Outline

1. Overview

2. Semantics of W-types

3. Semantics of inductive types

4. Semantics of HITs

5. From semantics to syntax

6. Adding HITs to a theory
My philosophy

- I want to use ("formal") type theory as an internal language for higher categories.
- Therefore, I want a type theory that has semantics in a wide class of categories (not just one "intended" model).
- Today, we are semantically motivated: but the "intended semantics" is a large class of "good model categories", which suffice (for instance) to represent every $\infty$-topos.
- In the distant future, it would be nice to be able to construct new models of HoTT inside HoTT. But for now, we use set-math (e.g. ZFC) as the metatheory.
The goal

Original Goal

Every good model category models higher inductive types.
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Every good model category models higher inductive types.

Basic idea is 5 years old. Why not published yet?

1. We are easily distracted.
2. There are a lot of details in making it precise.
3. Easy to construct models of particular HITs; harder to say what a general “HIT” is.
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Definition

The W-type of \((x : A) \vdash B(x)\) type is inductively generated by

- \(\text{sup} : \prod_{(x : A)}(B(x) \to W_{A,B}) \to W_{A,B}\)

or equivalently

- \(\text{sup} : \left(\sum_{(x : A)}(B(x) \to W_{A,B})\right) \to W_{A,B}\)
### Semantics of W-types

#### Definition

The W-type of $(x : A) \vdash B(x)$ type is inductively generated by

- $\text{sup} : \prod_{(x : A)} (B(x) \to W_{A,B}) \to W_{A,B}$

or equivalently

- $\text{sup} : \left( \sum_{(x : A)} (B(x) \to W_{A,B}) \right) \to W_{A,B}$

#### Theorem (Classical)

$W_{A,B}$ is the initial algebra for the polynomial endofunctor

$$P_{A,B}(X) \equiv \left( \sum_{(x : A)} (B(x) \to X) \right)$$
Now in category theory

**Definition**

The **polynomial endofunctor** associated to an exponentiable map \( f : B \to A \) is

\[
\begin{align*}
C & \xrightarrow{B^*} C/B \\
& \xrightarrow{\Pi_f} C/A \xrightarrow{\Sigma} C
\end{align*}
\]

**Definition**

An **algebra** for an endofunctor \( S : C \to C \) is an object \( X \) equipped with a map \( SX \to X \).

How can we construct an initial algebra for an endofunctor?
Some categorical technology

Some categorical technology


**Theorem (Kelly)**

Let $\mathcal{A}$ be a cocomplete category with two cocomplete factorization systems $(\mathcal{E}, \mathcal{M})$ and $(\mathcal{E}', \mathcal{M}')$, let $\mathcal{A}$ be $\mathcal{E}$- and $\mathcal{E}'$-cowellpowered, let $S$ be a well-pointed endofunctor, and for some regular cardinal $\alpha$, let $S$ preserve the $\mathcal{E}'$-tightness of $(\mathcal{M}, \alpha)$-cones. Then $S$-$\text{Alg}$ is constructively reflective in $\mathcal{A}$.
The high technology: user-friendly version

Theorem (Kelly?)

Let $C$ be a locally presentable category. Then:

- Every accessible endofunctor of $C$ generates an algebraically-free monad.
- Every small diagram of accessible monads on $C$ has an algebraic colimit.
Review about monads

- Monad = endofunctor $T$ with $\mu : TT \to T$, $\eta : \text{Id} \to T$, axioms
- $T$-algebra = object $X$ with $TX \to X$, axioms
- The forgetful functor $U_T : T\text{-Alg} \to C$ has a left adjoint

\[ F_TX = (TX, \mu_X : TTX \to TX). \]

and in particular an initial object $F_T(\emptyset)$. 
Review about monads

- Monad = endofunctor $T$ with $\mu : TT \to T$, $\eta : \text{Id} \to T$, axioms
- $T$-algebra = object $X$ with $TX \to X$, axioms
- The forgetful functor $U_T : T\text{-Alg} \to C$ has a left adjoint
  \[ F_T X = (TX, \mu_X : TTX \to TX). \]
  and in particular an initial object $F_T(\emptyset)$.
- The assignation $T \mapsto T\text{-Alg}$ is a fully faithful embedding

\[ \text{Monads}^{\text{op}} \hookrightarrow \text{Cat}/C. \]

i.e. we have

\[ \text{Monads}(T_1, T_2) \cong \text{Cat}/C(T_2\text{-Alg}, T_1\text{-Alg}) \]
Free monads

