The meaning of synthetic topology

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Outline

The ubiquity of topology

A bold claim

Topology is not just one mathematical subject among many.
It is a foundational feature of all mathematics.

Old point of view	New point of view
Everything in math is a set.	Everything in math is a space.
A space is a set equipped with a topology.	A set is a space whose topology is discrete.
We give sets a topology when we think it would be useful.	Constructions on discrete spaces lead unavoidably to non-discrete ones.

Origins of topology

Topologies arise naturally, usually from universal constructions.

Example

Even if A and B are discrete, the function-space B^A may not be.

- $2^{\mathbb{N}}$ is Cantor space.
- $\mathbb{N}^{\mathbb{N}}$ is Baire space.
- $\mathbb{R}$ is Cauchy sequences (in $\mathbb{Q}^{\mathbb{N}}$) or Dedekind cuts (in $2^{\mathbb{Q}}$).
- The completion of a ring (formal power series, in $R^{\mathbb{N}}$) is local.
- The profinite completion (sequences of elements of quotients).
- The Zariski spectrum of a ring.

The times, they are a'changing

From *Interview with Yuri Manin* (by Mikhail Gelfand), AMS Notices, October 2009:

> [A]fter Cantor and Bourbaki . . . set theoretic mathematics resides in our brains. . . . we start with the discrete sets of Cantor, upon which we impose something more in the style of Bourbaki.
>
> But fundamental psychological changes also occur. Nowadays . . . the place of old forms and structures . . . is taken by some geometric, right-brain objects. Instead of sets, clouds of discrete elements, we envisage . . . spaces. . .
>
> . . . [T]here is an ongoing reversal in the collective consciousness of mathematicians: the. . . homotopical picture of the world becomes the basic intuition, and if you want to get a discrete set, then you pass to the set of connected components. . . Cantor's problems of the infinite recede to the background: from the very start, our images are so infinite that if you want to make something finite out of them, you must divide them by another infinity.

Outline

Analytic topology

Recall the claim

Topology is a foundational feature of all mathematics.

The lightweight approach

Define a notion of “space” that can be used in place of bare sets when doing most of mathematics.

Question

What’s wrong with ordinary topological spaces?

Function spaces

Problem #1

Topological spaces are not cartesian closed: there is no well-behaved "space of continuous functions" B^A.

In particular, there is no well-behaved "power-space" 2^A.

Traditional solution

For a category $\mathcal{C}$ of "test spaces", A is $\mathcal{C}$-generated if it has the final topology induced by all the continuous maps $X \to A$ for $X \in \mathcal{C}$.

$\mathcal{C} =$ compact Hausdorff spaces	$\rightsquigarrow$	compactly generated spaces
$\mathcal{C} = \{ \mathbb{R}^n \mid n \in \mathbb{N} \}$	$\rightsquigarrow$	Δ-generated spaces
$\mathcal{C} = \{ \mathbb{N}_\infty \}$	$\rightsquigarrow$	sequential spaces

All these categories of spaces are cartesian closed.

Continuous bijections

Problem #2

Not every continuous bijection is a homeomorphism.

That this is actually a problem is more easily seen in algebra.

Example

Let $\flat\mathbb{R}$ denote the real numbers with the discrete topology. Then $\flat\mathbb{R} \to \mathbb{R}$ is a noninvertible homomorphism of topological abelian groups with trivial kernel and cokernel.

Thus, topological abelian groups are not an abelian category.

All the cartesian closed categories of spaces have the same problem.

The stuff of topology

Idea

A $\mathcal{C}$-probed set is like a $\mathcal{C}$-generated space, but more than one "map" $X \to A$ can have the same underlying function on points.

Then $\mathbb{R}/\flat\mathbb{R}$ has one point, but nontrivial "topology".

Definition

Fix a Grothendieck topology on $\mathcal{C}$. A $\mathcal{C}$-probed set is a sheaf on $\mathcal{C}$: a functor $\mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$ satisfying "gluing" axioms.

