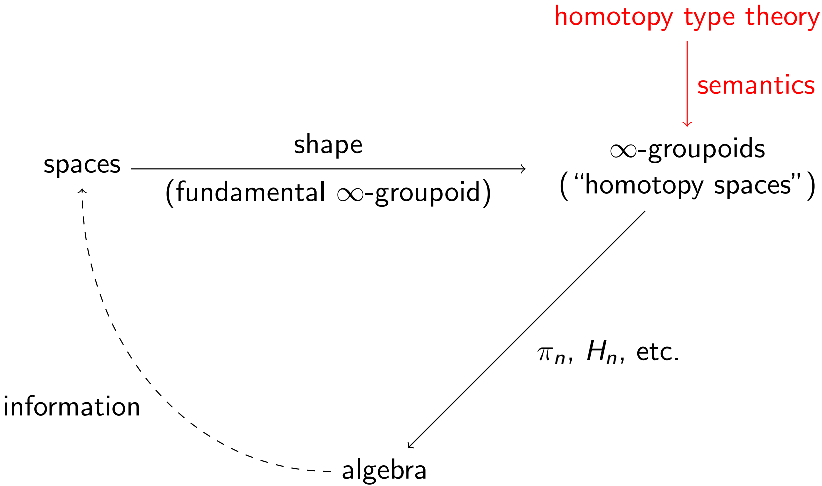
Classical algebraic topology



Modern algebraic topology



Homotopy type theory



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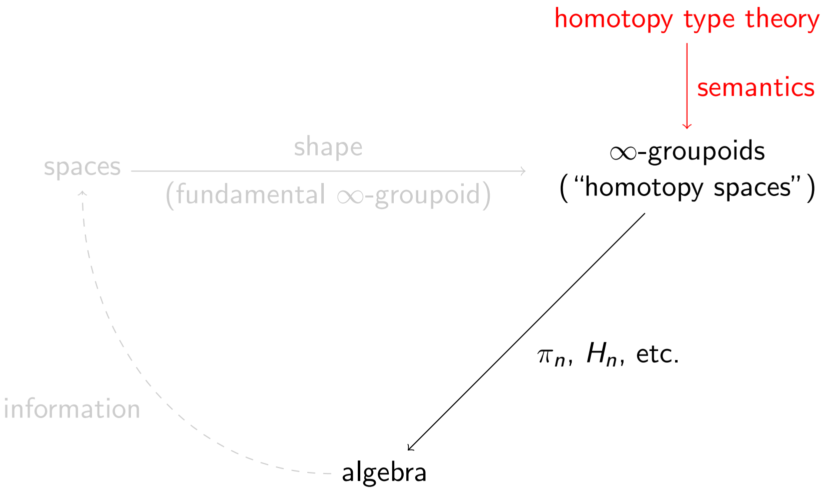
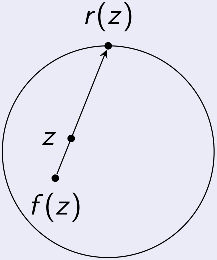


Brouwer's fixed-point theorem (classical version)

Theorem

Any continuous map $f : \mathbb{D}^2 \to \mathbb{D}^2$ *has a fixed point.*

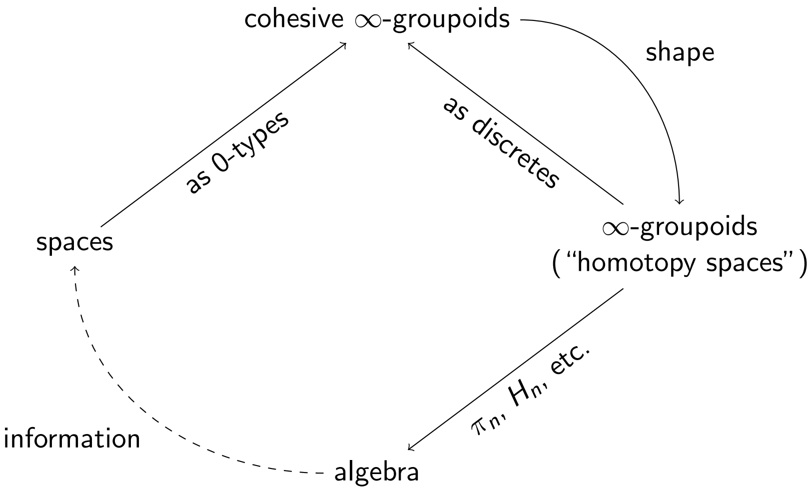
Proof.