**Definition**

Every monad has an underlying endofunctor; this defines a functor

\[
\text{monads on } \mathcal{C} \rightarrow \text{endofunctors on } \mathcal{C}.
\]

A **free monad** on an endofunctor \( S \) is the value at \( S \) of a (partially defined) left adjoint to this:

\[
\text{Monads}(\overline{S}, T) \cong \text{Endofrs}(S, T)
\]

\[
\begin{array}{ccc}
S & \rightarrow & \overline{S} \\
\downarrow & & \downarrow \\
&T &
\end{array}
\]
## Algebraically-free monads

### Definition

A monad $\overline{S}$ is **algebraically-free** on $S$ if we have an equivalence of categories over $\mathcal{C}$:

$$\overline{S} \text{ monad-algebras} \xrightarrow{\sim} S \text{ endofunctor-algebras}$$

### Theorem (Kelly?)

*Every algebraically-free monad is free, and the converse holds if $\mathcal{C}$ is locally small and complete.*
Semantics of W-types, again

Idea

- Given $(x : A) \vdash B(x)$ type
- It interprets as a fibration $f : B \to A$, hence exponentiable
- The associated polynomial endofunctor $S_f$ is accessible
- By Kelly’s theorem, it generates an algebraically-free monad $T_f$
- Define $W_{A,B} = T_f(\emptyset)$.
Semantics of W-types, again

### Idea

- Given \((x : A) \vdash B(x)\) type
- It interprets as a fibration \(f : B \to A\), hence exponentiable
- The associated polynomial endofunctor \(S_f\) is accessible
- By Kelly’s theorem, it generates an algebraically-free monad \(T_f\)
- Define \(W_{A,B} = T_f(\emptyset)\).

### Subtleties (ignore for now)

- Pullback-stability
- Fibrancy in homotopical models
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Semantics of inductive types

Example

Consider the inductive type $H$ generated by

- $\sup_1 : \prod_{(x:A)}(B(x) \to H) \to H$
- $\sup_2 : \prod_{(x:C)}(D(x) \to H) \to H$

Expect $H$ to be initial among objects $X$ equipped with two maps

$\left(\sum_{(x:A)}(B(x) \to X)\right) \to X$ and $\left(\sum_{(x:C)}(D(x) \to X)\right) \to X$.

Questions

1. Given endofunctors $S_1, S_2$, is there a monad $T$ whose algebras have (unrelated) $S_1$-algebra and $S_2$-algebra structures?
2. Given monads $T_1, T_2$, is there a monad $T$ whose algebras have (unrelated) $T_1$-algebra and $T_2$-algebra structures?
Algebraic colimits of monads

**Definition**

An algebraic colimit of a diagram $D : J \to \text{Monads}$ is a monad $T$ with an equivalence of categories over $C$:

$$T\text{-Alg} \xrightarrow{\sim} \text{lim}_{j \in J} D_j\text{-Alg}$$

This is a limit in $\text{Cat}_{/C}$, so it means that

$T$-algebra structures on $X \iff$ compatible families of $D_j$-algebra structures on $X$.

**Theorem**

Every algebraic colimit is a colimit in the category of monads, and the converse holds if $C$ is locally small and complete.
Idea

- Each constructor yields a polynomial endofunctor, hence an algebraically-free monad $T_c$
- Take the algebraic coproduct $T = \sum_c T_c$ of all these monads
- The inductive type is $T(\emptyset)$.

Remark 1

The domains of constructors can be more general than in a $W$-type, but they can be reduced easily to that form using $\Sigma$-types.