For $A : \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$, have $A(X) =$ "the set of maps $X \to A$".

$\{\, \mathbb{R}^n \mid n \in \mathbb{N} \,\}$, open covers	$\rightsquigarrow$	Δ-topological sets
$\{\, \mathbb{N}_\infty \,\}$, canonical covers	$\rightsquigarrow$	consequential sets
compacta, finite closed covers	$\rightsquigarrow$	condensed sets*
$\{\, 2^{\mathbb{N}} \,\}$, finite closed covers	$\rightsquigarrow$	countably condensed sets

$\mathcal{C}$-generated spaces embed fully-faithfully in $\mathcal{C}$-probed sets.

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An intuitive topological topos

Arguably the easiest kind of $\mathcal{C}$-probed set to understand is:

Definition (∼Johnstone 1979)

A consequential set is a set A equipped with

- For each sequence $x_0, x_1, x_2, \ldots$ and point y, a set of "proofs that (x_n) converges to y" (possibly empty).

There must be specified proofs that:

1. Every constant sequence $x, x, x, x, \ldots$ converges to x.
2. If a sequence (x_n) converges to x, so does any subsequence.
3. If every subsequence of (x_n) contains a further subsequence converging to x, then (x_n) converges to x.

satisfying coherence axioms.

Origins of topology

Definition

In a discrete consequential set, the only witnesses of convergence are the specified ones that $x, x, x, x, \ldots$ converges to x.

(And trivial variants, like $y, x, x, x, \ldots$ converging to x.)

The discrete sets are an embedded copy of the category of sets, closed under colimits, but not limits or function-spaces.

Example

- If A and B are discrete, B^A has pointwise convergence.
- A limit of discrete sets, like the profinite completion, has componentwise convergence.
- $\mathbb{R}$, as Cauchy sequences or Dedekind cuts, automatically has its usual topology.

Exactness

- The monomorphisms of consequential sets are injective on points and proofs.
- The epimorphisms of consequential sets are surjective on points and induce a "quotient structure" on proofs.

Example

$\flat\mathbb{R} \to \mathbb{R}$ is mono but not epi.

Consequential sets are balanced: any monic epic is an iso.

Theorem

Consequential abelian groups are an abelian category.

In particular, $\flat\mathbb{R} \to \mathbb{R}$ is the kernel of the quotient map $\mathbb{R} \to \mathbb{R}/\flat\mathbb{R}$.

Outline

Synthetic topology

Recall the claim

Topology is a foundational feature of all mathematics.

To take this more seriously, we change the meanings of words so:

- "Set" means "space"
- "Function" means "continuous function"
- All constructions of "sets" automatically yield spaces.

Actually, in traditional formal foundations like ZFC, "set" is already an undefined term, given meaning by its axioms.

We just change the axioms a bit.

Synthetic topology

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Synthetic vs analytic

Theorem

There is a formal interpretation function that automatically "compiles" any mathematical argument into an internal statement in any topos, such as $\mathcal{C}$-probed sets.

By analogy:

Proofs in group theory	$\rightsquigarrow$	True in any group
Proofs in ring theory	$\rightsquigarrow$	True in any ring
Proofs in mathematics	$\rightsquigarrow$	True in any topos

Therefore:

- Even a ZFC devotee can reason about spaces synthetically.
- $\mathcal{C}$-probed sets can inform our choice of axioms for spaces.

An example of internal reasoning

Theorem

Any injective map of abelian groups is the kernel of its quotient.

Proof.

Given $\phi : H \hookrightarrow G$, the quotient G/H is the set of equivalence classes $[g] = \{ g' \mid g' - g \in H \}$. Thus, if $[g] = [0]$, then $g \in [0]$ and so $g - 0 = g \in H$. □

This is a proof in ordinary mathematics.