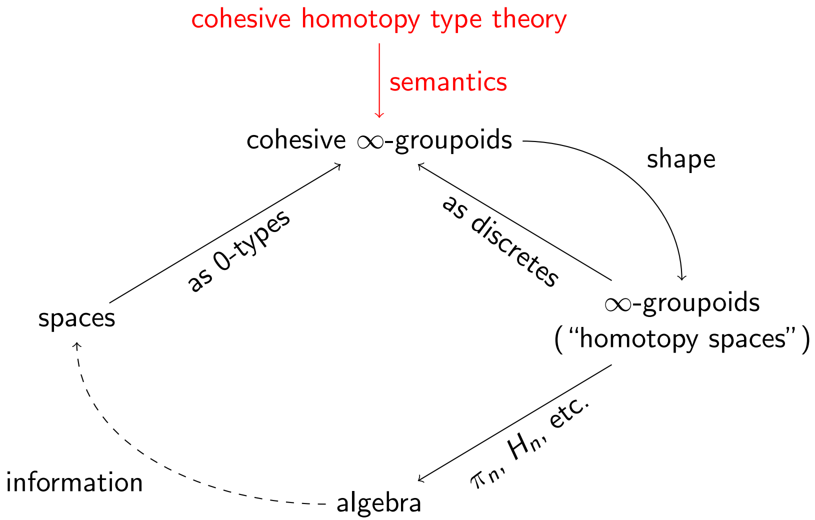
Suppose $f : \mathbb{D}^2 \to \mathbb{D}^2$ is continuous with no fixed point. For any $z \in \mathbb{D}^2$, draw the ray from $f(z)$ through z to hit $\partial\mathbb{D}^2 = \mathbb{S}^1$ at $r(z)$. Then r is continuous, and retracts $\mathbb{D}^2$ onto $\mathbb{S}^1$. Hence $\pi_1(\mathbb{S}^1) = \mathbb{Z}$ is a retract of $\pi_1(\mathbb{D}^2) = 0$, a contradiction. □

- $\mathbb{D}^2$ and $\mathbb{S}^1$ are spaces "up to homeomorphism".
- We still have to "do homotopy theory" with them.

Cohesive algebraic topology



Cohesive homotopy type theory



Cohesive ∞-groupoids

Idea

A continuous ∞-groupoid is an ∞-groupoid with compatible topologies on the set of k-morphisms for all k.

Example

- An ordinary topological space of objects, with only identity k-morphisms for $k > 0$.
- An ordinary ∞-groupoid, with the discrete topology in all dimensions.
- An ordinary ∞-groupoid with the *indiscrete* topology.
- The delooping of a topological group G, with one object, with G as the *space* of 1-morphisms, and only k-identities for $k > 1$.

(Cohesive ∞-groupoids, really)

Technicality

To get a good ∞-category, instead of ∞-groupoids internal to topological spaces, we use sheaves on the site of cartesian spaces $\{\mathbb{R}^n\}_{n\in\mathbb{N}}$ with the Grothendieck topology of open covers.

(There are other interesting sites too: cohesion is more general than $\mathbb{R}^n$-detected continuity.)

Outline

What is cohesive HoTT?

Answer #1

We "expand the universe" of HoTT to include cohesive ∞-groupoids in addition to ordinary ones.

Answer #2

We realize that the HoTT we've been doing all along *might as well* have been talking about cohesive ∞-groupoids in addition to ordinary ones.

Adding homotopy to type theory

Ordinary type theory (for a mathematican)

- Intuition: *types as sets, terms as functions*.

