Homotopy type theory in higher toposes

An invitation to synthetic mathematics

Michael Shulman[1]

[1]University of San Diego

University of California, Los Angeles
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Outline

1. Towards synthetic mathematics
2. Type theory
3. Homotopy type theory
4. Higher topos models
5. Conclusion

Set-theoretic mathematics is analytic

Set-theoretic mathematics is reductive or analytic: breaking things into tiny pieces.

Amazing discovery (Cantor, Zermelo, Bourbaki, . . .)

All of mathematics can be built out of sets.

Set theory provides for mathematicians:

- A relative guarantee of consistency.
- A common language for communication.
- A powerful toolbox of general axioms.

But it has its discontents

From *Interview with Yuri Manin* (by Mikhail Gelfand), AMS Notices, October 2009:

> ...after Cantor and Bourbaki ...set theoretic mathematics resides in our brains. When I first start talking about something, I explain it in terms of Bourbaki-like structures ...we start with the discrete sets of Cantor, upon which we impose something more in the style of Bourbaki.
>
> But fundamental psychological changes also occur. ...the place of old forms and structures ...is taken by some geometric, right-brain objects.
>
> ...there is an ongoing reversal in the collective consciousness of mathematicians: the...homotopical picture of the world becomes the basic intuition, and if you want to get a discrete set, then you pass to the set of connected components...

Modern mathematics is synthetic

Modern mathematics is synthetic: putting things back together.

- Quantum groups
- Topological data analysis
- Computational number theory
- ...

Certain structures do not "belong" to one discipline, but are latent in all mathematical objects. For example:

1. All groups are "really" topological. Pontryagin duality, profinite completion, etc. force nontrivial topologies upon us.
2. All sets "really" have homotopical structure. Classifying spaces and stacks force nontrivial homotopy upon us.
3. All objects are "really" computational. Representing them in a computer forces us to account for this.

NB: "Topology" = up to homeomorphism; "homotopy" = up to weak homotopy equivalence (= Kan complexes = ∞-groupoids = anima)

A new toolbox

But even defining these synthesized structures piece-by-piece from sets is complicated. In particular, it's

- Hard to teach to undergraduates, and
- Hard for a computer to check.

Homotopy type theory is a formal system for synthetic mathematics. Its basic objects, called types, behave somewhat like sets, but also carry many latent structures, such as:

- Homotopical structure
- Computational structure
- Topological structure
- Smooth/algebraic structure
- Orbifold structure
- Group actions
- Sheaf structure
- Graph structure
- Abelian/stable structure
- Quantum structure
- Stochastic structure
- Infinitesimal structure
- Time-varying structure
- ∞-categorical structure

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What are types?

Like a set, a type has elements. We write $a : A$, not $a \in A$.

Unlike for ZFC-sets, most elements are not themselves types, and most elements belong to only one type.

- The elements of $A \times B$ are pairs (a, b) with $a : A$ and $b : B$.
- The elements of $A \to B$ are functions from A to B.
- The elements of $A \sqcup B$ are the elements of A and the elements of B, "tagged" with their origin.
- The elements of $\{ x : A \mid \phi(x) \}$ are elements of A "tagged" with a witness of the truth of $\phi(x)$.

Intuition

A type is a set that might have topological, homotopical, computational, etc. structure.

Type-theoretic mathematics

We can build all the basic objects of mathematics with types.

- $\mathbb{N}$ is a basic type with rules for induction and recursion.
- $\mathbb{Q} = \{\, (a, b) : \mathbb{N} \times \mathbb{N} \mid b > 0 \text{ and } \gcd(a, b) = 1 \,\}$.
- $\mathbb{R} = \{\, (L, R) : \mathcal{P}\mathbb{Q} \times \mathcal{P}\mathbb{Q} \mid \ldots \}$
- $GL(2, \mathbb{R}) = \left\{\, A : \mathbb{R}^{2\times 2} \;\middle|\; a_{00}a_{11} - a_{10}a_{01} \neq 0 \,\right\}$
- $\mathbb{S}^1 = \left\{\, (x, y) : \mathbb{R} \times \mathbb{R} \;\middle|\; x^2 + y^2 = 1 \,\right\}$
- ...