The internalizer compiles it to a proof that any monomorphism of abelian group objects in a topos is the kernel of its quotient.

But some things are weird

The axiom of choice is false in spaces.

Example

$\mathbb{R} \to \mathbb{S}^1$ is surjective, but has no (continuous) section.

So is the law of excluded middle ("for any P, either P or not-P").

Example

If we could claim "$x < 0$ or $x \geq 0$ for all $x \in \mathbb{R}$", then we could define a discontinuous function $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases}$$

Enter modalities

You might be surprised how much we can do without AC and LEM, but sometimes we really do need them.

Idea: use discreteness

The function

$$f(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases}$$

is continuous $\flat\mathbb{R} \to \mathbb{R}$, where $\flat\mathbb{R}$ is $\mathbb{R}$ retopologized discretely.

We augment mathematics with a modality $\flat$ that retopologizes any set discretely. We have $\flat\flat A = \flat A$, and a coercion $\flat A \to A$.

Using the modality

Now we can assume:

Discrete excluded middle

For any* property P of points $x \in A$, and any $x \in \flat A$, we have either $P(x)$ or not $P(x)$.

Discrete axiom of choice

For any* surjective $f : A \twoheadrightarrow B$, there is a map $s : \flat B \to A$ such that $f(s(x)) = x$ for any $x \in \flat B$.

We can also access other topological properties using the modality.

Definition

A set A is connected if any map from A to a discrete set $\flat B$ is constant, i.e. $(\flat B)^A \cong \flat B$.

For instance, $\mathbb{R}$ is connected.

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Synthetic topological proofs

Theorem (The Intermediate Value Theorem)

Let $f \in \flat(\mathbb{R}^{\mathbb{R}})$ and $c \in \flat\mathbb{R}$. If there are $a, b \in \mathbb{R}$ such that $f(a) < c < f(b)$, then there exists $x \in \mathbb{R}$ such that $f(x) = c$.

NB: x can't be chosen to vary continuously with f and c.

Proof.

Since $\flat(\mathbb{R}^{\mathbb{R}}) \times \flat\mathbb{R}$ is discrete, we can use LEM and hence proof by contradiction. Thus, suppose not; then we can define $g : \mathbb{R} \to 2$ by

$$g(x) = \begin{cases} 0 & \text{if } f(x) < c \\ 1 & \text{if } f(x) > c \end{cases}$$

Since 2 is discrete and $\mathbb{R}$ is connected, g is constant.
But $g(a) = 0$ and $g(b) = 1$, a contradiction. □

The point of synthetic topology

Conclusion

Synthetic topological foundations looks like ordinary mathematics, but we have to mark explicitly where things are not continuous, rather than explicitly introducing topology when needed.

This is good: continuity is more common than discontinuity.

We can also add other modalities:

- $\sharp A$ is A retopologized indiscretely: have $\flat \dashv \sharp$.
- $\int A$ is the shape or fundamental ∞-groupoid of A: have $\int \dashv \flat$.
- Smooth structure, super structure, etc. also have modalities.

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Outline

Classifying spaces

A classifying space is a representing object for some contravariant functor. For example:

$$\{\text{ vector bundles over } X \} \cong \mathrm{Map}(X, BO)$$
$$\{\text{ double covers of } X \} \cong \mathrm{Map}(X, B2).$$

Synthetically, we can define classifying spaces with just sets of sets!

$$B2 = \{ A \mid A \text{ has cardinality } 2 \}$$

An $f : X \to B2$ sends each $x \in X$ to the fiber over it; the total space of the double cover is

$$E_f = \{ (x, y) \mid y \in f(x) \}.$$

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Univalence, part 1

Suppose $f, g : X \to B2$ classify isomorphic double covers:

$$\begin{array}{ccc} E_f & \xrightarrow{\cong} & E_g \\ & \searrow \quad \swarrow & \\ & X & \end{array}$$

Then $f(x) \cong g(x)$ for all $x \in X$.

