Unbounded quantifiers and strong axioms in topos theory

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The motivating question

What is the topos-theoretic counterpart of the strong set-theoretic axioms of Separation, Replacement, and Collection?

Outline

1. The Question
   - Two kinds of set theory
   - Strong axioms
   - Internalization
2. The Answer
   - Stack semantics
   - Logic in stack semantics
   - Strong axioms

Basic fact

Elementary topos theory is equiconsistent with a weak form of set theory.

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Following Lawvere and others, I prefer to view this as an equivalence between two kinds of set theory.

"Membership-based" or "material" set theory

Data:

- A collection of *sets*.
- A *membership* relation $\in$ between sets.

Sample axioms:

- Extensionality: if $x \in a \iff x \in b$ for all x, then $a = b$.
- Pairing: for all x and y, the set $\{x, y\}$ exists.
- Power set: for all sets a, the set $P(a) = \{x \mid x \subseteq a\}$ exists.
- ...

Example: ZFC (Zermelo-Fraenkel set theory with Choice).

“Structural” or “categorical” set theory

Data:

- A collection of *sets*.
- A collection of *functions*.
- Composition and identity operations on functions.

Sample axioms:

- Sets and functions form a category **Set**.
- **Set** has finite limits and power objects.
- *Well-pointedness*: The terminal set 1 is a generator.
- ...

Example: Lawvere’s ETCS (Elementary Theory of the Category of Sets).

Note: An element of a set A is a function $1 \to A$.

The basic comparison

Theorem (Cole, Mitchell, Osius)

ETCS is equiconsistent with BZC.

BZC = Bounded Zermelo set theory with Choice
= ZFC without Replacement, and with Separation restricted to formulas with bounded quantifiers (“$\forall x \in a$” rather than “$\forall x$”)

Proof (Sketch).

1. BZC⇒ETCS: The category of sets and functions in BZC satisfies ETCS.
2. ETCS⇒BZC: The collection of “pointed well-founded trees” in an ETCS-category satisfies BZC. □

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NB: This has nothing to do with the "internal logic" (yet)!

Why do we care?

Why is this equivalence useful?

1. We can use either one as a foundation for mathematics, according to taste.
2. Some constructions are easier or more intuitive in one context or the other.

Material World		Structural World
forcing	$\leftrightarrow$	sheaves
ultrapowers	$\leftrightarrow$	filterquotients
realizability models	$\leftrightarrow$	realizability toposes
??	$\leftrightarrow$	free toposes
??	$\leftrightarrow$	glued toposes
??	$\leftrightarrow$	exact completion
$V = L$	$\leftrightarrow$	??

Outline

The strong axioms

What's missing from BZC?

- Unbounded Separation: For any set a and formula $\varphi(x)$, the set $\{x \in a \mid \varphi(x)\}$ exists.

  NB: "Unbounded" means "not bounded by a set." That is, we allow quantifiers such as "for all sets" rather than "for all *elements* of the set a."

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- Replacement/Collection:

$$\forall a.\ (\forall x \in a.\ \exists y.\ \varphi(x, y)) \Rightarrow$$
$$\exists b.\ (\forall x \in a.\ \exists y \in b.\ \varphi(x, y)) \wedge (\forall y \in b.\ \exists x \in a.\ \varphi(x, y))$$

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  What that really means: If for every $x \in a$ there exists a y_x such that some property $\varphi(x, y_x)$ holds, then we can collect these y's into a family of sets $b = \{y_x\}_{x\in a}$.

Why unbounded separation?

- Any universal mapping property in a large category involves unbounded quantifiers. E.g. if $\{F_x\}_{x \in a}$ is a family of diagrams $F_x \colon D \to C$, then forming

$$\{x \in a \mid F_x \text{ has a limit}\}$$

  uses unbounded separation.
- To justify mathematical induction for some property φ, we form the set $\{n \mid \varphi(n)\}$ and show that it is all of $\mathbb{N}$. This only works if we have separation for φ.

Why replacement/collection?

- To construct objects recursively, such as free algebras: for an operation T and a starting point X, we consider the sequence

$$X, TX, T^2X, T^3X, \ldots$$

  BZC can construct each T^nX, but not the entire sequence or its limit.
- In particular, cardinal numbers above $\aleph_\omega$ are unreachable without replacement/collection.
- In BZC we have to be careful when saying "**Set** is (co)complete." E.g. the family $\{P^n(\mathbb{N})\}_{n\in\mathbb{N}}$ may not have a coproduct (so we have to exclude it from the notion of "family").
- Even some "concrete" facts depend on replacement/collection, e.g. Borel Determinacy.