Latent structure

Unaugmented type theory is neutral mathematics, which respects all possible latent structures, but does not mandate any of them.

Example

$\mathbb{R}$ has a latent topology: we cannot define, in neutral mathematics, any discontinuous functions $\mathbb{R} \to \mathbb{R}$; but neither can we assert positively that every function $\mathbb{R} \to \mathbb{R}$ is continuous.

Example

Similarly, $\mathbb{N}$ is latently computable: we can't define noncomputable functions $\mathbb{N} \to \mathbb{N}$, nor prove they don't exist.

Formally, the logic of type theory is "constructive", lacking the law of excluded middle (P or not-P).

Neutral mathematics

Compare neutral geometry without the parallel postulate, which could become Euclidean, spherical, or hyperbolic by adding axioms. Similarly, we can further specify the behavior of types by assuming:

- Excluded middle, which excludes almost all latent structure.
- Brouwer's principle: all functions $\mathbb{R} \to \mathbb{R}$ are continuous.
- Church's thesis: all functions $\mathbb{N} \to \mathbb{N}$ are computable.
- ...

But anything we can do without these axioms, in neutral mathematics, automatically preserves all latent structures.

Slogan

- Analytic mathematics is additive: we start with sets and add the structure we want piece by piece.
- Synthetic mathematics is subtractive: types have all latent structures; axioms pare away the ones we don't want.

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Latent homotopical structure

Question

What axiom specifies types have nontrivial homotopy structure?

A basic family of types are the identity types

$$\mathtt{Id}_A(a, b),$$

whose elements are identifications of $a : A$ with $b : A$. Latently, they behave like homotopical paths (∞-groupoidal isos) from a to b.

Most types have no homotopy

Definition

A is (homotopically) discrete if $\mathtt{Id}_A(a,b)$ always has at most one element: no "information" is carried by a and b being equal.

Most types are discrete: $\mathbb{N}$, $\mathbb{Q}$, $\mathbb{R}$, etc.

Warning

$\mathbb{R}$ has latent *topology*, but is *homotopically* discrete. There are no *identifications* $\mathtt{Id}_A(1,2)$, though there are *topological paths*.

The universe

The universe $\mathcal{U}$ is a type whose elements are other types.
(Actually, a tower of universes $\mathcal{U}_1, \mathcal{U}_2, \ldots$ to avoid paradox.)

Univalence axiom (Voevodsky)

For $A : \mathcal{U}$ and $B : \mathcal{U}$, the elements of the identity type

$$\mathtt{Id}_{\mathcal{U}}(A, B)$$

are equivalences $A \simeq B$.

Two types can be equivalent in more than one way, e.g. $\mathbb{B} = 1 \sqcup 1$ has two automorphisms. Thus, univalence implies that $\mathcal{U}$ is not homotopically discrete: $\mathtt{Id}_{\mathcal{U}}(\mathbb{B}, \mathbb{B}) \simeq \mathbb{B}$.

Classifying spaces

In fact, $\mathcal{U}$ is the father of all classifying spaces.

- $BO(\mathbb{k}) = \{ V : \mathcal{U} \mid V \text{ is a } \mathbb{k}\text{-vector space} \}$ is a classifying space for vector bundles.
- $BG = \{ X : \mathcal{U} \mid X \text{ is a free transitive } G\text{-set} \}$ is a classifying space for principal G-bundles.
- $BAut(F) = \{ X : \mathcal{U} \mid \exists(g : X \cong F) \}$ is a classifying space for fibrations with fiber F.
- $BAb = \{ M : \mathcal{U} \mid M \text{ is an abelian group} \}$ is a classifying space for local systems, i.e. $M : X \to BAb$ is a local system on X.
- $BSp = \{ E : \mathcal{U} \mid E \text{ is a spectrum} \}$ is a classifying space for spectra, e.g. $E : A \to BSp$ is a parametrized spectrum over A.