But for $B2$ to really be a classifying space, we need $f = g$, that is $f(x) = g(x)$ for all x. Thus we need:

Axiom (Univalence, version 1)

For two sets A, B, we have $A = B$ if and only if $A \cong B$.

This is a better sort of classifying space than in classical topology: f is literally unique, not just up to homotopy.

Trivial double covers

Any X has a trivial double cover $X \times 2$, classified by a constant function $f : X \to B2$. Conversely:

- By induction, any double cover of a finite set is trivial.
- By the discrete axiom of choice, any double cover of a discrete set is trivial.
- Since 2 is discrete and finite, $B2$ is also discrete. Thus all maps $\mathbb{R} \to B2$ are constant, so any double cover of $\mathbb{R}$ is also trivial.

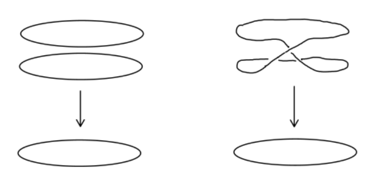
Nontrivial double covers

Example

We have $\mathbb{S}^1 = [0,1]/(0 \sim 1)$, so:

- $f : \mathbb{S}^1 \to B2$ consists of $g : [0,1] \to B2$ with $g(0) = g(1)$.
- $[0,1]$ is a retract of $\mathbb{R}$, so g is constant.
- But $g(0) = g(1)$ is an isomorphism $g(0) \cong g(1)$, which could be the identity or the "swap".

Thus, $\mathbb{S}^1$ has two double covers.



Univalence, part 2

For this to make sense, the construction of $f : \mathbb{S}^1 \to B2$ has to remember which proof of $g(0) = g(1)$ we gave.

Principle of proof-relevance

Any statement or theorem is represented by a space, which is nonempty just when that statement is true. For ordinary true statements, the set is just $*$, but more complicated statements can be "true in more than one way".

Axiom (Univalence, version 2)

For two sets A, B, the space $A = B$ is isomorphic to the set $\mathrm{Iso}(A, B)$ of isomorphisms between A and B.

Groupoids, higher groupoids, and stacks

For any space A and $a, b \in A$, we have another space $a = b$.
Then for $p, q \in (a = b)$, we have $p = q$, *ad infinitum*.

Each space is actually an ∞-groupoid.

Warning

If you know what an ∞-groupoid is, you might be used to thinking of them as "like topological spaces". For us, the ∞-groupoid "direction" is orthogonal to the "topological" one.

- $\mathbb{S}^1$ has a topological loop, but has no groupoid structure.
- $B2$ is topologically discrete, but is a groupoid, $(2 = 2) \cong 2$.

Analytically, a $\mathcal{C}$-probed ∞-groupoid is a stack on $\mathcal{C}$.

Connectedness and contractibility

- We said A is "connected" if any map $A \to B$ is constant, for discrete B.
- But we found a non-constant map $\mathbb{S}^1 \to B2$, where $B2$ is discrete. Thus $\mathbb{S}^1$ is not "connected"!

We need better terminology.

Definition

A space A is connected if any map $A \to B$ is constant when B is discrete and *0-truncated* (has no higher groupoid structure).

Definition

A space A is contractible if any map $A \to B$ is constant, for *any* discrete B.

Now $\mathbb{S}^1$ is connected but not contractible, while $\mathbb{R}$ is contractible.

Vector bundles

The space

$$B\mathsf{Vect} = \{\ V \mid V \text{ is a vector space}\ \}$$

classifies "continuous families of vector spaces", but $f : X \to B\mathsf{Vect}$ need not be "locally constant", as required for a bundle.

A better classifying space of vector bundles is $\flat B\mathsf{Vect}$.

Example

Any vector bundle over a contractible space is trivial.

Example

Over $\mathbb{S}^1$, there is a 1-dimensional Möbius bundle, constructed like the nontrivial double cover.