Remark 2

The initial monad (the empty algebraic coproduct) is $\text{Id}$, whose algebras are just objects. Thus, the empty type — the inductive type generated by no constructors — is the initial object.
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A first example

Example

Consider the propositional truncation $\parallel A \parallel$, generated by

- $A \to \parallel A \parallel$
- $\prod_{(x,y : \parallel A \parallel)} (x = y)$

- First constructor adds a point for every point of $A$
  $\leadsto$ constant endofunctor $S_1(X) = A$
- Second constructor adds a path for every two points of $\parallel A \parallel$
  $\leadsto$ endofunctor $S_2(X) = X \times X \times I$
- How do we control the endpoints of those paths?
The boundary endofunctor

Define another endofunctor

\[ \partial S_2(X) = X \times X \times 2 = (X \times X) + (X \times X). \]

- A \( \partial S_2 \)-algebra is a type with two binary operations.
- Every \( S_2 \)-algebra is a \( \partial S_2 \)-algebra via \( 2 \to I \) (i.e. “take the endpoints”).
- Every object is also a \( \partial S_2 \)-algebra via \([\pi_1, \pi_2]\) (i.e. \( (x, y) \mapsto x \) and \( (x, y) \mapsto y \)).
- The endpoints of the paths in an \( S_2 \)-algebra are correct iff these two \( \partial S_2 \)-algebra structures are the same.
Gluing intervals onto monads

So we are interested in the pullback category on the left:

\[
\begin{array}{ccc}
\bullet & \longrightarrow & \mathcal{C} \\
\downarrow & & \downarrow \\
S_2\text{-Alg} & \longrightarrow & \partial S_2\text{-Alg}
\end{array}
\hspace{1cm}
\begin{array}{ccc}
\overline{\partial S_2} & \longrightarrow & \text{Id} \\
\downarrow & & \downarrow \\
\overline{S_2} & \longrightarrow & T_2
\end{array}
\]

which corresponds to the algebraic colimit of monads on the right. We also need the $S_1$-algebra structure (a map from $A$), so:

**Conclusion**

$\|A\|$ is the initial $(\overline{S_1} + T_2)$-algebra.
Dependency between constructors

Example

The free group on $A$ is generated by

- $A \rightarrow FA$
- $(_ \cdot _) : FA \rightarrow FA \rightarrow FA$
- $\prod_{(x,y,z:FA)} (x \cdot (y \cdot z) = (x \cdot y) \cdot z)$
- $\ldots$

NB: The source and target of the path in the third constructor (associativity) refer to the second constructor (multiplication).

- $S_1(X) = A$
- $S_2(X) = X \times X$
- $S_3(X) = X \times X \times X \times I$, but then...
Dependency between constructors

\[ S_3(X) = X \times X \times X \times 1 \]
\[ \partial S_3(X) = X \times X \times X \times 2 = (X \times X \times X) + (X \times X \times X) \]

- Not every object is a \( \partial S_3 \)-algebra in the right way…
- …only the \( S_2 \)-algebras are!
- The functor \( S_2\text{-Alg} \to \partial S_3\text{-Alg} \) equips an \( S_2 \)-algebra (a “magma”) with the two ternary operations “\( x \cdot (y \cdot z) \)” and “\((x \cdot y) \cdot z\)”.
For $\|A\|$, instead of a pushout and then a coproduct, we could instead consider the outer pushout:

$$
\begin{align*}
\partial S_2 & \to \text{Id} \to S_1 \\
S_2 & \to T_2 \to S_1 + T_2
\end{align*}
$$

Similarly, for the free group we could incorporate $S_1$ from the beginning:

$$
\begin{align*}
\partial S_3 & \to S_2 \to S_1 + S_2 \\
S_3 & \to \bullet \to \bullet
\end{align*}
$$
**Cofibrations**

**Question:** What is special about $\mathbb{2} \to \mathbb{I}$ that makes this work?

\[
\begin{align*}
S(X) \times \mathbb{2} & \xrightarrow{u,v} X \\
S(X) \times \mathbb{I} & \quad \iff \quad S(X) \xrightarrow{u,v} X^\mathbb{I} \\
S(X) \times \mathbb{I} & \quad \iff \quad \prod_{s:S(X)} (u(s) =_X v(s))
\end{align*}
\]

**Answer:** $\mathbb{2} \to \mathbb{I}$ is a cofibration.
Other cofibrations

- If $C \to D$ is a cofib. & $X$ is fibrant, $X^D \to X^C$ is a fibration.
- Hence we have types (its fibers) of “(strict) extensions of a given map $C \to X$ to a map $D \to X$.”
- Other cofibrations give other kinds of constructors:

\[
\begin{align*}
\emptyset & \to \mathbf{1} & \rightsquigarrow & & X \\
2 & \to I & \rightsquigarrow & & x =_A y \\
\partial G_2 & \to G_2 & \rightsquigarrow & & p =_{x=y} q \\
\partial \Box^2 & \to \Box^2 & \rightsquigarrow & & \text{Square}(p, q, r, s) \\
\vdots & & & & \vdots
\end{align*}
\]
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Definition