Adding homotopy to type theory

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Homotopy type theory

- New intuition: *types as ∞-groupoids, terms as functors*.
- Detect their ∞-groupoid structure with the identity type.
- The old intuition is still present in the 0-types.
- Some types that already existed turn out "automatically" to have nontrivial ∞-groupoid structure (e.g. the universe is univalent).

Adding topology to type theory

Ordinary type theory

- Intuition: *types as sets, terms as functions*.

Synthetic topology

- New intuition: *types as spaces, terms as continuous maps*.
- Detect their topological structure in various ways.
- The old intuition is still present in the discrete spaces.
- Some types that already existed turn out "automatically" to have nontrivial topological structure (e.g. the real numbers $\mathbb{R}$ have their usual topology).

Cohesive HoTT

Cohesive HoTT

New intuition: *types as cohesive ∞-groupoids*.

Every type has both ∞-groupoid structure and cohesive/topological structure. Either, both, or neither can be trivial.

Example

- The higher inductive S^1 has nontrivial higher structure ($\Omega S^1 = \mathbb{Z}$), but is cohesively discrete (no topology).
- $\mathbb{S}^1 = \{ (x, y) : \mathbb{R}^2 \mid x^2 + y^2 = 1 \}$ has trivial higher structure (is a 0-type), but nontrivial cohesion (its "usual topology").

In a moment we will see that S^1 is the shape of $\mathbb{S}^1$.

What does cohesive HoTT look like?

- HoTT with extra stuff. Everything you know about synthetic homotopy theory is still true.
- Think of every type as having a cohesive structure (perhaps discrete), and every map as continuous.
- HITs like S^1 generally have discrete cohesion, whereas $\mathbb{R}$ and types built from it have "their usual topologies".
- Be careful with words:
  - A (-1)-truncated map (HoTT Book "embedding") need not be a subspace inclusion; call it a "mono".
  - There are "identifications" $p : x = y$ and "paths" $c : \mathbb{R} \to X$.
- Detect and operate on cohesive structure with "modalities" (in a moment).

We do need to stick to constructive logic

- The strong law of excluded middle $\prod_{A:\mathsf{Type}} A + \neg A$ is incompatible with univalence.
- The propositional law of excluded middle $\prod_{P:\mathsf{Prop}} P + \neg P$ is consistent with univalence.
- Even the *propositional* law of excluded middle is incompatible with cohesion.

Example

- Monos are "injective continuous maps".
- If $A \to B$ is a mono, its complement $\neg A \to B$ *is* a subspace.
- Their union as monos has the *disjoint union* topology: $A \cup \neg A \neq B$.

Outline

Axiomatic cohesion

Definition

- $\flat X$: the underlying ∞-groupoid of X retopologized discretely
- $\sharp X$: the underlying ∞-groupoid of X retopologized codiscretely

- Codiscrete types are a reflective subcategory, with reflector $\sharp$.

  "Every map into a codiscrete space is continuous"
- Discrete types are a coreflective subcategory, with coreflector $\flat$.

  "Every map out of a discrete space is continuous"

Axiomatic cohesion

Definition

- $\flat X$: the underlying ∞-groupoid of X retopologized discretely
- $\sharp X$: the underlying ∞-groupoid of X retopologized codiscretely
- $\int X$: the shape* of X, topologized discretely

* $\int$ is an "esh", IPA for a voiceless postalveolar fricative (English *sh*)

Axiomatic cohesion

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Discrete types are also reflective, with reflector $\int$:

$$(\textstyle\int X \to Y) \simeq (X \to Y)$$

whenever Y is discrete. Magically, this universal property characterizes the classical fundamental ∞-groupoid.

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The problem of discrete coreflection

- $\sharp$ and $\int$ are (idempotent, monadic) modalities in the sense of §7.7 of the Book. They internalize as functions Type $\to$ Type. Semantically, they act on all slice categories.
- E.g. a mono is "fiberwise codiscrete" iff it is a subspace inclusion, and fiberwise $\sharp$ reflects monos into subspaces.