Outline

A problem

"Theorem"

All spaces are discrete: $\flat A \cong A$ for all A.

Proof.

For the one-point space $*$, clearly $\flat * = *$.
For any $x \in A$ we have $f_x : * \to A$, hence $\flat f_x : \flat * = * \to \flat A$.
Thus $x \mapsto \flat f_x(*)$ is a map $A \to \flat A$. □

To avoid this, we have to forbid using $\flat$ in the "context" of $x \in A$ when A is non-discrete.

A brief introduction to type theory

Type theory lets us formalize this. At any point in an argument there is a context consisting of the variables currently assumed, along with the sets they belong to, e.g.

$$a \in A,\ b \in B,\ c \in C$$

Certain constructions extend the context. For instance, to construct a function $X \to Y$ we may assume some $x \in X$, extending the context to

$$a \in A,\ b \in B,\ c \in C,\ x \in X$$

and proceed to construct an element $y \in Y$ in this context, to be the value of $f(x)$.

Modal type theory

To handle $\flat$, we make a new way to extend the context: with a lock.

$$a \in A,\ b \in B,\ c \in C,\ 🔒$$

A variable occurring “behind a lock” can only be used if it belongs to a discrete set. This allows rules for $\flat$, like those for functions:

- To construct a function $X \to Y$, extend the context by $x \in X$ and construct an element of Y.
- To construct an element of $\flat X$, extend the context by 🔒 and construct an element of X.

This prevents the problematic “for any $x \in A$ we have $f_x : * \to A$, hence $\flat f_x : \flat * = * \to \flat A$” because x is behind a lock, hence unusable in $\flat f_x$ as A is not discrete.

Semantics of dependent type theory

Recall the compiler that interprets mathematics into toposes.

Syntactic structure	Categorical structure
Context $a \in A,\ b \in B$	Object $A \times B$
Element in that context $x \in X$ given $a \in A,\ b \in B$	Morphism $A \times B \to X$

The rule for constructing functions thus becomes the universal property of an exponential object:

$$\frac{f \in Y^X \text{ in context } a \in A, b \in B}{y \in Y \text{ in context } a \in A, b \in B, x \in X} \qquad \frac{A \times B \to Y^X}{A \times B \times X \to Y}$$

Semantics of modal dependent type theory

But what does 🔒 mean categorically?

$$\frac{x \in \flat X \text{ in context } a \in A, b \in B}{x \in X \text{ in context } a \in A, b \in B, 🔒} \qquad \frac{A \times B \to \flat X}{? \to X}$$

This looks like it should be a left adjoint to $\flat$.

Definition (Gratzer–Kavvos–Nuyts–Birkedal)

For any 2-category $\mathcal{M}$, there is a modal dependent type theory called MTT, using a lock $🔒_\mu$ that looks left adjoint to the modality associated to each morphism μ of $\mathcal{M}$.

Theorem (S.)

For any suitable diagram $\mathcal{C} : \mathcal{M} \to \mathrm{Cat}$, there is a diagram $\widehat{\mathcal{C}}$ with additional left adjoints $🔒_\mu$, and a compiler interpreting MTT into it.

Semantics of modal dependent type theory

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$$\frac{x \in \flat X \text{ in context } a \in A, b \in B}{x \in X \text{ in context } a \in A, b \in B, \text{🔒}} \qquad \frac{A \times B \to \flat X}{\textbf{?} \to X}$$

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The meaning of synthetic topology

Conclusion

There is a general syntactic way to extend mathematics with modalities such as $\flat$, and a compiler interpreting them into toposes.

Therefore:

- ZFC devotees can reason synthetically using $\flat$.
- Toposes can inform our choice of axioms for spaces with $\flat$.

Further reading:

- M. Shulman, Homotopy type theory: the logic of space, arXiv:1703.03007
- M. Shulman, Semantics of multimodal adjoint type theory, arXiv:2303.02572