A **HIT spec** consists of

- An **ordered list** of constructors.
- Each constructor has a **domain**, giving a polynomial endofunctor $S_n$.
- Each constructor has a **shape**, which is a cofibration $C_n \rightarrow D_n$ mapping the “boundary” into the “path”.
- Finally, each constructor has a **boundary**, which has something to do with $C_n$. 
The semantics of a HIT spec

- Each constructor yields a map of free monads

\[ S_n \times C_n \to S_n \times D_n \]

- Starting from \( T_0 = \text{Id} \), we build up monads successively:

\[
\begin{align*}
S_n \times C_n & \to T_{n-1} \\
S_n \times D_n & \to T_n
\end{align*}
\]

A monad built in this way we call a cell monad.

- A HIT with \( n \) constructors is \( T_n(\emptyset) \).
Question

What can the boundaries of a path-constructor be?

Answer

The semantics tells us! They have to be:

- monad morphisms $S_n \times C_n \rightarrow T_{n-1}$, or equivalently
- endofunctor maps $S_n \times C_n \rightarrow T_{n-1}$.

But what are those?
Free endofunctors

Suppose $S_n$ is polynomial on $(a : A_n) \vdash B_n(a)$:

$$S_n(X) = \sum_{a : A_n} X^{B_n(a)}$$

$$(S_n \times C_n)(X) = (\sum_{a : A_n} X^{B_n(a)}) \times C_n$$

$$= \sum_{(a, c) : A_n \times C_n} X^{B_n(a)}$$

• Internally, this is a coproduct of the functors $\lambda X. X^{B_n(a)}$. So $S_n \times C_n \to T_{n-1}$ consists of “a map $\lambda X. X^{B_n(a)} \to T_{n-1}$ for each $(a, c) : A_n \times C_n$.”
Suppose $S_n$ is polynomial on $(a : A_n) \vdash B_n(a)$:

$$S_n(X) = \sum_{a : A_n} X^{B_n(a)}$$

$$(S_n \times C_n)(X) = \left(\sum_{a : A_n} X^{B_n(a)}\right) \times C_n$$

$$= \sum_{(a, c) : A_n \times C_n} X^{B_n(a)}$$

- Internally, this is a coproduct of the functors $\lambda X.X^{B_n(a)}$.
  So $S_n \times C_n \to T_{n-1}$ consists of “a map $\lambda X.X^{B_n(a)} \to T_{n-1}$ for each $(a, c) : A_n \times C_n$”.

- But by (internal) Yoneda,

$$\text{NatTrans}(\lambda X.X^{B_n(a)}, T_{n-1}) = T_{n-1}(B_n(a)).$$

- So natural transformations $S_n \times C_n \to T_{n-1}$ are the same as

$$\prod_{(a, c) : A_n \times C_n} T_{n-1}(B_n(a))$$
Syntax for boundaries

\[ \Pi_{(a,c) : A_n \times C_n} T_{n-1}(B_n(a)) \]
Syntax for boundaries

\[ \Pi_{a:A_n} \left(C_n \rightarrow T_{n-1}(B_n(a)) \right) \]
Syntax for boundaries

\[ \prod_{a:A_n} \left( C_n \rightarrow T_{n-1}(B_n(a)) \right) \]

- \( T_{n-1}(\emptyset) \) is the HIT generated by the first \((n-1)\) constructors.
- \( T_{n-1}(B_n(a)) \) is the HIT generated by the first \((n-1)\) constructors and one extra constructor with domain \( B_n(a) \).