(Ignoring some subtleties here re: topology vs. homotopy.)

Synthetic homotopy theory

Treating types as homotopy-spaces, with univalence we can develop the basics of algebraic topology:

- Homotopy groups, generalized homology and cohomology
- Van Kampen and Hurewicz theorems
- Freudenthal suspension and Blakers–Massey theorems
- $\pi_n(S^n) = \mathbb{Z}$, $\pi_3(S^2) = \mathbb{Z}$, $\pi_4(S^3) = \mathbb{Z}/2$, ...
- Serre and Atiyah–Hirzebruch spectral sequences
- Localization at primes
- ...

(Learn more starting at `https://homotopytypetheory.org/book`)

Homotopy theory plus other latent structures

It's unknown quite how much of algebraic topology is valid in neutral mathematics assuming only univalence, but so far the answer seems to be: a lot.

Thus, all of this algebraic topology is also valid for any other latent structure that isn't ruled out by univalence!

Voevodsky (c. 2009) showed excluded middle (properly formulated) is compatible with univalence. But what about axioms for other latent structures?

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Topos models

One way to "extract" latent structure in types is to interpret type theory into analytically defined X-structured objects.

1. Isolate a small category $\mathscr{D}$ of "basic X-structured objects".
2. Consider presheaves of sets on $\mathscr{D}$.
3. Localize at some covers to get a Grothendieck topos.
4. Construct a "compiler" interpreting each type as an object of the topos, each function as a morphism, etc.

Then, anything we prove in neutral type theory automatically interprets to a statement about X-structured sets. We can also isolate special axioms that hold in X-structured sets, thereby specializing neutral type theory to a more focused language.

Higher topos models

To combine X-structure with homotopical structure, use homotopy sheaves instead.

1. Isolate a small category $\mathscr{D}$ of "basic X-structured objects".
2. Consider presheaves of homotopy-spaces on $\mathscr{D}$.
3. Localize at covers to get a Grothendieck–Rezk–Lurie ∞-topos.
4. Construct a "compiler" interpreting each type as an object of the ∞-topos, each function as a morphism, etc.

Examples of toposes

Informally, objects of a topos are “glued together” from basic ones.

Objects of topos	Basic objects
(Generalized) manifolds	open submanifolds $U \subseteq \mathbb{R}^n$
Sequential spaces	convergent sequences $\{0, 1, 2, \ldots, \infty\}$
Algebraic spaces	affine schemes
Sheaves on X	open subsets $U \subseteq X$
Combinatorial graphs	vertices and edges
Time-varying sets	elements that exist starting at a time t
G-sets	orbits G/H
Quantum systems	consistent classical observations
Computable sets	computable subsets $U \subseteq \mathbb{N}$
Random variables	measurable sets

More precisely...

$\mathscr{D}$ a small category, $\mathscr{S}$ the category of homotopy-spaces.
We want a homotopy theory of presheaves $\mathscr{D}^{\mathsf{op}} \to \mathscr{S}$.

To be precise, we'll work with $\mathscr{S} =$ simplicial sets, but to first approximation you can think of topological spaces.

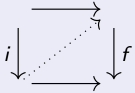
We'll put a homotopy-theoretic structure on the (strict) functor category $[\mathscr{D}^{\mathsf{op}}, \mathscr{S}]$ that carries information about homotopy-coherent functors and natural transformations.

Model categories

Definition (Quillen)

A model category is a complete and cocomplete category $\mathscr{M}$ with three classes of maps $\mathcal{F}$ (fibrations), $\mathcal{C}$ (cofibrations), and $\mathcal{W}$ (weak equivalences), such that

1. If two out of f, g, and gf are in $\mathcal{W}$, so is the third.
2. $\mathscr{M} = \mathcal{F} \circ (\mathcal{C} \cap \mathcal{W}) = (\mathcal{F} \cap \mathcal{W}) \circ \mathcal{C}$.
3. f is in $\mathcal{F}$ iff any square with $i \in \mathcal{C} \cap \mathcal{W}$ has a diagonal filler:



4. Similar characterizations of $\mathcal{C}$, $\mathcal{F} \cap \mathcal{W}$, and $\mathcal{C} \cap \mathcal{W}$.

($\mathcal{C} \cap \mathcal{W}$ = acyclic cofibrations and $\mathcal{F} \cap \mathcal{W}$ = acyclic fibrations)

Model categories

The homotopy theory lives in the objects that are fibrant ($X \to 1$ is a fibration) and cofibrant ($\emptyset \to X$ is a cofibration).