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- E.g. a mono is "fiberwise codiscrete" iff it is a subspace inclusion, and fiberwise $\sharp$ reflects monos into subspaces.
- $\flat$ is an "idempotent comonadic modality", but it cannot similarly be internalized or extended to slice categories.

Theorem

The only internal "coreflective subuniverses" are the "slice categories" Type$/U$ *for some* U : Prop*.*

The solution to discrete coreflection

First Solution

$\flat$ can only be applied in the empty context.

Semantically: discrete objects are a coreflective subcategory of the category of cohesive ∞-groupoids, but not of all its slice categories.

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Better Solution

$\flat$ can only be applied when everything in the context is discrete.

Semantically: discrete objects are a coreflective subcategory of the category of cohesive ∞-groupoids, considered as *indexed* over ordinary ∞-groupoids.

Modal type theory

$$\frac{x : \flat A \vdash C}{x : \flat A \vdash \flat C} \qquad \text{or} \qquad \frac{x :: A \mid \cdot \vdash C}{x :: A \mid y : B \vdash \flat C}$$

Technicality

Literally requiring types in the context to be of the form $\flat A$ breaks the admissibility of substitution. Instead we "judgmental-ize" it with a formalism of "crisp variables" $x :: A$ that semantically mean the same as $x : \flat A$.

Cf. Pfenning-Davies 2001, Reed 2009, Licata-Shulman 2016, Licata-Shulman-Riley 2017

Cohesive type theories

Type Theory	Conjectural Semantics
HoTT	∞-toposes
Spatial type theory ($\flat$ and $\sharp$)	local ∞-toposes
Cohesive type theory ($\flat$, $\sharp$, and $\int$)	cohesive ∞-toposes
Real-cohesive type theory ($\int$ generated by $\mathbb{R}$)	the ∞-topos of continuous ∞-groupoids

Real-cohesion

Let $\mathbb{R}$ be the Dedekind real numbers.

Axiom $R\flat$

A type A is discrete if and only if

$$\mathsf{const} : A \to (\mathbb{R} \to A)$$

is an equivalence. ("Every map $\mathbb{R} \to A$ is constant.")

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A type A is discrete if and only if

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is an equivalence. ("Every map $\mathbb{R} \to A$ is constant.")

In particular, if A is discrete then

$$(\mathbb{R} \to A) \simeq A \simeq (\mathbf{1} \to A)$$

so that $\int \mathbb{R} = \mathbf{1}$.

The shape of the circle

Theorem

$$\int \mathbb{S}^1 = S^1.$$

(Recall $\mathbb{S}^1 = \{ (x, y) : \mathbb{R}^2 \mid x^2 + y^2 = 1 \}$, while S^1 is the HIT.)

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Proof.

- S^1 is the coequalizer of two maps $\mathbf{1} \rightrightarrows \mathbf{1}$.
- $\mathbb{S}^1$ is the coequalizer of two maps $\mathrm{id}_{\mathbb{R}}, (+1) : \mathbb{R} \rightrightarrows \mathbb{R}$.
- $\int$ is a left adjoint, hence preserves colimits.
- $\int\mathbb{R} = \mathbf{1}$.

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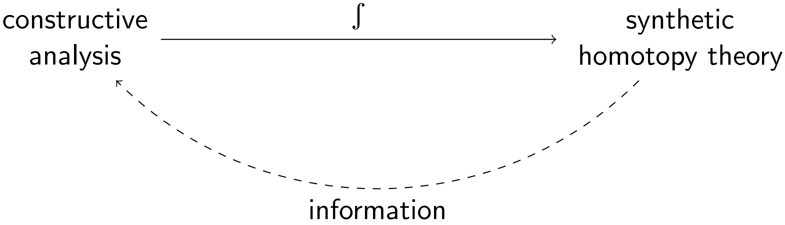
- $\int$ is a left adjoint, hence preserves colimits.
- $\int \mathbb{R} = \mathbf{1}$.
- Discrete types in the empty context are coreflective, hence closed under colimits; thus S^1 is discrete. □

Cohesive homotopy type theory



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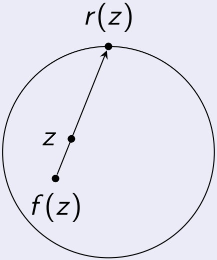

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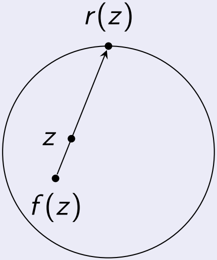
□

The real-cohesive version, first try

Theorem

Any function $f : \mathbb{D}^2 \to \mathbb{D}^2$ has a fixed point.