Structural quantifiers

- In material set theory, unbounded quantifiers are very natural, while bounded ones feel like an *ad hoc* restriction.
- In structural set theory, there are fundamentally two *kinds* of quantifier: those that range over *elements*, such as $x\colon 1 \to A$, and those that range over *sets*, such as A itself. The former are "bounded," the latter "unbounded."

(Functions $f\colon A \to B$ are equivalent to elements $1 \to B^A$, hence their quantifiers are also "bounded.")

This is not a problem yet, but it will return...

Structural strong axioms

- Structural Unbounded Separation: For any set A and formula $\varphi(x)$, there is a monic $[\![\varphi]\!] \rightarrowtail A$ such that for any $x\colon 1 \to A$, x factors through $[\![\varphi]\!]$ iff $\varphi(x)$ holds.

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- **Structural Collection:** For any set A and formula $\varphi(x, Y)$, if for every $x\colon 1 \to A$ there is a set Y with $\varphi(x, Y)$, then there is a $B \in \mathbf{Set}/A$ such that for every $x \in A$ we have $\varphi(x, x^*B)$.

$$\begin{array}{ccc} Y_x & \longrightarrow & B \\ \downarrow & \lrcorner & \downarrow \\ 1 & \xrightarrow[x]{} & A \end{array}$$

(In the absence of choice, we have to pass to a cover of A.)

Comparing strong axioms

Theorem (Osius, McLarty)

"ETCS + the structural strong axioms" is equivalent to ZFC.

Corollary

Any part of mathematics that can be developed in ZFC can also be developed in ETCS + SSA.

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So what's the problem?

Outline

Internalization

Problem 1

- A structural set theory must be a *well-pointed* topos (sets are determined by their “elements” $1 \to A$).
- But: Most constructions on toposes (sheaves, realizability, gluing) don’t preserve well-pointedness!

Internalization

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- But: Most constructions on toposes (sheaves, realizability, gluing) don’t preserve well-pointedness!

Solution

- Every topos (including non-well-pointed ones) has an internal logic or Mitchell-Bénabou language.
- When we say that (for instance) forcing in material set theory corresponds to sheaves in structural set theory, we mean that the *internal logic* of the topos of sheaves is the same as the logic of a forcing model.

Review of the internal logic

Idea

For any formula $\varphi(x)$, we can define a subobject $[\![\varphi]\!] \rightarrowtail A$, which we think of as the "set" $\{x \in A \mid \varphi(x)\}$.

Construction

We construct $[\![\varphi]\!]$ by mirroring the logical connectives with topos-theoretic structure. For instance:

"and"	$\leftrightarrow$	intersection of subobjects
"or"	$\leftrightarrow$	union of subobjects
"not"	$\leftrightarrow$	complement of subobjects

If φ has no free variables, it is valid if $[\![\varphi]\!] \rightarrowtail 1$ is all of 1.

Review of the internal logic

For quantifiers, we have

"there exists"	$\leftrightarrow$	image along projection
"for all"	$\leftrightarrow$	dual image along projection.

That is, $[\![\exists y \in B.\ \varphi(x,y)]\!] \rightarrowtail A$ is defined to be the image of the composite

$$[\![\varphi(x,y)]\!] \rightarrowtail A \times B \to A.$$

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This leads to...

Problem 2

- The internal logic only understands bounded quantifiers, since only they have a projection map.
- But: stating the strong axioms requires unbounded quantifiers!

Some possible solutions

1. Assume completeness or cocompleteness with respect to some external set theory, build a hierarchy of sets $V_\alpha = P(\sum_{\beta<\alpha} V_\beta)$, and interpret unbounded quantifiers as ranging over elements of these (Fourman 1980).
2. Introduce a "category of classes" containing the class of all sets as an object, so that unbounded quantifiers over sets become bounded quantifiers in the internal logic of the category of classes ("Algebraic set theory," Joyal-Moerdijk 1995 etc.).
3. Assert directly that we can perform recursive constructions for some externally described class of iterative operations (Taylor 1999).