Example

$\mathscr{M}$ = topological spaces:

- cofibrations $\approx$ relative cell complexes
- fibrations = Serre fibrations
- fibrant+cofibrant objects $\approx$ cell complexes.

Example

$\mathscr{M} = \mathscr{S}$ (simplicial sets):

- cofibrations = monos
- fibrations = Kan fibrations
- fibrant+cofibrant objects = Kan complexes.

Interpreting type theory in a model category

If all objects are cofibrant, we can interpret type theory by:

type A	$\rightsquigarrow$	fibrant object A
element $a : A$	$\rightsquigarrow$	morphism $a : 1 \to A$
function $f : A \to B$	$\rightsquigarrow$	morphism $f : A \to B$
family of types $\{B(x)\}_{x:A}$	$\rightsquigarrow$	fibration $B \twoheadrightarrow A$
element family $\{b(x) : B(x)\}_{x:A}$	$\rightsquigarrow$	section $A \to B \twoheadrightarrow A$
identity type $\{\mathtt{Id}_A(a, b)\}_{a:A,b:A}$	$\rightsquigarrow$	path object $PA \twoheadrightarrow A \times A$

If the model category is sufficiently nice, we can interpret all the rules of type theory this way.

The injective model structure

Theorem (Heller, Lurie)

The category $[\mathscr{D}^{\mathsf{op}}, \mathscr{S}]$ has an injective model structure such that:

- *The weak equivalences and cofibrations are levelwise.*
- *In particular, all objects are cofibrant.*
- *It is combinatorial, right proper, simplicially enriched, and simplicially locally cartesian closed.*

Theorem (Awodey–Warren, van den Berg–Garner, Lumsdaine–Shulman, etc.)

The injective model structure interprets all of homotopy type theory except (possibly) the universe type $\mathcal{U}$.

Universes in model categories

Definition

A universe in a model category (relative to some cardinal κ) is a fibration $\pi : \widetilde{U} \to U$ with κ-small fibers such that every fibration with κ-small fibers is a pullback of π.

Under general assumptions, the pullback is unique up to homotopy; so U is a classifying space for κ-small fibrations.

Theorem (Voevodsky)

$\mathscr{S}$ has universes for all sufficiently large κ.

Here the types have no structure except homotopy. Mathematics in $\mathscr{S}$ is classical, with all non-homotopical latent structure excised.

Universes in presheaves

Note that $[\mathscr{D}^{\mathsf{op}}, \mathscr{S}] = [\mathscr{C}^{\mathsf{op}}, \mathrm{Set}]$ where $\mathscr{C} = \mathscr{D} \times \mathbb{\Delta}$.

Definition

If $\mathscr{M} = [\mathscr{C}^{\mathsf{op}}, \mathrm{Set}]$ is a presheaf category, define $U \in \mathscr{M}$ where $U(c)$ is the “set” of κ-small fibrations over $\mathscr{C}(-, c)$. The functorial action is by pullback. Similarly, define $\widetilde{U}$ and $\pi : \widetilde{U} \to U$.

Theorem

Every κ-small fibration is a pullback of π.

This takes a bit of work to make precise:

- $U(c)$ must be a set containing at least one representative for each *isomorphism class* of such κ-small fibrations,
- Chosen cleverly to make pullback *strictly* functorial.

But the real problem is that π itself may not be a fibration!

Injective fibrations

So. . . what are the fibrations in the injective model structure?