Thus, the boundary of a constructor \( \prod_{a:A_n} \prod_{f:B_n(a) \rightarrow W} \cdots \) is

- For each \( a : A_n \),
- \( \ldots \) a \( C_n \)-shaped picture (pair of points, parallel paths, etc.)
- \( \ldots \) in the HIT generated by the previous constructors and new symbols “\( f(b) \)” for all \( b : B_n(a) \).
### Examples

<table>
<thead>
<tr>
<th>Expression</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td><code>loop : base = base</code></td>
<td>“base” is a term in the HIT generated by “base” only</td>
</tr>
<tr>
<td><code>merid : \prod_x N = S</code></td>
<td>( N ) and ( S ) are terms in the HIT generated by ( N ) and ( S ) only</td>
</tr>
<tr>
<td><code>surf : p \cdot q = q \cdot p</code></td>
<td>( p \cdot q ) and ( q \cdot p ) are terms in the HIT generated by ( b ), ( p : b = b ), and ( q : b = b )</td>
</tr>
<tr>
<td>( \prod_{x,y:</td>
<td></td>
</tr>
</tbody>
</table>
A more complicated example

In the localization $L_f(A)$ at $f : P \rightarrow Q$, we see:

$$\prod_{x : P} \prod_{g : P \rightarrow L_f(A)} \text{ext}(g, f(x)) = g(x)$$

Here $A = P$, $B(a) = P$, and both $\text{ext}(g, f(x))$ and $g(x)$ are (assuming $x : P$) terms in the HIT $W'$ generated by $\text{ext} : \prod_{g : P \rightarrow W'}(Q \rightarrow L_f(A))$ and $g : P \rightarrow W'$. 
• In general, we expect some “grammar” describing what the boundary of a constructor can be.

• We are leveraging the type theory itself to be this grammar: the boundary simply consists of terms in a particular type.
Definition

Inductively, a HIT spec $W$ is either empty, or consists of:

- A HIT spec $W'$ (the previous constructors).
- A constructor domain $(a : A) \vdash B(a)$.
- A constructor shape, which is a cofibration $C \to D$.
- A constructor boundary $\prod_{a : A} \left( C \to W'_a \right)$, where $W'_a$ is the HIT generated by $W'$ together with a map $B(a) \to W'_a$. 
Definition

Given a HIT spec $W$, a $W$-algebra is a type $X$ together with:

- A $W'$-algebra structure (inductively), and...
- For each $a : A$ and $f : B(a) \rightarrow X$, the $W'$-algebra structure and $f$ make $X$ a $W'_a$-algebra. So by recursion we have $W'_a \rightarrow X$, hence a boundary composite $C \rightarrow W'_a \rightarrow X$. The additional data is an extension of this to $D$:

\[
\begin{array}{c}
C \\
\downarrow \\
D
\end{array} \rightarrow 
\begin{array}{c}
W'_a \\
\downarrow \\
X
\end{array} 
\]
Definition

Given a HIT spec $\mathcal{W}$, a $\mathcal{W}$-algebra is a type $X$ together with:

- A $\mathcal{W}'$-algebra structure (inductively), and...
- For each $a : A$ and $f : B(a) \rightarrow X$, the $\mathcal{W}'$-algebra structure and $f$ make $X$ a $\mathcal{W}'_a$-algebra. So by recursion we have $\mathcal{W}'_a \rightarrow X$, hence a boundary composite $C \rightarrow \mathcal{W}'_a \rightarrow X$. The additional data is an extension of this to $D$.

- **Intro**: “$\mathcal{W}$ is a $\mathcal{W}$-algebra”.
- **Elim**: “Any dependent $\mathcal{W}$-algebra over $\mathcal{W}$ has a section.”
- **Comp**: “The section is a $\mathcal{W}$-algebra map.”
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What are these cofibrations, anyway?

We could either

1. Fix a particular set of cofibrations that exist in models of interest, like $\partial G_n \to G_n$ or $\partial I^n \to I^n$.

2. Extend the type theory with a judgment for cofibrations, and a “type of extensions” of a given function along such a cofibration.
   - In the case $2 \to I$ this will behave like cubical identity types.
   - NB: objects like $I$ are not usually fibrant; put them in a separate context (like cubical “dimension variables”) or use HTS-style “pretypes”.
The type of extensions along a cofibration

\[
\begin{align*}
\Gamma \vdash i : A \rightarrowtail B & \quad \Gamma, y : B \vdash C \text{ type} & \quad \Gamma, x : A \vdash d : C[i(x)/y] \\
\Gamma \vdash \text{Extn}_{i,y}.C(x.d) \text{ type} & \\
\Gamma, y : B \vdash c : C & \quad \Gamma, x : A \vdash c[i(x)/y] \equiv d \\
\Gamma \vdash \lambda y. c : \text{Extn}_{i,y}.C(x.d) & \\
\Gamma \vdash f : \text{Extn}_{i,y}.C(y)(x.d(x)) & \quad \Gamma \vdash b : B \\
\Gamma \vdash f \circ b : C(b) & \quad f \circ i(a) \equiv d[a/x] \\
\end{align*}
\]

(plus \(\beta, \eta\))

- For \(1 + 1 \rightarrowtail \mathbf{1}\), reproduces cubical identity types.
- Semantically, represents the pullback corner map (Leibniz cotensor) of a cofibration against a fibration.
What's up with that induction?