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3. Assert directly that we can perform recursive constructions for some externally described class of iterative operations (Taylor 1999).
4. Extend the internal logic so that it can speak about quantifiers over objects.

Outline

Kripke-Joyal semantics

aka Yoneda-fication of the internal logic

A subobject $[\![\varphi]\!] \rightarrowtail A$ is determined by knowing the morphisms $U \to A$ which factor though it. That is, by the subfunctor $\mathbf{S}(-, [\![\varphi]\!])$ of the representable $\mathbf{S}(-, A)$.

Definition

Given $a \colon U \to A$, we say U **forces** $\varphi(a)$, written $U \Vdash \varphi(a)$, if a factors through $[\![\varphi]\!]$.

We think of $U \Vdash \varphi(a)$ as saying "$\varphi(a(u))$ holds for all $u \in U$" or "φ holds at the generalized element $a \in_U A$."

Note: If φ has no free variables, it is valid if and only if $1 \Vdash \varphi$.

Kripke-Joyal semantics

The inductive construction of $[\![\varphi]\!]$ translates directly into inductive properties of forcing. For example:

$$U \Vdash (\varphi(a) \wedge \psi(a)) \iff U \Vdash \varphi(a) \text{ and } U \Vdash \psi(a)$$

$$U \Vdash (\varphi(a) \vee \psi(a)) \iff U = V \cup W, \text{ where } V \Vdash \varphi(a) \text{ and } W \Vdash \psi(a).$$

$$U \Vdash (\exists y \in B.\ \varphi(x,y)) \iff \text{there exist } p\colon V \twoheadrightarrow U \text{ and } b\colon V \to B \text{ such that } V \Vdash \varphi(pa, b).$$

In this way we can describe the subfunctor $\mathbf{S}(-, [\![\varphi]\!])$ directly. The previous construction of $[\![\varphi]\!]$ then becomes a *proof* that this functor is representable.

The stack semantics

We generalize Kripke-Joyal semantics to formulas in the language of categories.

Before we had maps $a\colon U \to A$, regarded as generalized elements of A parametrized by U. Now we need a notion of "generalized object of $\mathbf{S}$ parametrized by U," for which we use simply objects of $\mathbf{S}/U$.

Thus, we consider formulas φ in the language of categories with parameters in $\mathbf{S}/U$, and define the relation "$U \Vdash \varphi$" by precise analogy with Kripke-Joyal semantics. For instance:

$$U \Vdash (\exists Y.\, \varphi(x, Y)) \iff \text{there exist } p\colon V \twoheadrightarrow U \text{ and } B \in \mathbf{S}/V \text{ such that } V \Vdash p^*\varphi(B).$$

Here $p^*\varphi$ denotes the result of pulling back all parameters along $p\colon V \to U$, giving a formula in $\mathbf{S}/V$.

We say φ is valid if $1 \Vdash \varphi$.

Comparison to internal logic

For any formula φ in the internal language, we can define a formula $\hat{\varphi}$ in the language of categories, by replacing every variable x of type A by an arrow-variable $x \colon 1 \to A$.

Theorem

Let $\varphi(x)$ be a formula in the usual internal logic, with x of type A. Then for any $a \colon U \to A$, we have

$$U \Vdash \varphi(a) \quad \Longleftrightarrow \quad U \Vdash U^* \hat{\varphi}(a)$$

(Kripke-Joyal) *(stack semantics)*

(On the right-hand side, we regard a as $1_U \to U^* A$ in $\mathbf{S}/U$.)

Therefore, the stack semantics generalizes the internal logic. (Actually, this generalization is at least implicit in many places in the literature.)

Classifying objects

Definition

For a formula φ in $\mathbf{S}/U$, a classifying object for φ is a monic $[\![\varphi]\!] \rightarrowtail U$ such that for any $p\colon V \to U$, we have

$$V \Vdash p^*\varphi \iff p \text{ factors through } [\![\varphi]\!].$$

Corollary

Every bounded formula (having only quantifiers over elements $1 \to A$) has a classifying object, and $[\![\hat{\varphi}]\!] = [\![\varphi]\!]$.

But we don't *need* classifying objects in order to discuss validity in the stack semantics.

Outline

Soundness

Theorem

The stack semantics is sound for intuitionistic reasoning. That is, if $U \Vdash \varphi$ and we can prove that φ implies ψ with intuitionistic reasoning, then necessarily $U \Vdash \psi$.