Why levelwise isn't enough

When is $X \in [\mathscr{D}^{\mathrm{op}}, \mathscr{S}]$ injectively fibrant? We want to lift in

$$\begin{array}{ccc} A & \xrightarrow{g} & X \\ {\scriptstyle i}\big\downarrow{\scriptstyle \sim} & \nearrow & \\ B & & \end{array}$$

where $i : A \to B$ is a levelwise acyclic cofibration.
If X is levelwise fibrant, then for all $d \in \mathscr{D}$ we have a lift

$$\begin{array}{ccc} A_d & \xrightarrow{g_d} & X_d \\ {\scriptstyle i_d}\big\downarrow{\scriptstyle \sim} & \nearrow{\scriptstyle h_d} & \\ B_d & & \end{array}$$

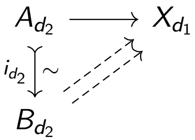
but these may not fit together into a natural transformation $B \to X$.

Naturality up to homotopy

Naturality would mean that for any $\delta : d_1 \to d_2$ in $\mathscr{D}$ we have $X_\delta \circ h_{d_2} = h_{d_1} \circ B_\delta$. This may not hold, but we do have

$$X_\delta \circ h_{d_2} \circ i_{d_2} = X_\delta \circ g_{d_2} = g_{d_1} \circ A_\delta = h_{d_1} \circ i_{d_1} \circ A_\delta = h_{d_1} \circ B_\delta \circ i_{d_2}.$$

Thus, $X_\delta \circ h_{d_2}$ and $h_{d_1} \circ B_\delta$ are both lifts in the following:

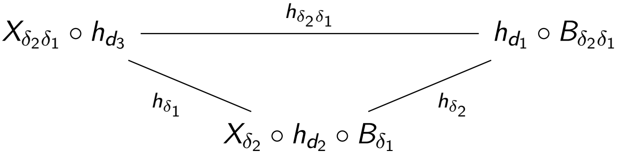


Since lifts between acyclic cofibrations and fibrations are unique up to homotopy, we do have a homotopy

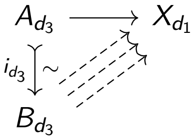
$$h_\delta : X_\delta \circ h_{d_2} \sim h_{d_1} \circ B_\delta.$$

Coherent naturality

Similarly, given $d_1 \xrightarrow{\delta_1} d_2 \xrightarrow{\delta_2} d_3$, we have a triangle of homotopies



whose vertices are lifts in the following:



Thus, homotopy uniqueness of lifts gives us a 2-simplex filler.

Homotopy coherent natural transformations

For $X, Y \in [\mathscr{D}^{\mathrm{op}}, \mathscr{S}]$, a homotopy coherent natural transformation $h : X \rightsquigarrow Y$ consists of:

- For every $d \in \mathscr{D}$, a morphism $h_d : X_d \to Y_d$.
- For every $d_1 \xrightarrow{\delta} d_2$ in $\mathscr{D}$, a homotopy $h_\delta : \Delta[1] \to \mathscr{S}(X_{d_2}, Y_{d_1})$ between $Y_\delta \circ h_{d_2}$ and $h_{d_1} \circ X_\delta$, such that h_{id_d} is constant.
- For every $d_1 \xrightarrow{\delta_1} d_2 \xrightarrow{\delta_2} d_3$ in $\mathscr{D}$, a 2-simplex $h_{\delta_1,\delta_2} : \Delta[2] \to \mathscr{S}(X_{d_3}, Y_{d_1})$ whose boundaries involve h_{δ_1}, h_{δ_2}, and $h_{\delta_2\delta_1}$, satisfying similar constancy conditions.
- And so on.

Thus, for a levelwise-fibrant Y to be injectively fibrant, we must be able to rectify homotopy-coherent transformations into Y.