**Question**

HITs are defined inductively. Where does that induction happen?

**Answer #1**

In the metatheory.

I.e. given any type theory containing some HITs, we can choose one of them, choose a domain, shape, and boundary to determine a new constructor, and obtain a new type theory containing one more HIT.
A theory containing HITs

Answer #2

By defining a new judgment form inside the theory.

I.e. we have a judgment for “HIT specs”, whose rule is “add a new constructor”, and a rule that any HIT spec gives a HIT.

But now

Any judgment form in the theory must be interpreted by something in the semantics. What is a “semantic HIT spec”?
A theory containing HITs

Answer #2

By defining a new judgment form inside the theory.

I.e. we have a judgment for “HIT specs”, whose rule is “add a new constructor”, and a rule that any HIT spec gives a HIT.

But now

Any judgment form in the theory must be interpreted by something in the semantics. What is a “semantic HIT spec”? 

...a monad.
### A type theory with monads

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- Substitution for type variables in monads
  ⇒ all monads are indexed

- Substitution for monad variables in algebras
  ⇒ a monad map $T \to S$ gives a functor $S$-$\text{Alg} \to T$-$\text{Alg}$

- Elimination of monad formers into all judgments
  ⇒ monad colimits are algebraic colimits
Part II

All the lies I just told you
Outline

7 Fibrancy

8 Pullback-stability

9 Fibrancy in boundaries

10 Closure of universes
Problem #1: Fibrancy

<table>
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<td>$T(\emptyset)$ may not be fibrant, hence may not represent a type.</td>
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- In non-recursive cases (e.g. empty type, coproduct type) we can just fibrantly replace it.
- But in recursive cases, its fibrant replacement may no longer be a $T$-algebra: the newly added fillers need a free $T$-action, which may produce new horns that need fillers, etc.
Let $R$ be the fibrant replacement monad (Garner).

Let $T_R = T + R$.

- A $T_R$-algebra is a $T$-algebra with an unrelated $R$-algebra structure.
- In particular, every $T_R$-algebra is fibrant!

Theorem (L-S)

$T_R(\emptyset)$ satisfies the $T$-algebra induction principle: any fibration $p : Y \to T_R(\emptyset)$ that is a $T$-algebra map has a $T$-algebra section.
Outline

1. Fibrancy
2. Pullback-stability
3. Fibrancy in boundaries
4. Closure of universes
Problem #2: Pullback-stability

Everything in a model of type theory must be strictly stable under pullback.

- Use indexed endofunctors, monads, free monads, colimits of monads. Everything is pullback-stable up to iso...
- ...except $R$, which is not an indexed monad at all!
- Local universes mumbo-jumbo: form $T + R$ in the “universal case.”
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Problem #3: Boundaries can’t use fibrancy

**Problem**

*R* is not an indexed monad, but our algebraic colimits have to work on indexed monads.

- Therefore, we have to coproduct with *R* at the **very end** to obtain our HIT.
- Therefore, in the middle, the "previous constructors" HIT is **not fibrant**.
- Therefore, we can’t use "fibrant operations" (like path concatenation and eliminators) in boundaries of constructors.
**Problem #3: Boundaries can’t use fibrancy**

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<td>Hub and spoke does <strong>not</strong> help.</td>
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<td>Use cofibrations like $\partial \Box \to \Box$ whose domains implicitly involve “concatenations”</td>
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Outline

- Fibrancy
- Pullback-stability
- Fibrancy in boundaries
- Closure of universes
Problem #4: Closure of universes

Even in the best model categories, we don’t know how to show that any universes are closed under HITs.

- A universe $\tilde{U} \to U$ classifies fibrations with “small fibers”.
- For closure of $U$, the “universal HIT” over something built from $U$ must be classified by $U$.
- But the (algebraic) fibrant replacement of a map with small fibers over a large base may no longer have small fibers!