Proof.

Induction on formulas, as usual. □

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The real question: what does $U \Vdash \varphi$ mean in terms of **S**?

It reduces to the internal logic when φ is bounded, but what about when φ contains unbounded quantifiers?

Universal properties

Theorem

If φ asserts some universal property, then $1 \Vdash \varphi$ iff that universal property is true in all slice categories $\mathbf{S}/U$ and is preserved by pullback.

Examples

- $1 \Vdash$ "P is a product of A and B" iff P is, in fact, a product of A and B, since products are preserved by pullbacks.
- $1 \Vdash$ "$\mathbf{S}$ is a topos" is always true, since each $\mathbf{S}/U$ is a topos and each $f^* : \mathbf{S}/U \to \mathbf{S}/V$ is logical.
- $1 \Vdash$ "$\mathbf{S}$ has a natural numbers object" iff $\mathbf{S}$ in fact has a NNO.

Non-universal properties

For other types of formulas φ, the meaning of $1 \Vdash \varphi$ can be quite different. For example:

- $1 \Vdash$ "A is projective" $\iff$ A is *internally* projective.
- $1 \Vdash$ "All epimorphisms split (AC)" $\iff$ $\mathbf{S}$ satisfies the *internal* axiom of choice (IAC).
- $1 \Vdash$ "$\mathbf{S}$ is Boolean" $\iff$ $\mathbf{S}$ is Boolean.
- $1 \Vdash$ "$\mathbf{S}$ is two-valued" $\iff$ $\mathbf{S}$ is Boolean.

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- $1 \Vdash$ "$\mathbf{S}$ is two-valued" $\iff$ $\mathbf{S}$ is Boolean.
- $1 \Vdash$ "$\mathbf{S}$ is well-pointed" ... always!

In particular, the stack semantics of any topos models a structural set theory. If **S** satisfies IAC and has a NNO, then its stack semantics models ETCS; otherwise it models an intuitionistic structural set theory.

Outline

Collection revisited

Theorem

The structural axiom of Collection is always validated by the stack semantics of any topos.

Idea of Proof.

Collection says that if for all $x \in A$, we have a Y with $\varphi(x, Y)$, then we have a family $\{Y_x\}_{x \in A}$ such that $\varphi(x, Y_x)$ for all x. But in the stack semantics (as in the internal logic), quantifiers over "elements" $x \in A$ actually range over all *generalized* elements $U \to A$, including the universal one $1_A \colon A \to A$. And saying that there exists a Y with $\varphi(x, Y)$, for $x = 1_A$, is essentially the desired conclusion. □

(Similar facts about forcing semantics have been observed elsewhere, e.g. Awodey-Butz-Simpson-Streicher.)

Separation revisited

Theorem

The following are equivalent.

1. $1 \Vdash$ *"**S** satisfies structural Unbounded Separation".*
2. *Every formula has a classifying object.*
3. *(If **S** is well-pointed) **S** satisfies the structural axioms of Separation and Collection.*

Definition

If **S** satisfies the above properties, we call it autological: it can describe itself (its stack semantics) in terms of its own logic (the subobjects $[\![\varphi]\!]$).

Corollary

*If **S** is an autological topos with an NNO satisfying IAC, then its stack semantics models ETCS+SSA.*

Examples

Some autological toposes

- The topos of sets in any model of ZF (or IZF).
- Any Grothendieck topos (over an autological base).
- Any filterquotient of an autological Boolean topos.
- The gluing of two autological toposes along a “definable” lex functor.
- Realizability toposes, such as the effective topos.

NB: Being autological is an *elementary* property (albeit not a finitely axiomatizable one).

Independence proofs

Example

We can describe forcing models of material set theory in categorical language as follows.

1. Start with a material set theory, such as ZF.
2. Build its topos of sets, a structural set theory.
3. Pass to some topos of sheaves on some site.
4. The stack semantics of the topos of sheaves is again a structural set theory.
5. Reconstruct a material set theory using "well-founded trees" in this stack semantics.

The fact that toposes of sheaves remain autological ensures that strong axioms are preserved by this sequence.

Back to the motivating question

Question

What is the topos-theoretic counterpart of the strong set-theoretic axioms of Separation, Replacement, and Collection?

One Answer

The property of being autological, i.e. of satisfying the structural strong axioms in the stack semantics.