The coherent morphism coclassifier

Theorem

There is a coclassifier of coherent transformations: an object $\mathsf{C}^{\mathscr{D}}(Y)$ *with a natural bijection*

$$\frac{\textit{coherent natural } X \rightsquigarrow Y}{\textit{strict natural } X \to \mathsf{C}^{\mathscr{D}}(Y)}$$

Also called a cobar construction, $\mathsf{C}^{\mathscr{D}}(Y)$ is the totalization of the cosimplicial object

$$GU \rightrightarrows GUGUY \overset{\rightarrow}{\rightrightarrows} GUGUGUY \quad \cdots$$

where G is right adjoint to the forgetful $U : [\mathscr{D}^{\mathrm{op}}, \mathscr{S}] \to \mathscr{S}^{\mathrm{ob}\mathscr{D}}$.

Injective fibrancy

The (strictly natural) identity $X \rightsquigarrow X$ corresponds to a canonical $\nu_X : X \to \mathsf{C}^{\mathscr{D}}(X)$, which is always a levelwise acyclic cofibration.

Theorem (S.)

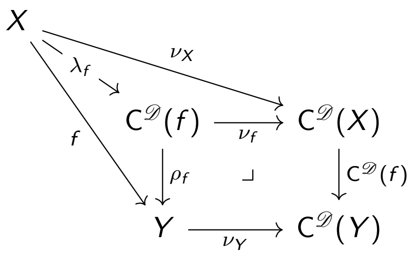
$X \in [\mathscr{D}^{\mathsf{op}}, \mathscr{S}]$ is injectively fibrant if and only if it is levelwise fibrant and $\nu_X : X \to \mathsf{C}^{\mathscr{D}}(X)$ has a retraction $r : \mathsf{C}^{\mathscr{D}}(X) \to X$.

Proof.

$\Rightarrow$: Lift against ν_X.
$\Leftarrow$: Construct a homotopy-coherent lift, hence a strict map into $\mathsf{C}^{\mathscr{D}}(X)$, then compose with r. $\square$

Injective fibrations

Given $f : X \to Y$, define a factorization by pullback:



Theorem (S.)

$f : X \to Y$ is an injective fibration if and only if it is a levelwise fibration and λ_f has a retraction $r : C^{\mathscr{D}}(f) \to X$ over Y.

This characterization is not "cofibrantly generated". The injective model structure is cofibrantly generated, but we still don't know anything about the generating acyclic cofibrations.

Universes for injective fibrations

Define a semi-algebraic injective fibration to be a levelwise fibration *equipped with* a retraction of λ_f.

Definition

In $[\mathscr{D}^{\mathsf{op}}, \mathscr{S}] = [\mathscr{C}^{\mathsf{op}}, \mathrm{Set}]$ where $\mathscr{C} = \mathscr{D} \times \mathbb{\Delta}$, define $U(c)$ to be the "set" of semi-algebraic κ-small injective fibrations over $\mathscr{C}(-, c)$. Similarly define $\widetilde{U}$ and $\pi : \widetilde{U} \to U$.

Semi-algebraicity structures can be glued together to make a universal one over U; thus we have:

Theorem (S.)

$\pi : \widetilde{U} \to U$ is a fibration, and U is a universe. Thus, $[\mathscr{D}^{\mathsf{op}}, \mathscr{S}]$ interprets all of homotopy type theory, with univalence.

Sheaf universes

Finally, given a left exact localization $L_S[\mathscr{C}^{op}, \mathscr{S}]$:

1. By Anel–Biedermann–Finster–Joyal (2021), lexness of S-localization is pullback-stable.
2. By Rijke–S.–Spitters, any $f : X \twoheadrightarrow Y$ has a "classifier of locality structures".
3. Define a semi-algebraic local fibration to be a semi-algebraic injective fibration equipped with a locality structure.
4. Now use the same approach.

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Unifying homotopy theory

Synthetic homotopy theory is neutral homotopy theory: valid in all ∞-toposes. This includes many domains of "classical" interest in algebraic topology, such as:

- Equivariant homotopy theory
- Parametrized homotopy theory
- Global homotopy theory
- Homotopy theory of sheaves
- Motivic homotopy theory
- ...

Working abstractly with model categories or ∞-toposes is one way to do neutral homotopy theory. Homotopy type theory is another:

- Can work intuitively with types that have elements and behave much like classical spaces.
- Also a programming language; can be formalized in a computer.

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