Attempted proof.



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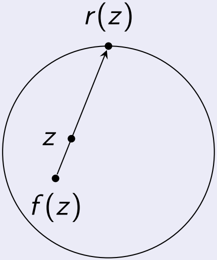

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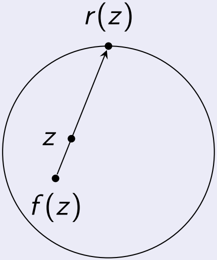

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□

Problems

There are two problems with this:

1. It's a proof by contradiction of a positive statement: the sort that's disallowed in constructive mathematics. But *cohesive* homotopy type theory is incompatible with excluded middle.
2. Even disregarding that, the assumption "f has no fixed point" tells us only that $f(z) \neq z$ for all z, whereas constructively, in order to draw the line connecting two points we need them to be *apart* (have a positive distance), not merely *unequal*.

Classicality axioms for cohesion

Flat excluded middle ($\flat$LEM)

For all $P : \flat\mathrm{Prop}$ we have $P + \neg P$.

"We can use proof by contradiction in a fully discrete context."

Analytic Markov's Principle (AMP)

For $x, y : \mathbb{R}$, if $x \neq y$ then $|x - y| > 0$.

"Disequality implies apartness."

Both hold in the topos of continuous ∞-groupoids.

Digression: Omniscience principles in real-cohesion

	Cauchy reals (standard)	Dedekind reals (analytic)
LPO: $\forall x(x = 0 \vee x \neq 0)$	✓	✗
LLPO: $\forall x(x \leq 0 \vee x \geq 0)$	✓	✗
MP: $\forall x(x \neq 0 \vee \lvert x \rvert > 0)$	✓	?

✓ = provable from $\flat$LEM

✗ = disprovable from $\flat$LEM

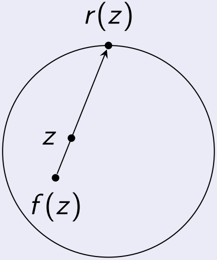
? = consistent with $\flat$LEM; maybe provable?

The real-cohesive version, second try

Theorem (Using $\flat$LEM and AMP)

Any function $f : \flat(\mathbb{D}^2 \to \mathbb{D}^2)$ has a fixed point.

Proof.



Since the context is discrete, we may use proof by contradiction. Suppose f has no fixed point. Then for any $z : \mathbb{D}^2$, we have $f(z) \neq z$, hence $d(z, f(z)) > 0$. So we can draw the ray from $f(z)$ through z to hit $\partial\mathbb{D}^2 = \mathbb{S}^1$ at $r(z)$. Then r retracts $\mathbb{D}^2$ onto $\mathbb{S}^1$. Hence $\int\mathbb{S}^1$ is a retract of $\int\mathbb{D}^2$. But $\mathbb{D}^2$ is a retract of $\mathbb{R}^2$, hence $\int\mathbb{D}^2$ is contractible, while $\int\mathbb{S}^1 = S^1$, which is not contractible. □

The real-cohesive version, second try

Recall, all types are "spaces" and all operations are "continuous", while $\flat A$ means A retopologized discretely. Thus,

Theorem

Any function $f : \flat(\mathbb{D}^2 \to \mathbb{D}^2)$ *has a fixed point.*

means intuitively that

Any function $f : \mathbb{D}^2 \to \mathbb{D}^2$ has a fixed point, but such fixed points cannot be selected *continuously* as a function of f.

This is certainly true: a small deformation in f can cause its fixed point to "jump" discontinuously.

A constructive real-cohesive version

Theorem (NOT using $\flat$LEM or AMP)

For any function $f : \flat(\mathbb{D}^2 \to \mathbb{D}^2)$ *and* $\varepsilon > 0$*, there exists a point* $z : \mathbb{D}^2$ *with* $d(z, f(z)) < \varepsilon$*.*

A constructive real-cohesive version

Theorem (NOT using $\flat$LEM or AMP)

For any function $f : \flat(\mathbb{D}^2 \to \mathbb{D}^2)$ and $\varepsilon > 0$, there exists a point $z : \mathbb{D}^2$ with $d(z, f(z)) < \varepsilon$.

Proof.

- WLOG ε is rational. Since $\mathbb{Q}$ is discrete, so is the context.
- $U = \{ z \mid d(z, f(z)) > \frac{\varepsilon}{2} \}$ and $V = \{ z \mid d(z, f(z)) < \varepsilon \}$. Then $\mathbb{D}^2 = U \cup V$, since $\forall x : \mathbb{R}(x < \varepsilon \vee x > \frac{\varepsilon}{2})$.

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- "Ray from $f(z)$ to z" defines a retraction $r : U \to \mathbb{S}^1$.
- Thus S^1 is a retract of $\int U$, so $\int U$ contains a nontrivial loop.

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- "Ray from $f(z)$ to z" defines a retraction $r : U \to \mathbb{S}^1$.
- Thus S^1 is a retract of $\int U$, so $\int U$ contains a nontrivial loop.
- $\mathbb{D}^2 = U \cup V = U \sqcup^{U \cap V} V$, so $\int U \sqcup^{\int(U \cap V)} \int V$ is contractible.

Recall: the van Kampen theorem

Theorem

For P the pushout of $f : A \to B$ and $g : A \to C$, and $u, v : P$,

$$\|u = v\|_0 \simeq \mathsf{code}(u, v).$$

$\mathsf{code}(\mathsf{inl}(b), \mathsf{inl}(b'))$ is a set-quotient of the type of sequences

$$b \overset{p_0}{\underset{B}{\rightsquigarrow}} f(x_1)\ ,\ g(x_1) \overset{q_0}{\underset{C}{\rightsquigarrow}} g(y_1)\ ,\ f(y_1) \overset{p_1}{\underset{B}{\rightsquigarrow}} \cdots \overset{p_n}{\underset{B}{\rightsquigarrow}} b'$$

(or $b \overset{p_0}{\underset{B}{\rightsquigarrow}} b'$ when $n = 0$) by an equivalence relation generated by

$$(\dots, q_k, y_k, \mathsf{refl}_{f(y_k)}, y_k, q_{k+1}, \dots) = (\dots, q_k \cdot q_{k+1}, \dots)$$
$$(\dots, p_k, x_k, \mathsf{refl}_{g(x_k)}, x_k, p_{k+1}, \dots) = (\dots, p_k \cdot p_{k+1}, \dots).$$

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Corollary

If $p : b =_B b'$ and $p' : b =_B b'$ get identified in P, then $\|C\|$.

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- $U = \{ z \mid d(z, f(z)) > \frac{\varepsilon}{2} \}$ and $V = \{ z \mid d(z, f(z)) < \varepsilon \}$. Then $\mathbb{D}^2 = U \cup V$, since $\forall x : \mathbb{R}(x < \varepsilon \vee x > \frac{\varepsilon}{2})$.
- By shrinking f near $\mathbb{S}^1$, we may assume $\mathbb{S}^1 \subseteq U$.
- "Ray from $f(z)$ to z" defines a retraction $r : U \to \mathbb{S}^1$.
- Thus S^1 is a retract of $\int U$, so $\int U$ contains a nontrivial loop.
- $\mathbb{D}^2 = U \cup V = U \sqcup^{U \cap V} V$, so $\int U \sqcup^{\int(U \cap V)} \int V$ is contractible. ($\int$ preserves colimits in discrete context.)
- By the van Kampen theorem, we have $\|\int V\|$, hence $\|V\|$. □